INTERIOR CURVATURE ESTIMATES FOR HYPERSURFACES OF PRESCRIBING SCALAR CURVATURE IN DIMENSION THREE

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Abstract. We prove a priori interior curvature estimates for hypersurfaces of prescribing scalar curvature equations in $\mathbb{R}^3$. The method is motivated by the integral method of Warren and Yuan in [24]. The new observation here is that we construct a "Lagrangian" graph which is a submanifold of bounded mean curvature if the graph function of a hypersurface satisfies a scalar curvature equation.

1. Introduction

We study the regularity thoery of a hypersurface $M^n \subseteq \mathbb{R}^{n+1}$ with positive scalar curvature. In hypersurface geometry, the Gauss equation tells us

$$R_g = \sigma_2(\kappa) := \sum_{1\leq i_1<i_2\leq n} \lambda_{i_1}\lambda_{i_2}$$

where $\kappa(x) = (\lambda_1(x), \cdots, \lambda_n(x))$ are principal curvatures of the hypersurface.

Suppose M^n is a C^1 graph $X = (x, u(x))$ over $x \in B_r \subseteq \mathbb{R}^n$. In this setting, the scalar curvature equation which we study is

$$\sigma_2(\kappa(x)) = f(X(x), \nu(x)) > 0 \tag{1.1}$$

where ν is a normal of the given hypergraph over a ball $B_r \subset \mathbb{R}^n$. This is a second order elliptic PDE depending on graph function u. In dimension 2, it is Monge-Ampere equation

$$\det(u_{ij}) = f(x, u, \nabla u). \tag{1.2}$$

Our study of the scalar curvature equation is motivated by isometric embedding problems. A famous isometric embedding problem is Weyl problem. The problem of realizing, in three-dimensional Euclidean space, a regular metric of positive curvature given on a sphere. The Weyl problem was finally solved by Nirenberg [17] and Pogorelov [18] independently. Their solving the problem of Weyl by a continuity method where obtaining C^2 estimate to the scalar curvature equation is important to the method.

Motivated by the Weyl problem, E. Heinz [11] derived a purely interior estimate for the equation (1.2) in dimension two. If u satisfies the equation (1.2) in $B_r \subseteq \mathbb{R}^2$ with positive f, then

$$\sup_{B_{\frac{r}{2}}}|D^2u| \leq C(|u|_{C^1(B_r)}, |f|_{C^2(B_r)}, \inf_{B_r} f). \tag{1.3}$$

And this type of estimate turns out to be very useful when one study the isometric embedding problem for surfaces with boundary or for non-compact surfaces. But

Heinz's interior C^2 estimate is false when $n \geq 3$ by Pogorelov [19] even for the convex solutions to the equation $\det D^2 u = 1$.

The second motivation is from the studying of fully nonlinear partial differential equation theory itself. Caffarelli-Nirenberg-Spruck started to study σ_k-Hessian operators and established existence of Dirichlet problem for σ_k equations in their seminal work [2]. Here the σ_k-Hessian operators are $k-$th elementary symmetric functions for $1 \leq k \leq n$. The key to the existence of Dirichlet problem is by establishing the following C^2 estimates

$$\sup_{\bar{\Omega}} |D^2 u| \leq C(|u|_{C^1(\bar{\Omega})}, f, \varphi, \partial\Omega).$$

Although there are C^2 estimates to σ_k-Hessian equations for boundary value problems, there are no interior C^2 estimates to σ_k-Hessian equations in general. Because Pogorelov's counter-examples were extended by J. Urbas in [23] to $k \geq 3$. The best we can expect is the Pogorelov type interior C^2 estimates with homogeneous boundary data which were derived in [19, 5]. So the interior regularities for solutions to the following σ_2-Hessian equations

$$\sigma_2(D^2 u) \quad = \quad f(x, u, Du) > 0 \tag{1.4}$$

and prescribing scalar curvature equations

$$\sigma_2(\kappa(x)) \quad = \quad f(X(x), \nu(x)) > 0$$

are longstanding problems.

A major breakthrough was made by Warren-Yuan [24]. In $\mathbb{R}^3$, they obtained C^2 interior estimate for the equation

$$\sigma_2(D^2 u) \quad = \quad 1. \tag{1.5}$$

Recently in [15], McGonagle-Song-Yuan proved interior C^2 estimate for convex solutions of the above equation in any dimensions. Using a different argument, Guan-Qiu [7] proved the same estimates for more general equations (1.4) and (1.1) with certain convexity constraints. Moreover, we proved interior curvature estimate for isometrically immersed hypersurfaces in $\mathbb{R}^{n+1}$ with positive scalar curvature in [7].

In this paper, we completely solve this problem for scalar equations in dimension three.

Theorem 1. *Suppose M is a smooth graph over $B_{10} \subset \mathbb{R}^3$ with positive scalar curvature. It is a solution of equation (1.1). Then we have*

$$\sup_{x \in B_{\frac{1}{2}}} |\kappa(x)| \leq C \tag{1.6}$$

where C depends only on $||M||_{C^1(B_{10})}$, $\|f\|_{C^2(B_{10}\times\mathbb{S}^2)}$ and $\|\frac{1}{f}\|_{L^\infty(B_{10}\times\mathbb{S}^2)}$.

Analogously we proved the interior C^2 estimates to sigma-2 equations (1.4) in a recent paper [20].

Theorem 2. [20] *Let u be a solution to (1.4) on $B_{10} \subset \mathbb{R}^3$. Then we have*

$$\sup_{B_{\frac{1}{2}}} |D^2 u| \leq C \tag{1.7}$$

where C depends only on $\|f\|_{C^2(B_{10}\times\mathbb{R}\times\mathbb{R}^3)}$, $\|\frac{1}{f}\|_{L^\infty(B_{10}\times\mathbb{R}\times\mathbb{R}^3)}$ and $||u||_{C^1(B_{10})}$.

In order to introduce our idea, let us briefly review the ideas for attacking these problems so far. In two dimensional case, Heinz used Uniformization theorem to transform this interior estimate for Monge-Ampere equation into the regularity of an elliptic system and univalent of this mapping, see also [10, 14] for more details. Another interesting proof using only maximum principle was given by Chen-Han-Ou in [4]. Our new quantity in [7] can also give a new proof of Heinz. The restriction for this method is that we need some convexity conditions which are not the case in the higher dimensions.

In $\mathbb{R}^3$, a key observation made in [24] is that the equation (1.5) is exactly a special Lagrangian equation which stems from the special Lagrangian geometry [9]. And an important property for the special Lagrangian equation is that a Lagrangian graph $(x, Du) \subset \mathbb{R}^3 \times \mathbb{R}^3$ is a minimal submanifold which has mean value inequality and sobolev inequality. So Warren and Yuan proved interior C^1 estimate for the special Lagrangian submanifold which in turn proved interior C^2 estimate for the special Lagrangian equation. Our new observation is that the graph (X, ν), where X is a position vector of a hypersurface satisfying the equation (1.4), can be viewed as a submanifold in $\mathbb{R}^4 \times \mathbb{R}^4$ with bounded mean curvature. Then applying a similar argument of Michael-Simon [16], see also Hoffman-Spruck [12], we have a mean value inequality in order to remove the convexity condition in [7]. Finally, we apply a modified argument of Warren-Yuan in [24] to get the estimate.

At last, we remark that the arguments are higher co-dimensional analogous to the original integral proof by Bombieri-De Giorgi-Miranda [1] for the gradient estimate for co-dimension one minimal graph and by Ladyzhenskaya and Ural'Tseva [13] for general prescribed mean curvature equations. Here we use a method similar to Trudinger's simplified proof in [21, 22], see also Chapter 16 of the book [6].

The higher dimensional cases for these equations are still open to us.

2. Preliminary Lemmas

We first introduce some definitions and notations.

Definition 3. For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, the k- th elementary symmetric function $\sigma_k(\lambda)$ is defined as

$$\sigma_k(\lambda) := \sum_{1 \le i_1 < i_2 \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

We also define the linearized operator of σ_k to be

$$\sigma_k^{ii} \quad := \quad \frac{\partial \sigma_k(\lambda)}{\partial \lambda_i}.$$

These definitions can be extended to symmetric matrices where $\lambda = (\lambda_1, \cdots, \lambda_n)$ are the corresponding eigenvalues of the symmetric matrices.

For example, in $\mathbb{R}^3$

$$\sigma_2(D^2u) \quad := \quad \sigma_2(\lambda(D^2u)) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

or

$$\sigma_2(D^2u) \quad := \quad \frac{(u_{11} + u_{22} + u_{33})^2 - u_{11}^2 - u_{22}^2 - u_{33}^2 - u_{12}^2 - u_{13}^2 - u_{23}^2}{2}.$$

Definition 4. For $1 \le k \le n$, we denote Γ_k by

$$\Gamma_k := \{\lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}.$$

The following algebraic lemma is from Lemma 2.1 in [?].

Lemma 5. *Suppose $\lambda \in \Gamma_2$. Then there is a constant $c > 0$ depending only on n such that for any i from 1 to n*

$$\sigma_2^{ii}(\lambda) \geq \frac{c\sigma_2(\lambda)}{\sigma_1(\lambda)}. \tag{2.1}$$

If $\lambda_1 \geq \cdots \geq \lambda_n$, then there exist $c_1 > 0$ and $c_2 > 0$ depending only on n such that

$$\sigma_2^{11}(\lambda)\lambda_1 \geq c_1\sigma_2(\lambda) \tag{2.2}$$

and for any $j \geq 2$

$$\sigma_2^{jj}(\lambda) \geq c_2\sigma_1(\lambda). \tag{2.3}$$

Proof. For our purpose, we only give a proof in dimension 3. It is not hard to see that (2.1) follows from (2.2) and (2.3).

First we claim that if $\lambda \in \Gamma_2$, then there is

$$\sigma_2^{33} \geq \sigma_2^{22} \geq \sigma_2^{11} \quad > \quad 0.$$

From $\lambda_1 \geq \lambda_2 \geq \lambda_3$, it is obvious that

$$\sigma_2^{33} \geq \sigma_2^{22} \geq \sigma_2^{11}.$$

In $\mathbb{R}^3$, we have

$$\sigma_2^{11}\sigma_2^{22} \quad = \quad (\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) = \lambda_3^2 + \sigma_2 > 0$$

Combining $\sigma_2^{11} + \sigma_2^{22} + \sigma_2^{33} = 2\sigma_1 > 0$, we obtain

$$\sigma_2^{33} \geq \sigma_2^{22} \geq \sigma_2^{11} \quad > \quad 0.$$

For (2.2), we consider two cases.

Case 1: $-\lambda_2\lambda_3 \geq 0$. It is easy to see

$$(\lambda_2 + \lambda_3)\lambda_1 \quad = \quad \sigma_2 - \lambda_2\lambda_3 \geq \sigma_2.$$

Case 2: $\lambda_2\lambda_3 \geq 0$. We have

$$\lambda_2\lambda_3 \quad \leq \quad \frac{(\lambda_2 + \lambda_3)^2}{4}.$$

Because of $\lambda_2 + \lambda_3 > 0$, we see that

$$\lambda_2\lambda_3 \quad \leq \quad \frac{\lambda_1(\lambda_2 + \lambda_3)}{2}.$$

Then there is

$$\begin{aligned}\sigma_2 = (\lambda_2 + \lambda_3)\lambda_1 + \lambda_2\lambda_3 \quad &\leq \quad \frac{3\lambda_1(\lambda_2 + \lambda_3)}{2} \\ &= \quad \frac{3}{2}\sigma_2^{11}\lambda_1.\end{aligned}$$

So we have proved (2.2) for $c_1 = \frac{2}{3}$.

For (2.3), we only need to show $\lambda_1 + \lambda_3 \geq c_2\sigma_1$. We also divide into the following two cases.

Case 1: $\lambda_3 \geq 0$. This is obvious.

Case 2: $\lambda_3 < 0$. From $\sigma_2 \geq 0$, we have

$$\begin{aligned}-\lambda_3^2 \quad &\geq \quad -(\lambda_2 + \lambda_3)(\lambda_1 + \lambda_3) \\ &\geq \quad -(\lambda_1 + \lambda_3)^2.\end{aligned}$$

The above inequality implies

$$\lambda_1 + 2\lambda_3 \quad \geq \quad 0.$$

Thus we prove the inequality (2.3) for $c_2 = \frac{1}{3}$. □

For scalar curvature equation (1.1) with positive scalar curvature, we may assume that M is admissible in the following definition without loss of generality.

Definition 6. A C^2 hypersurface M is called admissible if at every point $X \in M$ its principal curvature satisfies

$$\kappa \in \Gamma_2.$$

Moreover, it follows from Lemma 5 that $\sigma_2^{ij} := \frac{\partial \sigma_2(\lambda(h_{ij}))}{\partial h_{ij}}$ is positive definite when $\lambda(h_{ij}) \in \Gamma_2$.

So the curvature estimates can be reduced to the estimate of mean curvature H due to the following fact

$$\max|\lambda_i| \leq H = \sigma_1(\kappa). \tag{2.4}$$

In the rest of this article, we will denote C to be universal constants under control (depending only on $\|f\|_{C^2}$, $\|\frac{1}{f}\|_{L^\infty}$ and $\|M\|_{C^1}$) which may change line by line.

Suppose that a hypersurface M in $\mathbb{R}^{n+1}$ can be written as a graph over $B_r \subseteq \mathbb{R}^n$. At any point of $x \in B_1$, the principal curvature $\kappa = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of the graph $M = (x, u(x))$ satisfy a equation

$$\sigma_2(\kappa) = f(X, \nu) > 0 \tag{2.5}$$

where X is the position vector of M, and ν a normal vector on M.

Sometimes we may choose an orthonormal frame $\{e_1, e_2, \cdots, e_n, \nu\}$ in $\mathbb{R}^{n+1}$. Denote ν be a normal on M such that $H > 0$. We collect the following fundamental formulas of a hypersurface in $\mathbb{R}^{n+1}$:

$$\begin{aligned} X_{ij} &= -h_{ij}\nu \quad (Gauss\ formula) \\ \nu_i &= h_{ij}e_j \quad (Weingarten\ formula) \\ h_{ijk} &= h_{ikj} \quad (Codazzi\ equation) \\ R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} \quad (Gauss\ equation) \end{aligned}$$

where R_{ijkl} is the curvature tensor. We also have the following commutator formula:

$$h_{ijkl} - h_{ijlk} \quad = \quad h_{im}R_{mjkl} + h_{mj}R_{mikl}. \tag{2.6}$$

Combining Codazzi equation, Gauss equation and (2.6), we have

$$h_{iikk} \quad = \quad h_{kkii} + \sum_m (h_{im}h_{mi}h_{kk} - h_{mk}^2 h_{ii}). \tag{2.7}$$

Lemma 7. *Suppose the scalar curvature of hypersurface M satisfies equation (2.5) in $\mathbb{R}^n$. In orthonormal coordinate, we have the following equations*

$$\sigma_2^{kl}h_{kli} = \nabla f(e_i) \tag{2.8}$$

and

$$\begin{aligned} &\sigma_2^{kl}h_{iikl} + \sum_{k\neq l} h_{kki}h_{lli} - \sum_{k\neq l} h_{kli}h_{kli} \\ &-2f\sum_k h_{ki}^2 + (f\sigma_1 - 3\sigma_3)h_{ii} \quad = \quad \nabla^2 f(e_i, e_i). \end{aligned} \tag{2.9}$$

If $f = f(X,\nu)$, then there are estimates

$$|\nabla f| \le C(1+H) \tag{2.10}$$

and

$$-C(1+H)^2 + \sum_k h_{ij}^k d_\nu f(e_k) \le \nabla^2 f(e_i,e_j) \le C(1+H)^2 + \sum_k h_{ij}^k d_\nu f(e_k) \tag{2.11}$$

where C depends on $\|f\|_{C^2}$,$\|M\|_{C^1}$.

Proof. Taking first and second derivatives of the equation $\sigma_2(\kappa) = f$, we get (2.8) and

$$\sigma_2^{kl} h_{klii} + \sum_{k\neq l} h_{kki}h_{lli} - \sum_{k\neq l} h_{kli}h_{kli} = \nabla^2 f(e_i,e_i).$$

Using (2.7), we have

$$\sigma_2^{kl} h_{klii} \quad = \quad \sigma_2^{kl} h_{iikl} - \sum_m \sigma_2^{kl}(h_{im}h_{mi}h_{kl} - h_{mk}h_{ml}h_{ii}).$$

Then we obtain (2.9) by the following elementary identities

$$\sigma_2^{kl} h_{kl} = 2f \tag{2.12}$$

and

$$\sum_m \sigma_2^{kl} h_{mk}h_{ml} = \sigma_1\sigma_2 - 3\sigma_3. \tag{2.13}$$

Through direct computations using (Gauss formula) and (Weingarten formula), we have

$$\nabla f(e_i) = d_X f(e_i) + h_i^k d_\nu f(e_k)$$

and

$$\begin{aligned} \nabla^2 f(e_i,e_j) \quad = \quad & d_X^2 f(e_i,e_j) + h_j^k d_{X,\nu}^2 f(e_i,e_k) - h_{ij} d_X f(\nu) + h_i^k d_{\nu,X}^2 f(e_k,e_j) \\ & + h_i^k h_j^l d_\nu^2 f(e_k,e_l) - h_i^k h_{kj} d_\nu f(\nu) + h_{ij}^k d_\nu f(e_k). \end{aligned} \tag{2.14}$$

Here, d_X and d_X^2 represent the first and second derivatives with respect to the first argument of f, while d_ν and d_ν^2 represent the first and second derivatives with respect to the second argument of f, and $d_{\nu,X}^2$ represents the mixed derivative.

By (2.4), Codazzi equation and the above two identities, we get the estimates (2.10) and (2.11). □

We recall some elementary facts about a hypersurface. Denote $W = \sqrt{1+|Du|^2}$. The first fundamental form and the second fundamental form can be written in local coordinate as $g_{ij} = \delta_{ij} + u_i u_j$ and $h_{ij} = \frac{u_{ij}}{W}$. The inverse of the first fundamental form is $g^{ij} = \delta_{ij} - \frac{u_i u_j}{W^2}$. The Weingarten curvature is $h_i^j = D_i(\frac{u_j}{W})$.

Definition 8. Newton transformation tensor is defined as

$$[T_k]_i^j := \frac{1}{k!}\delta_{jj_1\cdots j_k}^{ii_1\cdots i_k} h_{j_1}^{i_1}\cdots h_{j_k}^{i_k}.$$

The corresponding $(2,0)$-tensor is defined as

$$[T_k]^{ij} := [T_k]_k^i g^{kj}.$$

From this definition one can easily show a divergence free identity

$$\sum_j D_j[T_k]_i^j = 0. \tag{2.15}$$

Lemma 9. *For any $1 \leq k < n$, there is a family of elementary relations between σ_k opertators and Newton transformation tensors*

$$[T_k]_i^j \quad = \quad \sigma_k \delta_i^j - [T_{k-1}]_i^l h_l^j \tag{2.16}$$

or

$$[T_k]_i^j = \sigma_k \delta_i^j - [T_{k-1}]_l^j h_i^l. \tag{2.17}$$

For $k = n$, we have

$$[T_{n-1}]_l^j h_i^l \quad = \quad \sigma_n \delta_i^j.$$

Moreover, the $(2,0)$-tensor of T_k is symmetry such that

$$[T_k]^{ij} = [T_k]^{ji}. \tag{2.18}$$

Proof. We only prove the first one, because the second one is similar. From Definition 3, it is easy to check that

$$\sigma_k(\kappa) = \frac{1}{k!}\delta_{j_1 \cdots j_k}^{i_1 \cdots i_k} h_{j_1}^{i_1} \cdots h_{j_k}^{i_k}. \tag{2.19}$$

By the definition and (2.19), we obtain (2.16) as follows:

$$\begin{aligned}
[T_k]_i^j &= \frac{1}{k!}\delta_{jj_1 \cdots j_k}^{ii_1 \cdots i_k} h_{j_1}^{i_1} \cdots h_{j_k}^{i_k} \\
&= \frac{1}{k!}\delta_{j_1 \cdots j_k}^{i_1 \cdots i_k} h_{j_1}^{i_1} \cdots h_{j_k}^{i_k} \delta_i^j - \frac{1}{(k-1)!}\delta_{j_1 j_2 \cdots j_k}^{ii_2 \cdots i_k} h_{j_1}^{j} h_{j_2}^{i_2} \cdots h_{j_k}^{i_k} \\
&= \sigma_k \delta_i^j - [T_{k-1}]_i^k h_k^j.
\end{aligned}$$

By $[T_n]_i^j = 0$, we get

$$0 = [T_n]_i^j \quad = \quad \sigma_n \delta_i^j - [T_{n-1}]_i^k h_k^j.$$

For $k = 1$, the symmetry of the $(2,0)$-tensor of T_1 come from the symmetry of h. Inductively, we assume the symmetry of $(2,0)$-tensor T_k is true when $k = m$. From (2.16), we have

$$\begin{aligned}
[T_{m+1}]^{ij} &= [T_{m+1}]_l^i g^{lj} = \sigma_{m+1}\delta_l^i g^{lj} - [T_m]_l^p h_p^i g^{lj} \\
&= \sigma_{m+1} g^{ij} - [T_m]^{pj} h_p^i.
\end{aligned}$$

On the other hand, by (2.17) we have

$$\begin{aligned}
[T_{m+1}]^{ji} &= [T_{m+1}]_l^j g^{li} = \sigma_{m+1}\delta_l^j g^{li} - [T_m]_p^j h_l^p g^{li} \\
&= \sigma_{m+1} g^{ji} - [T_m]_p^j h^{pi} \\
&= \sigma_{m+1} g^{ji} - [T_m]^{jp} h_p^i.
\end{aligned}$$

From the symmetry of g and T_m, we have proved (2.18). □

Lemma 10. *If u satisfies the scalar equation (1.1) in $\mathbb{R}^3$, then the following integral is bounded as below*

$$\int_{B_r(x_0)}(\sigma_1 f-\sigma_3)dx\le C$$

where C depends only on $\|f\|_{L^\infty(B_{r+1}(x_0))}$.

Proof. Due to the scalar curvature equation (1.1), we can prove that $f\sigma_1-\sigma_3$ is nonnegative. In fact, we denote $G_{ij}:=fg_{ij}+h_i^l h_{lj}$. Because $f>0$ and linear algebra, we know that the matrix $[G_{ij}]$ and its inverse $[G^{ij}]$ are positive definite. We are going to verify that $G^{ij}=\frac{\sigma_2^{ij}}{f\sigma_1-\sigma_3}$. By the scalar curvature equation (1.1) in $\mathbb{R}^3$, we have

$$\begin{aligned}\sigma_2^{ip}G_{pj}&=\sigma_2^{ip}(fg_{pj}+h_p^k h_{kj})\\&=f\sigma_1\delta_j^i-fh_j^i+\sigma_2 g^{ik}h_{kj}-[T_2]_k^i h_j^k\\&=f\sigma_1\delta_j^i-\sigma_3\delta_j^i.\end{aligned}$$

This gives us $G^{ij}=\frac{\sigma_2^{ij}}{f\sigma_1-\sigma_3}$. We know that $[\sigma_2^{ij}]$ is also a positive definite matrix. So we have

$$f\sigma_1-\sigma_3>0. \tag{2.20}$$

Denote $\phi\in C_0^\infty(B_{r+1}(x_0))$ a non-negative function with $|D\phi|+|D^2\phi|\le C$. We assume that $\phi\equiv 1$ in $B_r(x_0)$ and $0\le\phi\le 1$ in $B_{r+1}(x_0)$.

Thus we have

$$\int_{B_r(x_0)} f\sigma_1-\sigma_3 dx \quad\le\quad \int_{B_{r+1}(x_0)}\phi^2(f\sigma_1-\sigma_3)dx.$$

For the first part of the above integral, it has

$$\begin{aligned}\int_{B_{r+1}(x_0)}\phi^2 f\sigma_1 &\le C(\|f\|_{L^\infty})\int_{B_{r+1}(x_0)}\phi^2 div(\frac{Du}{W})dx\\&= C\int_{B_{r+1}(x_0)}-\sum_i(\phi^2)_i\frac{u_i}{W}dx\le C.\end{aligned} \tag{2.21}$$

Then we estimate the second term and using (2.15)

$$\begin{aligned}-\int_{B_{r+1}(x_0)}\phi^2\sigma_3 dx &= -\frac{1}{3}\int_{B_{r+1}(x_0)}\sum_i\phi^2[T_2]_i^j D_j(\frac{u_i}{W})dx\\&=\frac{2}{3}\int_{B_{r+1}(x_0)}\sum_i\phi[T_2]_i^j\phi_j\frac{u_i}{W}dx.\end{aligned}$$

Using (2.16), we continue our estimate

$$\int_{B_{r+1}}\sum_i\phi[T_2]_i^j\phi_j\frac{u_i}{W}dx = \int_{B_{r+1}}\sum_i\phi\phi_i\frac{u_i}{W}\sigma_2 dx-\int_{B_{r+1}}\sum_{i,j}\phi[T_1]_i^k\phi_j\frac{u_i}{W}D_k(\frac{u_j}{W})dx.$$

Due to the scalar curvature equation, we have the bound for the first term

$$\int_{B_{r+1}}\sum_i\phi\phi_i\frac{u_i}{W}\sigma_2 dx \quad\le\quad C(\|f\|_{L^\infty}).$$

For the second term, we do integration by parts and use (2.15)

$$\begin{aligned} -\int_{B_{r+1}} \sum_{i,j} \phi[T_1]_i^k \phi_j \frac{u_i}{W} D_k(\frac{u_j}{W}) dx &= \int_{B_{r+1}} \sum_{i,j} [T_1]_i^k (\phi\phi_j)_k \frac{u_i}{W}\frac{u_j}{W} dx \\ &\quad + \int_{B_{r+1}} \sum_{i,j} [T_1]_i^k D_k(\frac{u_i}{W}) \phi\phi_j \frac{u_j}{W} dx \\ &\leq C \int_{B_{r+1}} div(\frac{Du}{W}) dx + C(\|f\|_{L^\infty}). \end{aligned}$$

We have used (2.4) and the scalar curvature equation (1.1) in the above inequality. The term $\int_{B_{r+1}} div(\frac{Du}{W})dx$ can be estimated the same as (2.21).

In conclusion, we have

$$\int_{B_r(x_0)} f\sigma_1 - \sigma_3 dx \quad \leq \quad C.$$

The above constants C are universal constants under control (depending only on $\|f\|_{L^\infty}$), which are different from line by line. $\square$

3. An important differential inequality

Let us consider the quantity of $b(x) := \log \sigma_1$. In dimension three, we have a very important differential inequality.

Lemma 11. *For admissible solutions of the equations (1.1) in $\mathbb{R}^3$, we have*

$$\sigma_2^{ij} b_{ij} \geq \frac{1}{100}\sigma_2^{ij} b_i b_j - C(f\sigma_1 - \sigma_3) + g^{ij} b_i d_\nu f(e_j) \tag{3.1}$$

where C depends only on $\|f\|_{C^2}$, $\|\frac{1}{f}\|_{L^\infty}$ and $\|u\|_{C^1}$.

Remark. Our choice of $b(x)$ is different from $\log\sqrt{1+\lambda_1^2}$ as in [24] or $\log u_{11}$ as in [8] and [3]. We compute $\log \sigma_1$ in this paper, because it allows us to avoid discussions of viscosity solutions and, at the same time, has sufficient and better concavity than $\log u_{11}$. We are uncertain whether the corresponding higher-dimensional inequalities (3.1) hold or not. This is one of the challenges in generalizing our theorem to higher dimensions.

Proof. It is similar as we did in Lemma 3 of [20]. For simplicity, we may choose an orthonormal frame and assume that $\{h_{ij}\}$ is diagonal at a fixed point p. Thus we have at p

$$\sigma_2^{kl} b_k b_l \quad = \quad \sigma_2^{kl} \frac{\sum_i h_{iik}}{\sigma_1} \frac{\sum_j h_{jjl}}{\sigma_1}$$

and

$$\sigma_2^{kl} b_{kl} \quad = \quad \frac{\sum_i \sigma_2^{kl} h_{iikl}}{\sigma_1} - \frac{\sigma_2^{kl} \sum_i h_{iik} \sum_j h_{jjl}}{\sigma_1^2}.$$

Using Lemma 7, we get

$$\begin{aligned} A := \sigma_2^{kl} b_{kl} - \epsilon \sigma_2^{kl} b_k b_l \quad \geq \quad & \frac{\sum\limits_i (\sum\limits_{k\neq l} h_{kli}^2 - \sum\limits_{k\neq l} h_{kki} h_{lli})}{\sigma_1} \\ & - \frac{(1+\epsilon)\sigma_2^{kk} (\sum\limits_i h_{iik})^2}{\sigma_1^2} \\ & + \frac{2f \sum_{j,i} h_{ji}^2 - (f\sigma_1 - 3\sigma_3)\sigma_1}{\sigma_1} \\ & + \sum_i \frac{\nabla^2 f(e_i, e_i)}{\sigma_1}. \end{aligned}$$

By (1.1) and (2.20), we have

$$f\sigma_1 - \sigma_3 \quad = \quad \sqrt{(f+\lambda_1^2)(f+\lambda_2^2)(f+\lambda_3^3)} \geq C \frac{(1+\sigma_1)^2}{\sigma_1}.$$

Due to

$$\frac{2f \sum_{j,i} h_{ji}^2}{\sigma_1} \quad \geq \quad 0$$

and

$$\begin{aligned} \sum_i \frac{\nabla^2 f(e_i, e_i)}{\sigma_1} \quad & \geq \quad -\frac{C(1+\sigma_1)^2}{\sigma_1} + g^{ij} b_i d_\nu f(e_j) \\ & \geq \quad -C(\|f\|_{C^2}, \|M\|_{C^1}, \|\frac{1}{f}\|_{L^\infty})(f\sigma_1 - \sigma_3) + g^{ij} b_i d_\nu f(e_j), \end{aligned}$$

we have

$$\begin{aligned} A \quad \geq \quad & \frac{\sum\limits_i (\sum\limits_{k\neq l} h_{kli}^2 - \sum\limits_{k\neq l} h_{kki} h_{lli})}{\sigma_1} \\ & - \frac{(1+\epsilon)\sigma_2^{kk} (\sum\limits_i h_{iik})^2}{\sigma_1^2} \\ & - C(f\sigma_1 - \sigma_3) + g^{ij} b_i d_\nu f(e_j). \end{aligned}$$

We use (2.8) to substitute terms with h_{iii} in A,

$$\begin{aligned} A \quad \geq \quad & \frac{6h_{123}^2}{\sigma_1} + \frac{2\sum_{k\neq l} h_{kll}^2}{\sigma_1} + \sum_{k\neq l} \frac{2h_{kkl}}{\sigma_1} (\frac{\sum\limits_{i\neq l} \sigma_2^{ii} h_{iil} - f_l}{\sigma_2^{ll}}) \\ & - \frac{2h_{113}h_{223} + 2h_{112}h_{332} + 2h_{221}h_{331}}{\sigma_1} \\ & - \frac{(1+\epsilon)\sigma_2^{kk} (\sum_{i\neq k} h_{iik} - \frac{\sum_{i\neq k} \sigma_2^{ii} h_{iik}}{\sigma_2^{kk}} + \frac{f_k}{\sigma_2^{kk}})^2}{\sigma_1^2} \\ & - C(f\sigma_1 - \sigma_3) + g^{ij} b_i d_\nu f(e_j). \end{aligned}$$

Due to symmetry, we only need to give the lower bound of the terms which contain h_{221} and h_{331}. We denote these terms by A_1 as below

$$\begin{aligned} A_1 \quad := \quad & \frac{2(\sigma_2^{11}+\sigma_2^{22})h_{221}^2}{\sigma_1\sigma_2^{11}} + \frac{2(\sigma_2^{11}+\sigma_2^{33})h_{331}^2}{\sigma_1\sigma_2^{11}} - \frac{2(h_{221}+h_{331})f_1}{\sigma_1\sigma_2^{11}} \\ & + \frac{2(\sigma_2^{22}+\sigma_2^{33}-\sigma_2^{11})h_{221}h_{331}}{\sigma_1\sigma_2^{11}} \\ & - \frac{(1+\epsilon)[(\lambda_2-\lambda_1)h_{221}+(\lambda_3-\lambda_1)h_{331}+f_1]^2}{\sigma_1^2\sigma_2^{11}}. \end{aligned}$$

Then we use Cauchy-Schwarz inequality and Lemma 5 to get

$$\begin{aligned} -\frac{(1+\epsilon)[(\lambda_2-\lambda_1)h_{221}+(\lambda_3-\lambda_1)h_{331}+f_1]^2}{\sigma_1^2\sigma_2^{11}} \quad \geq \\ -\frac{(1+2\epsilon)[(\lambda_2-\lambda_1)h_{221}+(\lambda_3-\lambda_1)h_{331}]^2}{\sigma_1^2\sigma_2^{11}} - [1+\epsilon+\frac{(1+\epsilon)^2}{\epsilon}]\frac{f_1^2}{\sigma_1^2\sigma_2^{11}}. \end{aligned}$$

Due to (2.10) and Lemma 5, we have

$$-\frac{f_1^2}{\sigma_1^2\sigma_2^{11}} \quad \geq \quad -\frac{C(\|f\|_{C^1},\|M\|_{C^1},\|\frac{1}{f}\|_{L^\infty})\sigma_1^2}{\sigma_1^2\sigma_2^{11}} \geq -C\sigma_1.$$

Thus we have

$$\begin{aligned} -\frac{(1+\epsilon)[(\lambda_2-\lambda_1)h_{221}+(\lambda_3-\lambda_1)h_{331}+f_1]^2}{\sigma_1^2\sigma_2^{11}} \quad \geq \\ -\frac{(1+2\epsilon)[(\lambda_2-\lambda_1)h_{221}+(\lambda_3-\lambda_1)h_{331}]^2}{\sigma_1^2\sigma_2^{11}} - \frac{C}{\epsilon}\sigma_1. \end{aligned} \tag{3.2}$$

Similarly, we have

$$\begin{aligned} -\frac{2(h_{221}+h_{331})f_1}{\sigma_1\sigma_2^{11}} \quad & \geq \quad -\frac{2\epsilon^2\sigma_1(h_{221}+h_{331})^2}{\sigma_1\sigma_2^{11}} - \frac{f_1^2}{2\epsilon^2\sigma_2^{11}\sigma_1^2} \\ & \geq \quad -\frac{2\epsilon^2(h_{221}+h_{331})^2}{\sigma_2^{11}} - \frac{C}{\epsilon^2}\sigma_1. \end{aligned} \tag{3.3}$$

Then we substitute (3.2) and (3.3) into A_1 to get

$$\begin{aligned} A_1 \quad \geq \quad & \frac{2\sigma_2^{11}+2\sigma_2^{22}}{\sigma_1\sigma_2^{11}}h_{221}^2 + \frac{2\sigma_2^{11}+2\sigma_2^{33}}{\sigma_1\sigma_2^{11}}h_{331}^2 \\ & + \frac{4\lambda_1}{\sigma_1\sigma_2^{11}}h_{221}h_{331} - \frac{2\epsilon^2(h_{221}+h_{331})^2}{\sigma_2^{11}} \\ & - \frac{(1+2\epsilon)[(\lambda_2-\lambda_1)h_{221}+(\lambda_3-\lambda_1)h_{331}]^2}{\sigma_1^2\sigma_2^{11}} \\ & - \frac{C}{\epsilon^2}\sigma_1. \end{aligned}$$

We will show the Claim 12 and Claim 13 in the below. □

Claim 12. For any $\epsilon \leq \frac{2}{3}$, we have

$$\frac{2\sigma_2^{11}+2\sigma_2^{22}}{\sigma_1\sigma_2^{11}}h_{221}^2 + \frac{2\sigma_2^{11}+2\sigma_2^{33}}{\sigma_1\sigma_2^{11}}h_{331}^2 + \frac{4\lambda_1}{\sigma_1\sigma_2^{11}}h_{221}h_{331} \geq \frac{2\epsilon(h_{221}+h_{331})^2}{\sigma_2^{11}}.$$

Proof. This claim follows from the following elementary inequality

$$\begin{aligned}
& (\sigma_2^{11}+\sigma_2^{22}-\epsilon\sigma_1)(\sigma_2^{11}+\sigma_2^{33}-\epsilon\sigma_1)-(\lambda_1-\epsilon\sigma_1)^2 \\
= & (1-\epsilon)^2\sigma_1^2+(1-\epsilon)\sigma_1(\lambda_2+\lambda_3)+\lambda_2\lambda_3-(\lambda_1-\epsilon\sigma_1)^2 \\
= & 3(1-\epsilon)f+(2-3\epsilon)(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2).
\end{aligned}$$

If we assume $2-3\epsilon\geq 0$, then the above quantity is nonnegative. □

Claim 13. For any $\delta\leq\frac{1}{20}$, we have

$$\begin{aligned}
& \frac{2\sigma_2^{11}+2\sigma_2^{22}}{\sigma_1\sigma_2^{11}}h_{221}^2+\frac{2\sigma_2^{11}+2\sigma_2^{33}}{\sigma_1\sigma_2^{11}}h_{331}^2+\frac{4\lambda_1}{\sigma_1\sigma_2^{11}}h_{221}h_{331} \\
\geq & \frac{(1+\delta)[(\lambda_2-\lambda_1)h_{221}+(\lambda_3-\lambda_1)h_{331}]^2}{\sigma_1^2\sigma_2^{11}}.
\end{aligned}$$

Proof. We compute the coefficient in front of $\frac{h_{221}^2}{\sigma_1^2\sigma_2^{11}}$

$$\begin{aligned}
2(\sigma_2^{11}+\sigma_2^{22})\sigma_1-(1+\delta)(\lambda_1-\lambda_2)^2 &= (1-\delta)\lambda_1^2+(1-\delta)\lambda_2^2+4\lambda_3^2+6f+2\delta\lambda_1\lambda_2 \\
&= (1-\delta)(\lambda_1+\frac{\delta}{1-\delta}\lambda_2)^2+\frac{1-2\delta}{1-\delta}\lambda_2^2+4\lambda_3^2+6f.
\end{aligned}$$

Similarly, the coefficient in front of $\frac{h_{331}^2}{\sigma_1^2\sigma_2^{11}}$ is

$$(1-\delta)(\lambda_1+\frac{\delta}{1-\delta}\lambda_3)^2+\frac{1-2\delta}{1-\delta}\lambda_3^2+4\lambda_2^2+6f.$$

We also compute the coefficient of $\frac{2h_{221}h_{331}}{\sigma_1^2\sigma_2^{11}}$

$$\begin{aligned}
2\lambda_1\sigma_1-(1+\delta)(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) &= (1-\delta)\lambda_1^2+(3+\delta)f-2(2+\delta)\lambda_2\lambda_3 \\
&= (1-\delta)(\lambda_1+\frac{\delta}{1-\delta}\lambda_2)(\lambda_1+\frac{\delta}{1-\delta}\lambda_3) \\
&\quad -\frac{4-3\delta}{1-\delta}\lambda_2\lambda_3+3f.
\end{aligned}$$

It is easy to see that for any small δ

$$[\frac{1-2\delta}{1-\delta}\lambda_2^2+4\lambda_3^2][\frac{1-2\delta}{1-\delta}\lambda_3^2+4\lambda_2^2] \quad\geq\quad [-\frac{4-3\delta}{1-\delta}\lambda_2\lambda_3]^2$$

and

$$(6f)^2 \quad\geq\quad (3f)^2.$$

We have proved that the coefficient matrix in front of h_{221}^2, h_{331}^2 and $2h_{221}h_{331}$ is positive definite. So we complete the proof of this claim.

Then we choose $\epsilon=\frac{1}{100}$, such that

$$1+\delta\geq\frac{1+2\epsilon}{1-\epsilon}$$

where δ is small constant in the Claim 13.

In all, we have proved that

$$\begin{aligned}
A &= \sum_{i=1}^3 A_i-C(f\sigma_1-\sigma_3)+g^{ij}b_id_\nu f(e_j) \\
&\geq -C(f\sigma_1-\sigma_3)+g^{ij}d_\nu f(e_i)b_j.
\end{aligned}$$

□

4. Mean value inequality.

In this section we prove a mean value type inequality. So we can transform the pointwise estimate into the integral estimate which is easier to deal with. It is unclear for higher dimensional scalar curvature equations. This is the second difficulty to generalize our theorem in higher dimensions.

Theorem 14. *Suppose u are admissible solutions of equations (1.1) on $B_{10} \subset \mathbb{R}^3$, then we have for any $y_0 \in B_2$*

$$\sup_{B_1} b = b(y_0) \leq C \int_{B_1(y_0)} b(x)(\sigma_1 f - \sigma_3) dx \tag{4.1}$$

where C depends only on $\|f\|_{C^2}$, $\|\frac{1}{f}\|_{L^\infty}$ and $\|u\|_{C^1}$.

Proof. Because the graph $X^\Sigma = (X, \nu) = (x_1, x_2, x_3, u, \frac{u_1}{W}, \frac{u_2}{W}, \frac{u_3}{W}, -\frac{1}{W})$ where u satisfies equation (1.1) can be viewed as a three dimensional smooth submanifold in $(\mathbb{R}^4 \times \mathbb{R}^4, f(X,\nu) \sum_{i=1}^{i=4} dx_i^2 + \sum_{i=1}^{i=4} dy_i^2)$. To illustrate the key observation, we first consider the simplest case $f = 1$. Then we will give all the details for general cases.

We shall see it is a submanifold with bounded mean curvature when $f = 1$.

In fact, we have

$$X_i^\Sigma = (X_i, \nu_i) = (X_i, h_i^k X_k)$$

and

$$G_{ij} = < X_i^\Sigma, X_j^\Sigma >_{\mathbb{R}^4 \times \mathbb{R}^4} = g_{ij} + \sum_k h_i^k h_{kj}.$$

We have proved in Lemma 10 that

$$G^{ij} = \frac{\sigma_2^{ij}}{\sigma_1 - \sigma_3}.$$

Then we show that the mean curvature is bounded as follows:

$$\begin{aligned} |\mathscr{H}_1| &= |< G^{ij}(D_j X_i^\Sigma - (\Gamma_{ji}^k)^\Sigma X_k^\Sigma), \nu_1^\Sigma >| \\ &\leq |G^{ij}(D_j X_i^\Sigma - \Gamma_{ji}^k X_k^\Sigma)| + |< G^{ij}(\Gamma_{ji}^k X_k^\Sigma - (\Gamma_{ji}^k)^\Sigma X_k^\Sigma), \nu_1^\Sigma >| \\ &\leq |G^{ij}(D_j X_i^\Sigma - \Gamma_{ji}^k X_k^\Sigma)| \end{aligned}$$

where $(\Gamma_{ji}^k)^\Sigma$ and Γ_{ji}^k are Christoffel symbols corresponding to G_{ij} and g_{ij} and ν_1^Σ is any one of unit normals of X^Σ. The second inequality is because of $<X_k^\Sigma, \nu_1^\Sigma>= 0$.

So the mean curvature vector can be estimated as following

$$\begin{aligned} |\mathscr{H}| &\leq 3|G^{ij}(D_j X_i^\Sigma - \Gamma_{ji}^k X_k^\Sigma)| \\ &= 3|\frac{\sigma_2^{ij}}{\sigma_1 - \sigma_3}(-h_{ij}\nu, h_{ij}^k X_k - h_i^k h_{kj}\nu)| \\ &\leq 3|\frac{(-2\sigma_2\nu, -(\sigma_1 - 3\sigma_3)\nu)}{\sigma_1 - \sigma_3}| \leq C. \end{aligned}$$

In the derivation above, we have utilized Equation (2.8). Employing an argument similar to that found in Lemma 3.2 and Theorem 3.4 in [16], where Michael-Simon's mean value inequalities are proven for subharmonic functions on bounded mean curvature submanifolds, we arrive at the estimate (4.1) for the scalar curvature equation when $f = 1$.

For $f = f(X, \nu)$, we give all the details of the proof of these mean value inequalities. First we know from Lemma 11

$$\sigma_2^{ij} b_{ij} \geq -C(f\sigma_1 - \sigma_3) + g^{ij} b_i d_\nu f(e_j).$$

Let χ be a non-negative and non-decreasing function in $C^1(\mathbb{R})$ with support in the interval $(0, \infty)$. We set

$$\psi(r) := \int_r^\infty t\chi(\rho - t)dt$$

where $0 < \rho < 10$, and $r^2 := f(X(x), \nu(x))|X(x) - X(y_0)|^2 + 2 - 2(\nu(x), \nu(y_0))$.

Let us denote

$$\mathfrak{B}_\rho = \{x \in B_{10}(y_0) : \quad f(X(x), \nu(x))|X(x) - X(y_0)|^2 + 2 - 2(\nu(x), \nu(y_0)) \leq \rho^2\}.$$

We may assume that $(X(y_0), \nu(y_0)) = (0, E_4)$. In order to simplify the notation, we denote $f_i = \nabla f(e_i)$ and $f_{ij} = \nabla^2 f(e_i, e_j)$ in the following computations.

First, we have

$$2rr_i = f_i|X|^2 + 2(X, e_i)f - 2h_i^k(e_k, E_4) \tag{4.2}$$

and

$$\begin{aligned} 2r_ir_j + 2rr_{ij} &= f_{ij}|X|^2 + 2f_i(X, e_j) + 2f_j(X, e_i) + 2f\delta_{ij} \\ &\quad -2fh_{ij}(X, \nu) - 2h_{ij}^k(e_k, E_4) + 2h_i^k h_{kj}(\nu, E_4). \end{aligned} \tag{4.3}$$

Now we are going to compute the differential inequality of ψ,

$$\begin{aligned} \sigma_2^{ij}\psi_{ij} &= \sigma_2^{ij}(-r_ir\chi(\rho - r))_j \\ &= -\sigma_2^{ij}r_{ij}r\chi(\rho - r) - \sigma_2^{ij}r_ir_j\chi(\rho - r) + \sigma_2^{ij}r_ir_jr\chi'(\rho - r). \end{aligned}$$

From (4.2) and (4.3), we have

$$\begin{aligned} \sigma_2^{ij}\psi_{ij} &= -\chi(\rho - r)\sigma_2^{ij}[\frac{f_{ij}|X|^2}{2} + 2f_i(X, e_j) + f\delta_{ij} - fh_{ij}(X, \nu)] \\ &\quad +\chi(\rho - r)\sigma_2^{ij}[h_{ij}^k(e_k, E_4) - h_i^k h_{kj}(\nu, E_4)] \\ &\quad +\sigma_2^{ij}r_ir_jr\chi'(\rho - r). \end{aligned}$$

Then by (2.8), (2.12) and (2.13), we have

$$\begin{aligned} \sigma_2^{ij}\psi_{ij} &= -\frac{\chi}{2}|X|^2\sigma_2^{ij}f_{ij} - 2\chi\sigma_2^{ij}f_i(X, e_j) - 3(f\sigma_1 - \sigma_3)\chi + 2\chi f^2(X, \nu) \\ &\quad +\chi g^{kl}f_l(e_k, E_4) + \chi(f\sigma_1 - 3\sigma_3)(1 - (\nu, E_4)) \\ &\quad +\sigma_2^{ij}r_ir_jr\chi'. \end{aligned} \tag{4.4}$$

By (2.14), the first term on the right hand side of (4.4) is

$$\begin{aligned} -\frac{\chi}{2}|X|^2\sigma_2^{ij}f_{ij} &= -\frac{\chi}{2}|X|^2[\sigma_2^{ij}d_X^2 f(e_i, e_j) + 2\sigma_2^{ij}h_j^k d_{X,\nu}^2 f(e_i, e_k) \\ &\quad -\sigma_2^{ij}h_{ij}d_X f(\nu) + \sigma_2^{ij}h_i^k h_j^l d_\nu^2 f(e_k, e_l) \\ &\quad -\sigma_2^{ij}h_i^k h_{kj}d_\nu f(\nu) + \sigma_2^{ij}h_{ij}^k d_\nu f(e_k)]. \end{aligned}$$

Due to the equation (1.1) in $\mathbb{R}^3$, there is

$$(f+\lambda_1^2)(f+\lambda_2^2)(f+\lambda_3^3) = (f\sigma_1-\sigma_3)^2.$$

By (2.20), we have

$$f\sigma_1-\sigma_3 = \sqrt{(f+\lambda_1^2)(f+\lambda_2^2)(f+\lambda_3^3)}.$$

From the above identity, $f\sigma_1-\sigma_3$ has all the positive lower bounds which are needed in the proof. For example, we get the following inequalities for any $1\le i\neq j\le 3$

$$f\sigma_1-\sigma_3 \ge C(\|\frac{1}{f}\|_{L^\infty})(|\lambda_i\lambda_j|+1+\sigma_1) \tag{4.5}$$

and

$$f\sigma_1-\sigma_3 \ge |\sigma_3|. \tag{4.6}$$

From (4.5) and (4.6) , we have

$$-\frac{\chi}{2}|X|^2\sigma_2^{ij}f_{ij} \le C(\|f\|_{C^2},\|\frac{1}{f}\|_{L^\infty},\|u\|_{C^1})\chi r^2(f\sigma_1-\sigma_3).$$

We estimate similarly the second and fouth terms on the right hand side of (4.4)

$$-2\chi\sigma_2^{ij}f_i(X,e_j) \quad +2\chi f^2(X,\nu)\le C(\|f\|_{C^1},\|\frac{1}{f}\|_{L^\infty},\|u\|_{C^1})\chi r(f\sigma_1-\sigma_3).$$

It is obvious that

$$\sum_{k=1}^3(e_k,E_4)^2 = (1-(\nu,E_4))(1+(\nu,E_4)). \tag{4.7}$$

From the definition of r, we see

$$(1-(\nu,E_4)) \le \frac{r^2}{2} \tag{4.8}$$

and

$$|(e_k,E_4)| \le r. \tag{4.9}$$

From (4.8), (4.9), (4.5) and (4.6), we deal with the fifth and sixth terms of (4.4)

$$\chi g^{kl}f_l(e_k,E_4)+\chi(f\sigma_1-3\sigma_3)(1-(\nu,E_4)) \le C(\|f\|_{C^1},\|\frac{1}{f}\|_{L^\infty},\|u\|_{C^1})\chi(r+r^2)(f\sigma_1-\sigma_3).$$

In sum, we have

$$\sigma_2^{ij}\psi_{ij} \le -3\chi(f\sigma_1-\sigma_3)+C\chi(r^2+r)(f\sigma_1-\sigma_3)+\sigma_2^{ij}r_ir_jr\chi'. \tag{4.10}$$

We next claim that

$$\sigma_2^{ij}r_ir_j\le(f\sigma_1-\sigma_3)[1+C(\|f\|_{C^1},\|\frac{1}{f}\|_{L^\infty},\|u\|_{C^1})r]. \tag{4.11}$$

In fact, we may choose an orthonormal frame and assume that $\{h_{ij}\}$ is diagonal at the point in order to prove this claim. It is straightforward

$$\sum_i\frac{\sigma_2^{ii}[f_i|X|^2+2f(X,e_i)-2\sum_k h_{ki}(e_k,E_4)]^2}{4r^2}$$
$$\le C(\|f\|_{C^1},\|\tfrac{1}{f}\|_{L^\infty},\|u\|_{L^\infty})(f\sigma_1-\sigma_3)r+\sum_i\frac{\sigma_2^{ii}[f(X,e_i)-h_{ii}(e_i,E_4)]^2}{r^2}.$$

Moreover, we have the following elementary properties

$$(f\sigma_1-\sigma_3)\delta_{ij}-f\sigma_2^{ij} = \sigma_2^{kl}h_{ki}h_{lj} \tag{4.12}$$

and

$$
\begin{aligned}
f\sigma_2^{ii}h_{ii}^2(X,e_i)^2+2f\sigma_2^{ii}(X,e_i)(e_i,E_4)h_{ii}+f\sigma_2^{ii}(e_i,E_4)^2 &\geq \\
f\sigma_2^{ii}[h_{ii}(X,e_i)+(e_i,E_4)]^2. \tag{4.13}
\end{aligned}
$$

By (4.12), we estimate as below

$$
\begin{aligned}
\frac{\sum_i \sigma_2^{ii}[f(X,e_i)-h_{ii}(e_i,E_4)]^2}{r^2} &\leq \sum_i \frac{\sigma_2^{ii}f^2(X,e_i)^2-2\sigma_2^{ii}h_{ii}f(X,e_i)(e_i,E_4)+\sigma_2^{ii}h_{ii}^2(e_i,E_4)^2}{r^2}\\
&\leq \sum_i \frac{(f\sigma_1-\sigma_3)f(X,e_i)^2+(f\sigma_1-\sigma_3)(e_i,E_4)^2}{r^2}\\
&\quad -\sum_i \frac{\sigma_2^{ii}h_{ii}^2 f(X,e_i)^2+2\sigma_2^{ii}h_{ii}f(X,e_i)(e_i,E_4)+f\sigma_2^{ii}(e_i,E_4)^2}{r^2}.
\end{aligned}
$$

Then by (4.13), (4.7) and the definition of r, we obtain

$$
\begin{aligned}
\frac{\sum_i \sigma_2^{ii}[f(X,e_i)-h_{ii}(e_i,E_4)]^2}{r^2} &\leq \sum_i (f\sigma_1-\sigma_3)\frac{f(X,e_i)^2+(e_i,E_4)^2}{r^2}\\
&\leq (f\sigma_1-\sigma_3)\frac{f|X|^2+[(1-(\nu,E_4))(1+(\nu,E_4))]}{r^2}\\
&\leq (f\sigma_1-\sigma_3).
\end{aligned}
$$

So we have proved the claim (4.11).

We obtain from (4.11) and (4.10) that

$$\sigma_2^{ij}\psi_{ij}\leq(\sigma_1 f-\sigma_3)[-3\chi+C(r^2\chi+r\chi)+(1+Cr)r\chi'].$$

Then we mutiply both sides by b and take integral on the domain $\mathfrak{B}_{10}$

$$
\begin{aligned}
\int_{\mathfrak{B}_{10}} b\sigma_2^{ij}\psi_{ij}dM &\leq \rho^4\frac{d}{d\rho}(\int_{\mathfrak{B}_{10}}\frac{b\chi(\rho-r)}{\rho^3}(\sigma_1 f-\sigma_3)dM)\\
&\quad +C\int_{\mathfrak{B}_{10}} rb\chi(\rho-r)(\sigma_1 f-\sigma_3)dM\\
&\quad +C\int_{\mathfrak{B}_{10}} br^2\chi'(\sigma_1 f-\sigma_3)dM. \tag{4.14}
\end{aligned}
$$

By (3.1), we have

$$-C\int_{\mathfrak{B}_{10}}(\sigma_1 f-\sigma_3)\psi dM+\int_{\mathfrak{B}_{10}} g^{ij}d_\nu f(e_i)b_j\psi dM\leq\int_{\mathfrak{B}_{10}} b\sigma_2^{ij}\psi_{ij}dM. \tag{4.15}$$

Inserting (4.15) into (4.14), we get

$$
\begin{aligned}
-\frac{d}{d\rho}(\int_{\mathfrak{B}_{10}}\frac{b\chi(\rho-r)}{\rho^3}(\sigma_1 f-\sigma_3)dM) &\leq \frac{C\int_{\mathfrak{B}_{10}} rb\chi(\rho-r)(\sigma_1 f-\sigma_3)dM}{\rho^4}\\
&\quad +\frac{C\int_{\mathfrak{B}_{10}} br^2\chi'(\sigma_1 f-\sigma_3)dM}{\rho^4}\\
&\quad +\frac{C\int_{\mathfrak{B}_{10}}(\sigma_1 f-\sigma_3)\psi dM}{\rho^4}\\
&\quad -\frac{\int_{\mathfrak{B}_{10}} g^{ij}d_\nu f(e_i)b_j\psi dM}{\rho^4}.
\end{aligned}
$$

Because χ, χ' and ψ are all supported in $\mathfrak{B}_\rho$, we deal with right hand side of the above inequality term by term. For the first term, we have

$$\frac{\int_{\mathfrak{B}_{10}} rb\chi(\rho-r)(\sigma_1 f-\sigma_3)dM}{\rho^4}\leq\frac{\int_{\mathfrak{B}_{10}} b\chi(\rho-r)(\sigma_1 f-\sigma_3)dM}{\rho^3}. \tag{4.16}$$

Then for the second term, we integral from δ to R to get

$$\begin{aligned}\int_\delta^R\frac{\int_{\mathfrak{B}_{10}} br^2\chi'(\sigma_1 f-\sigma_3)dM}{\rho^4}d\rho &\leq \int_\delta^R\frac{\int_{\mathfrak{B}_{10}} b\chi'(\sigma_1 f-\sigma_3)dM}{\rho^2}d\rho\\ &\leq \frac{\int_{\mathfrak{B}_{10}} b\chi(\sigma_1 f-\sigma_3)dM}{\rho^2}|_\delta^R\\ &\quad+\int_\delta^R\frac{2\int_{\mathfrak{B}_{10}} b\chi(\sigma_1 f-\sigma_3)dM}{\rho^3}d\rho. \end{aligned}\tag{4.17}$$

For the third term, we use the definition of ψ to estimate

$$\frac{\int_{\mathfrak{B}_{10}} b(\sigma_1 f-\sigma_3)\psi dM}{\rho^4}\leq\frac{\int_{\mathfrak{B}_{10}} b\chi(\rho-r)(\sigma_1 f-\sigma_3)dM}{\rho^3}. \tag{4.18}$$

For the last term, we do integration by parts

$$-\frac{\int_{\mathfrak{B}_{10}} g^{ij}d_\nu f(e_i)b_j\psi dM}{\rho^4}=\frac{\int_{\mathfrak{B}_{10}}[g^{ij}d_\nu f(e_i)]_j b\psi dM-\int_{\mathfrak{B}_{10}} g^{ij}d_\nu f(e_i)r_j br\chi dM}{\rho^4}.$$

Then we use (4.2) and the definition of ψ to get

$$\begin{aligned}-\frac{\int_{\mathfrak{B}_{10}} g^{ij}d_\nu f(e_i)b_j\psi dM}{\rho^4} &\leq \frac{C(\|f\|_{C^1},\|\frac{1}{f}\|_{L^\infty},\|u\|_{C^1})\int_{\mathfrak{B}_{10}} b\sigma_1\psi dM-\int_{\mathfrak{B}_{10}} g^{ij}d_\nu f(e_i)r_j br\chi dM}{\rho^4}\\ &\leq \frac{C[\int_{\mathfrak{B}_{10}} b\sigma_1\psi dM+\int_{\mathfrak{B}_{10}}(\sigma_1 f-\sigma_3)br\chi dM]}{\rho^4}\\ &\leq \frac{C\int_{\mathfrak{B}_{10}} b\chi(\rho-r)(\sigma_1 f-\sigma_3)dM}{\rho^3}. \end{aligned}\tag{4.19}$$

We combine (4.16), (4.17), (4.18) and (4.19) with integrating from δ to R:

$$\begin{aligned}\int_{\mathfrak{B}_{10}}\frac{b(\sigma_1 f-\sigma_3)\chi(\delta-r)}{\delta^3}dM &\leq C\int_{\mathfrak{B}_{10}}\frac{b(\sigma_1 f-\sigma_3)\chi(R-r)}{R^3}dM\\ &\quad+C\int_\delta^R\frac{\int_{\mathfrak{B}_{10}} b(\sigma_1 f-\sigma_3)\chi(\rho-r)dM}{\rho^3}d\rho.\end{aligned}$$

Then using GrÃ¶nwall's inequality, we get

$$\int_{\mathfrak{B}_{10}}\frac{b(\sigma_1 f-\sigma_3)\chi(\delta-r)}{\delta^3}dM\leq C\int_{\mathfrak{B}_{10}}\frac{b(\sigma_1 f-\sigma_3)\chi(R-r)}{R^3}dM.$$

Letting χ approximate the characteristic function of the interval $(0,\infty)$, in an appropriate fashion, we obtain,

$$\frac{\int_{\mathfrak{B}_\delta} b(\sigma_1 f-\sigma_3)dM}{\delta^3}\leq C\frac{\int_{\mathfrak{B}_R} b(\sigma_1 f-\sigma_3)dM}{R^3}. \tag{4.20}$$

Because the graph (X,ν) where u satisfied equation (1.4) can be viewed as a three dimensional smooth submanifold in $(\mathbb{R}^4\times\mathbb{R}^4, f(\sum_{i=1}^4 dx_i^2)+\sum_{i=1}^4 dy_i^2)$ with volume form exactly $(\sigma_1 f - \sigma_3)dM$. Moreover, for a sufficient small $\delta > 0$, the geodesic ball with radius δ of this submanifold is comparable with $\mathfrak{B}_\delta$. Letting $\delta \to 0$, we finally get

$$b(y_0) \quad \leq \quad C\frac{\int_{\mathfrak{B}_R} b(\sigma_1 f - \sigma_3)dM}{R^3} \leq C\frac{\int_{B_R(y_0)} b(\sigma_1 f - \sigma_3)dx}{R^3}.$$

□

5. Proof of the theorem 1

Proof. From Theorem 14, we have at the maximum point x_0 of $\bar{B}_2(0)$

$$b(x_0) \quad \leq \quad C\int_{B_2(x_0)} \phi^2 b(\sigma_1 f - \sigma_3)dx \tag{5.1}$$

where $\phi \in C_0^\infty(B_2)$, $\phi \equiv 1$ in B_1 and $0 \leq \phi \leq 1$ in B_2.

We shall estimate the first part $\int_{B_2(x_0)} \phi^2 b\sigma_1 f dx$ in the above integral (5.1).

We do integration by parts

$$\begin{aligned}\int_{B_2(x_0)} \phi^2 b\sigma_1 f dx &\leq C\int_{B_2(x_0)} \phi^2 b\sigma_1 dx \\ &\leq C(\int_{B_2(x_0)} bdx + \int_{B_2(x_0)} |Db|dx) \\ &\leq C(\|u\|_{C^1}, \|f\|_{L^\infty})(1 + \int_{B_2(x_0)} |Db|dx). \qquad (5.2)\end{aligned}$$

Then we use (2.1) to get

$$\int_{B_2(x_0)} |Db|dx \leq C\int_{B_2(x_0)} \sqrt{\sigma_2^{ij} b_i b_j}\sqrt{\sigma_1}dx.$$

By Holder inequality, we have

$$\begin{aligned}\int_{B_2(x_0)} |Db|dx &\leq (\int_{B_2(x_0)} \sigma_2^{ij} b_i b_j dx)^{\frac{1}{2}}(\int_{B_2(x_0)} \sigma_1 dx)^{\frac{1}{2}} \\ &\leq C(\|u\|_{C^1})\int_{B_3(x_0)} \phi^2 \sigma_2^{ij} b_i b_j dx \qquad (5.3)\end{aligned}$$

where $\phi \in C_0^\infty(B_3)$, $\phi \equiv 1$ in B_2 and $0 \leq \phi \leq 1$ in B_3.

We recall the inequality (3.1)

$$\sigma_2^{ij} b_{ij} \geq \frac{1}{100}\sigma_2^{ij} b_i b_j - C(\sigma_1 f - \sigma_3) + g^{ij} d_\nu f(e_i) b_j. \tag{5.4}$$

We have an integral version of this inequality

$$\begin{aligned}\int_{B_{r+1}} -2\phi\sigma_2^{ij}\phi_i b_j dM \quad \geq \quad & c_0\int_{B_{r+1}} \phi^2 \sigma_2^{ij} b_i b_j dM \qquad (5.5) \\ & -C\int_{B_{r+1}} (\sigma_1 f - \sigma_3)\phi^2 dM \\ & +\int_{B_{r+1}} g^{ij} d_\nu f(e_i) b_j \phi^2 dM\end{aligned}$$

for any $r < 5$ with all non-negative $\phi \in C_0^\infty(B_{r+1})$.

Then using (5.5) and Lemma 10, we see that

$$\begin{aligned}
\int_{B_3(x_0)} \phi^2 \sigma_2^{ij} b_i b_j dx &\leq C(\|u\|_{C^1}) \int_{B_3(x_0)} \phi^2 \sigma_2^{ij} b_i b_j dM \\
&\leq C[-\int_{B_3(x_0)} \phi \sigma_2^{ij} \phi_i b_j dM + \int_{B_3(x_0)} \phi^2(\sigma_1 f - \sigma_3) dx \\
&\quad + \int_{B_3(x_0)} \phi^2 |Db| dM] \\
&\leq C(\|u\|_{C^1}, \|f\|_{L^\infty})(\int_{B_3(x_0)} \sqrt{\phi^2 \sigma_2^{ij} b_i b_j} \sqrt{\sigma_2^{kl} \phi_k \phi_l} dx \\
&\quad +1 + \int_{B_3(x_0)} \phi^2 \sqrt{\sigma_2^{ij} b_i b_j} \sqrt{\sigma_1} dx).
\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\int_{B_3(x_0)} \phi^2 \sigma_2^{ij} b_i b_j dx &\leq C(\epsilon \int_{B_3(x_0)} \phi^2 \sigma_2^{ij} b_i b_j dx + \frac{1}{\epsilon} \int_{B_3(x_0)} \sigma_2^{ij} \phi_i \phi_i dx + \frac{1}{\epsilon}) \\
&\leq C\epsilon \int_{B_3(x_0)} \phi^2 \sigma_2^{ij} b_i b_j dx + \frac{C}{\epsilon}.
\end{aligned}$$

We choose ϵ small with $C\epsilon \leq \frac{1}{2}$ such that

$$\int_{B_3(x_0)} \phi^2 \sigma_2^{ij} b_i b_j dx \leq C(\|f\|_{C^2}, \|\frac{1}{f}\|_{L^\infty}, \|u\|_{C^1}). \tag{5.6}$$

So far we have obtained the estimate for the first part of (5.1) by combining (5.2), (5.3), and (5.6). We have

$$\int_{B_2(x_0)} \phi^2 b f \sigma_1 dx \leq C((\|f\|_{C^2}, \|\frac{1}{f}\|_{L^\infty}, \|u\|_{C^1}). \tag{5.7}$$

The second part is to estimate $\int_{B_2(x_0)} -\phi^2 b \sigma_3 dx$. Thanks to the divergence free property (2.15), we do integration by parts as follows

$$\begin{aligned}
-\int_{B_2(x_0)} \phi^2 b \sigma_3 dx &= -\frac{1}{3} \int_{B_2} \sum_i \phi^2 b [T_2]_i^j D_j(\frac{u_i}{W}) dx \\
&= \frac{1}{3} \underbrace{\int_{B_2} \sum_i [T_2]_i^j (\phi^2)_j b \frac{u_i}{W} dx}_{I} + \frac{1}{3} \underbrace{\int_{B_2} \sum_i [T_2]_i^j \phi^2 b_j \frac{u_i}{W} dx}_{II}.
\end{aligned} \tag{5.8}$$

We estimate I by applying (2.16) and (2.15)

$$\begin{aligned}
I &= \int_{B_2}\sum_i(\sigma_2\delta_i^j-[T_1]_i^k h_k^j)(\phi^2)_j b\frac{u_i}{W}dx\\
&\le C\int_{B_2} bdx-\int_{B_2}\sum_{i,j}[T_1]_i^k D_k(\frac{u_j}{W})(\phi^2)_j b\frac{u_i}{W}dx\\
&\le C(\|f\|_{L^\infty},\|u\|_{C^1})+\int_{B_2}\sum_{i,j}[T_1]_i^k\frac{u_j}{W}(\phi^2)_{jk}b\frac{u_i}{W}dx\\
&\quad+\int_{B_2}\sum_{i,j}[T_1]_i^k\frac{u_j}{W}(\phi^2)_j b_k\frac{u_i}{W}dx+2\int_{B_2}\sum_j\sigma_2\frac{u_j}{W}(\phi^2)_j bdx\\
&\le C+C\int_{B_2}\sigma_1 bdx+\int_{B_2}\sum_{i,j}[T_1]^{kl}b_k g_{li}\frac{u_i}{W}\frac{u_j}{W}(\phi^2)_j dx. \qquad (5.9)
\end{aligned}$$

The second term of (5.9) can be estimated by the same argument as before. We only need to estimate the last term of (5.9). By Cauchy-Schwarz inequality and (5.6), there is

$$\begin{aligned}
\int_{B_2}\sum_{i,j}[T_1]^{kl}b_k g_{li}\frac{u_i}{W}(\frac{u_j}{W})(\phi^2)_j dx &\le 4\int_{B_2}\phi^2[T_1]^{ij}b_i b_j dx\\
&\quad+4\int_{B_2}\sum_{k,l}[T_1]^{ij}g_{ik}\frac{u_k}{W}g_{jl}\frac{u_l}{W}(\sum_p\frac{u_p\phi_p}{W})^2dx\\
&\le C(\|f\|_{C^2},\|\frac{1}{f}\|_{L^\infty},\|u\|_{C^1}). \qquad (5.10)
\end{aligned}$$

From (5.9) and (5.10) we obtain

$$I=\int_{B_2}\sum_i[T_2]_i^j(\phi^2)_j b\frac{u_i}{W}dx\le C(\|f\|_{C^2},\|\frac{1}{f}\|_{L^\infty},\|u\|_{C^1}). \qquad (5.11)$$

Now we deal with II by using (2.17)

$$\begin{aligned}
II &\le \int_{B_2}\sum_i(\sigma_2\delta_i^j-[T_1]_k^j h_i^k)\phi^2 b_j\frac{u_i}{W}dx\\
&\le C(\|f\|_{L^\infty})\int_{B_2}|Db|dx-\int_{B_2}\sum_i[T_1]^{jk}h_{ki}\phi^2 b_j\frac{u_i}{W}dx. \qquad (5.12)
\end{aligned}$$

As before, the first term of (5.12) is already estimated by (5.3) and (5.6). We compute the second term of (5.12)

$$\begin{aligned}
-\int_{B_2}\sum_i[T_1]^{jk}h_{ki}\phi^2 b_j\frac{u_i}{W}dx &\le 2\int_{B_2}\phi^2[T_1]^{ji}b_j b_i dx\\
&\quad+2\int_{B_2}\sum_{k,l}[T_1]^{ij}h_{ik}\frac{u_k}{W}h_{jl}\frac{u_l}{W}\phi^2dx\\
&\le 2\int_{B_2}\phi^2[T_1]^{ji}b_j b_i dx\\
&\quad+2\int_{B_2}\sum_{i,j}\sigma_2\frac{u_iu_j}{W^2}h_{ij}\phi^2dx-2\int_{B_2}\sigma_3\frac{|Du|^2}{W^2}\phi^2dx.
\end{aligned}$$

By Lemma 10, we have

$$-\int_{B_2} \sigma_3 \frac{|Du|^2}{W^2}\phi^2 dx \leq \int_{B_2} (f\sigma_1 - \sigma_3)\frac{|Du|^2}{W^2}\phi^2 dx \quad \leq \quad C(\|f\|_{L^\infty}).$$

And by (5.6), we get the estimate

$$II = \int_{B_2} \sum_i [T_2]_i^j \phi^2 b_j \frac{u_i}{W} dx \leq C(\|f\|_{C^2}, \|\frac{1}{f}\|_{L^\infty}, \|u\|_{C^1}). \tag{5.13}$$

With the estimate (5.11) and (5.13) for I and II , we get

$$\int_{B_1(x_0)} -b\sigma_3 dx \leq C(\|f\|_{C^2}, \|\frac{1}{f}\|_{L^\infty}, \|u\|_{C^1}). \tag{5.14}$$

Finally, combining (5.7) and (5.14), we obatin the estimate

$$\log \sigma_1(x_0) \leq C(\|f\|_{C^2}, \|\frac{1}{f}\|_{L^\infty}, \|u\|_{C^1}).$$

□

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