UNIFORMIZATION OF PLANAR DOMAINS BY EXHAUSTION

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Abstract. We study the method of finding conformal maps onto circle domains by approximating with finitely connected subdomains. Every domain $D \subset \hat{\mathbb{C}}$ admits *exhaustions*, i.e., increasing sequences of finitely connected subdomains D_j whose union is D. By Koebe's theorem, each D_j admits a conformal map f_{D_j} from D_j onto a circle domain $f_{D_j}(D_j)$. Assuming $f_{D_j} \to f$, our goal is to find out if $f(D)$ is also a circle domain.

We present a countably connected D with an exhaustion (D_j) so that (f_{D_j}) has a limit whose image is *not a circle domain*, and a domain Ω with an exhaustion (Ω_j) so that (f_{Ω_j}) has a limit whose image has *uncountably many non-point complementary components*.

On the other hand, we prove that every exhaustion (D_j) of a countably connected D admits a *refinement* so that the image of the corresponding limit map is a circle domain. Our result extends the He-Schramm theorem on the uniformization of countably connected domains and provides a new proof.

1. Introduction

1.1. **Background.** The long-standing *Koebe conjecture* [15] predicts that every domain $D \subset \hat{\mathbb{C}}$ admits a conformal map onto a *circle domain*, i.e., a domain whose set of complementary components consists of closed disks and points. See [10] for an overview. Koebe himself proved this to be the case for finitely connected domains, cf. [7, Theorem 5.1]. Koebe's theorem has been extended to cover finitely connected targets with varying boundary shapes, the most general results being those by Brandt [5] and Harrington [9]. See [20] for further information.

A major breakthrough was made by He and Schramm [10], who showed that the Koebe conjecture holds for countably connected domains. Soon after Schramm [19] introduced the *transboundary extremal length* (or *transboundary modulus*), and applied it to give a simplified proof to the He-Schramm theorem as well as a generalization to uncountably connected "cofat" domains. See also [11], [12], [13]. Recently, results related to the Koebe conjecture have been established in [2], [14], [16], [18], [21], and [22].

The proofs by He-Schramm and Schramm apply approximation of a given domain from *outside* by a decreasing sequence of finitely connected domains together with Koebe's theorem to construct a sequence of conformal maps whose limit has circle domain image. In this paper, we study a modification of this method where a given domain is approximated from *inside* by *exhaustions*, i.e., increasing sequences of finitely connected subdomains.

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Our approach is motivated by the fact that exhaustions offer more flexibility than approximations from outside. They can potentially be applied to gain a better understanding of the Koebe conjecture and related problems. The challenge is in finding exhaustions with the desired properties among all the exhaustions of a given domain.

Theorems 1.1 and 1.2 below show that an arbitrary exhaustion does not work in general; the image of the limit map is not always a circle domain. However, our main result, Theorem 1.3, shows that any exhaustion of a countably connected domain admits a *refinement* so that the image of the corresponding limit map is a circle domain. We now describe our results in detail.

1.2. **Main results.** An *exhaustion* Φ of a domain $D \subset \hat{\mathbb{C}}$ is a sequence of domains $D_j \subset D$, each bounded by finitely many disjoint Jordan curves in D, such that

$$D_j \subset D_{j+1} \text{ for all } j = 1, 2, \dots \text{ and } D = \cup_j D_j.$$

We fix disjoint points $a_0, a_1, a_2 \in D_1$. Then by Koebe's theorem there are unique conformal maps $f_j : D_j \to \tilde{D}_j$ onto circle domains $\tilde{D}_j \subset \hat{\mathbb{C}}$ so that $f_j(a_k) = a_k$ for $k = 0, 1, 2$. Sequence (f_j) has a subsequence converging locally uniformly to a conformal $f : D \to f(D)$. We denote

$$\mathcal{F}_\Phi = \{f : D \to f(D) : f \text{ is the limit of a subsequence of } (f_j)\}.$$

If $\mathcal{F}_\Phi$ contains only one map f, i.e., if (f_j) converges, we denote $f = f_\Phi$. The use of this notation always contains the implicit assumption that $f_j \to f_\Phi$.

THEOREM 1.1. *There is a countably connected domain $D \subset \hat{\mathbb{C}}$ with exhaustion Φ such that $f_\Phi(D)$ is not a circle domain.*

We denote the set of complementary components of domain G by $\mathcal{C}(G)$. We say that $p \in \mathcal{C}(G)$ is *non-trivial* if $\operatorname{diam}(p) > 0$.

THEOREM 1.2. *There is a domain $D \subset \hat{\mathbb{C}}$ with exhaustion Φ such that $\mathcal{C}(f_\Phi(D))$ contains uncountably many non-trivial elements.*

Theorems 1.1 and 1.2 are in sharp contrast to [7, Theorem 2.1] on *slit domains*, i.e., domains whose sets of complementary components consist of vertical segments and points; if Φ is an exhaustion of D and if the targets $\tilde{D}_j$ above are slit domains so that $f_j \to f$, then $f(D)$ is always a slit domain.

In view of Theorems 1.1 and 1.2, in order to produce a limit map onto a circle domain it is necessary to modify, or refine, a given exhaustion. Let $\Phi = (D_j)$ and $\Phi' = (D'_j)$ be exhaustions of D. We say that Φ is a *refinement* of Φ', if every $p \in \mathcal{C}(D_j)$ is an element of $\mathcal{C}(D'_{j(p)})$ for some $j(p) \geqslant j$. Our main result reads as follows.

THEOREM 1.3. *Every exhaustion of a countably connected domain $D \subset \hat{\mathbb{C}}$ has a refinement Φ such that $f_\Phi(D)$ is a circle domain.*

Since every domain admits an exhaustion, Theorem 1.3 gives a new proof to the He-Schramm theorem. Our main tools are transfinite induction, which was also used by He-Schramm and Schramm, and Schramm's transboundary modulus.

2. Proof of Theorem 1.3

Let $G \subset \hat{\mathbb{C}}$ be a domain and $\hat{G} = \hat{\mathbb{C}}/\sim$, where

$$x \sim y \text{ if either } x = y \in G \text{ or } x, y \in p \text{ for some } p \in \mathcal{C}(G).$$

The corresponding quotient map is $\pi_G : \hat{\mathbb{C}} \to \hat{G}$. Identifying each $x \in G$ and $p \in \mathcal{C}(G)$ with $\pi_G(x)$ and $\pi_G(p)$, respectively, we have

$$\hat{G} = G \cup \mathcal{C}(G).$$

A homeomorphism $f : G \to G'$ has a homeomorphic extension $\hat{f} : \hat{G} \to \hat{G}'$.

Let $\Phi' = (D'_j)$ be an exhaustion of a countably connected domain D. We consider the following property: If $q_1 \in \mathcal{C}(D'_{j_1})$ and $q_2 \in \mathcal{C}(D'_{j_2})$, $j_1 \geqslant j_2$, and if $q_1 \cap q_2 \neq \emptyset$, then

$$\text{either } q_1 = q_2 \text{ or } q_1 \text{ lies in the interior of } q_2. \tag{1}$$

It is not difficult to see that any exhaustion Φ'' of D has a refinement Φ' satisfying (1). Since any refinement of Φ' is also a refinement of Φ'', we conclude that it suffices to prove Theorem 1.3 for exhaustions satisfying (1).

We prove Theorem 1.3 using transfinite induction (cf. [6]) and the following result. In this paper, *we allow closed disks to have zero diameter*. For instance, in the following proposition a disk $q \in \mathcal{C}(D)$ may be a point component.

Proposition 2.1. *Let $D \subset \hat{\mathbb{C}}$ be a countably connected domain. Fix an exhaustion $\Phi' = (D'_j)$ of D satisfying* (1)*, $p \in \mathcal{C}(D)$, and an open neighborhood U of p in $\hat{\mathbb{C}}$ such that $\overline{U} \in \mathcal{C}(D'_n)$ for some index n. Moreover, suppose every $f \in \mathcal{F}_{\Phi'}$ satisfies*

$$\hat{f}(q) \text{ is a disk for all } q \in \mathcal{C}(D) \setminus \{p\},\ q \subset U. \tag{2}$$

Then Φ' has a refinement $\Phi_p = (D_j(p))$ such that

$$D_j(p) \setminus U = D'_j \setminus U \quad \text{for all } j \in \mathbb{N} \quad \text{and} \tag{3}$$

(4) *if Φ is any refinement of Φ_p, then $\hat{g}(p)$ is a disk for all $g \in \mathcal{F}_\Phi$.*

2.1. Transfinite induction.

Suppose $D \subsetneq \hat{\mathbb{C}}$ is a countably connected domain. We lose no generality by assuming that the number of complementary components of D is infinite. We denote $E_0 = \hat{D} \setminus D$. For any compact non-empty $E \subset E_0$, let

$$E^* = \{p \in E : p \text{ is not isolated in } E\}.$$

By the Baire category theorem, $E^* \subsetneq E$. We can now use transfinite induction to define a well ordered set of subsets E_α of E_0 as follows: Given an ordinal $\alpha > 0$, we define

$$E_\alpha = \begin{cases} (E_\beta)^*, & \text{if } \alpha = \beta + 1 \text{ is a successor ordinal,} \\ \cap_{\beta<\alpha} E_\beta, & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

It follows that each E_α is compact and $E_\alpha \subsetneq E_\beta$ if $\alpha > \beta$ and $E_\beta \neq \emptyset$. There is an α_L so that E_{α_L} is finite and non-empty, thus $E_{\alpha_L+1} = \emptyset$.

We now show how Theorem 1.3 follows from Proposition 2.1.

Proposition 2.2. *Let $\Phi(0) = (D_j(0))$ be an exhaustion of D satisfying* (1)*. For every ordinal $0 \leqslant \alpha \leqslant \alpha_L + 1$ there is an exhaustion $\Phi(\alpha) = (D_j(\alpha))$ of D so that*

(i) *if $0 \leqslant \beta \leqslant \alpha$, then $\Phi(\alpha)$ is a refinement of $\Phi(\beta)$, and*
(ii) *if Φ is any refinement of $\Phi(\alpha)$, then*

$$\hat{f}(q) \text{ is a disk for all } f \in \mathcal{F}_\Phi \text{ and } q \in E_0 \setminus E_\alpha. \tag{5}$$

Suppose $\Phi(\alpha_L + 1) = (D_j)$ satisfies (5), and let (f_{j_k}) be a subsequence of the corresponding (f_j) converging to some f. Choosing $\Phi = (D_{j_k})$ and $f = f_\Phi$ shows that Theorem 1.3 follows from Proposition 2.2.

Proof of Proposition 2.2 assuming Proposition 2.1. First, we enumerate the elements $p = p(k) \in \mathcal{C}(D)$, and denote $p(k) \prec p(\ell)$ if $k < \ell$. This should not be confused with the ordering of the sets E_α. Each p belongs to $E_\alpha \setminus E_{\alpha+1}$ for exactly one $0 \leqslant \alpha \leqslant \alpha_L$. Fix such an α. Then each $p \in E_\alpha \setminus E_{\alpha+1}$ admits an open neighborhood $U_p \subset \hat{\mathbb{C}}$ so that $\overline{U}_p \in \mathcal{C}(D_j(0))$ for some j,

(6) $\pi_D(\overline{U}_p) \cap E_{\alpha+1} = \emptyset$, and

(7) $\overline{U}_p \cap \overline{U}_q = \emptyset$ if $q \in E_\alpha \setminus (E_{\alpha+1} \cup \{p\})$ or if $q \in E_0 \setminus E_\alpha$ satisfies $q \prec p$.

We apply transfinite induction. The claims of the proposition clearly hold for $\alpha = 0$ with the given exhaustion $\Phi(0)$. We assume that the claims hold for all $\beta < \alpha$ and verify them for α.

Let $\alpha = \beta + 1$ be a successor ordinal. By the induction assumption, (6) and (7), Condition (2) in Proposition 2.1 is satisfied with $\Phi' = \Phi(\beta)$, $p \in E_0 \setminus E_\beta$, and $U = U_p$. The proposition combined with our choice of U_p then gives a refinement $\Phi(\alpha) = (D_j(\alpha))$ of $\Phi(\beta) = (D_j(\beta))$ so that

$$D_j(\alpha) \setminus \bigcup_{p \in E_\beta \setminus E_\alpha} U_p = D_j(\beta) \setminus \bigcup_{p \in E_\beta \setminus E_\alpha} U_p \tag{8}$$

and so that (5) holds for all $p \in E_\beta \setminus E_\alpha$. Notice again that if Φ' is a refinement of Φ and if Φ'' is a refinement of Φ', then Φ'' is a refinement of Φ. The claims follow.

Now let $\alpha = \cap_{\beta<\alpha}\beta$ be a limit ordinal. We define $\Phi(\alpha) = (D_j(\alpha))$ as follows: first, let

$$D_j(\alpha) \setminus \Big(\bigcup_{p \in E_0 \setminus E_\alpha} U_p \Big) = D_j(0) \setminus \Big(\bigcup_{p \in E_0 \setminus E_\alpha} U_p \Big). \tag{9}$$

Fix $p \in E_0 \setminus E_\alpha$. Each $q \in E_0$ belongs to some $E_{\beta(q)} \setminus E_{\beta(q)+1}$. With this notation, we have $\beta(p) < \alpha$.

By (7) there are only finitely many $q \in E_{\beta(p)} \setminus E_\alpha$ such that

$$\overline{U}_p \cap \overline{U}_q \neq \emptyset. \tag{10}$$

Moreover, since each such $\overline{U}_q$ belongs to $\mathcal{C}(D_j(0))$, (6) and (7) show that

$$U_p \subset U_q \subset U_{q'} \tag{11}$$

if both $\overline{U}_q$ and $\overline{U}_{q'}$ satisfy (10) and $\beta(q) \leqslant \beta(q')$.

Among the elements q for which (10) holds, let $q(p)$ be the one with the maximal $\beta(q)$. Then $\beta(p) \leqslant \beta(q(p)) < \alpha$. We set

$$D_j(\alpha) \cap U_p = D_j(\beta(q(p))) \cap U_p, \quad p \in E_0 \setminus E_\alpha. \tag{12}$$

Then (9) and (12) define $\Phi(\alpha) = (D_j(\alpha))$. Furthermore, (8), (11), and the induction assumption show that $\Phi(\alpha)$ is a refinement of every $\Phi(\beta)$, $\beta \leqslant \alpha$, and that (5) holds. The proof is complete, modulo Proposition 2.1. □

2.2. **Transboundary modulus.** We will apply the following generalization of conformal modulus, first introduced by Schramm [19]. In addition to its importance in classical uniformization problems, this method has played a central role in recent developments on the uniformization of fractal metric spaces, cf. [1], [3], [4], [8], [17].

Let $G \subset \hat{\mathbb{C}}$ be a domain. The *transboundary modulus* $\operatorname{mod}(\Gamma)$ of a family Γ of paths in $\hat{G}$ is

$$\operatorname{mod}(\Gamma) = \inf_{\rho \in X(\Gamma)} \int_G \rho^2 \, dA + \sum_{p \in \mathcal{C}(G)} \rho(p)^2,$$

where $X(\Gamma)$ consists of all Borel functions $\rho : \hat{G} \to [0, \infty]$ for which

$$1 \leqslant \int_\gamma \rho \, ds + \sum_{p \in \mathcal{C}(G) \cap |\gamma|} \rho(p) \quad \text{for all } \gamma \in \Gamma.$$

Here $\int_\gamma \rho \, ds$ is the path integral of the restriction of γ to G. More precisely, this restriction is a countable union of disjoint paths γ_j, each of which maps onto a component of $|\gamma| \setminus \mathcal{C}(G)$, and we define

$$\int_\gamma \rho \, ds = \sum_j \int_{\gamma_j} \rho \, ds.$$

As noticed in [19], the transboundary modulus is a conformal invariant.

Lemma 2.3. *Suppose $f : G \to G'$ is conformal. Then for every path family Γ and $\hat{f}(\Gamma) = \{\hat{f} \circ \gamma : \gamma \in \Gamma\}$ we have*

$$\operatorname{mod}(\hat{f}(\Gamma)) = \operatorname{mod}(\Gamma).$$

The proof is a straightforward modification of the proof of the corresponding result for conformal modulus.

We will prove Proposition 2.1 by applying the following estimate. Given a domain $G \subset \hat{\mathbb{C}}$ and disjoint sets $A, B \subset \hat{\mathbb{C}}$, we denote

$$\begin{aligned} \Gamma(A, B; G) &= \{\text{paths in } \hat{G} \text{ joining } \pi_G(A) \text{ and } \pi_G(B)\}, \\ \operatorname{mod}(A, B; G) &= \operatorname{mod}(\Gamma(A, B; G)). \end{aligned}$$

Proposition 2.4. *Let $D \subset \hat{\mathbb{C}}$ be a countably connected domain. Fix an exhaustion $\Phi' = (D'_j)$ of D satisfying (1), $p \in \mathcal{C}(D)$, and an open neighborhood U of p in $\hat{\mathbb{C}}$ such that $\overline{U} \in \mathcal{C}(D'_n)$ for some index n. Moreover, suppose every $q \in \mathcal{C}(D) \setminus \{p\}$, $q \subset U$, is a disk. Then Φ' has a refinement $\Phi_p = (D_j(p))$ satisfying*

$$D_j(p) \setminus U = D'_j \setminus U \quad \textit{for all } j \in \mathbb{N}$$

so that if $\Phi = (D_j)$ is any refinement of Φ_p, then

$$\lim_{r \to 0} \limsup_{j \to \infty} \operatorname{mod}(S(z,r) \setminus p_j, \partial U; D_j) = 0 \tag{13}$$

for every $z \in p$, where p_j is the element of $\mathcal{C}(D_j)$ containing p.

Here and in what follows, if $z \in \mathbb{C}$ and $r > 0$ then $B(z,r)$ is the open euclidean disk with center z and radius r, circle $S(z,r)$ is the boundary of $B(z,r)$, and $\overline{B}(z,r) = B(z,r) \cup S(z,r)$. Also, in (13) $S(\infty, r) = S(0, 1/r)$.

We postpone the proof of Proposition 2.4 until Section 2.4. We next show that Proposition 2.1 follows from Proposition 2.4.

2.3. **From Proposition 2.4 to Proposition 2.1.** Fix $\Phi' = (D'_j)$, p, and U as in Proposition 2.1. Replacing D with $f(D)$ and Φ' with $(f(D'_j))$ for some $f \in \mathcal{F}_{\Phi'}$ if necessary, we may assume that the assumptions of Proposition 2.4 are valid. It then suffices to show that (13) implies (4) in Proposition 2.1: if Φ is any refinement of Φ_p, then $\hat{g}(p)$ is a disk for all $g \in \mathcal{F}_\Phi$.

Fix a refinement $\Phi = (D_j)$ of Φ_p. As before, let $f_j : D_j \to \tilde{D}_j$ be the associated conformal maps onto circle domains $\tilde{D}_j$. Fix $g \in \mathcal{F}_\Phi$. By passing to a subsequence if necessary, we may assume that $f_j \to g$.

Taking another subsequence if necessary, we may assume that $(\hat{f}_j(p_j))$ Hausdorff converges to a closed disk B, where p_j is the element of $\mathcal{C}(D_j)$ containing p (recall that B may have zero radius).

Since $f_j \to g$, we have $B \subset \hat{g}(p)$. We will prove that in fact $B = \hat{g}(p)$. This implies (4).

Applying suitable Möbius transformations if necessary, we may assume that U, $f_j(\partial U)$ and $\hat{f}_j(p_j)$ are all subsets of $\overline{B}(0,1)$. It is then understood that all distances in the rest of the proof are euclidean (instead of spherical).

Towards contradiction, suppose that $B \subsetneq \hat{g}(p)$. Then

$$\operatorname{dist}(w_0, B) \geqslant 2\delta \quad \text{for some } w_0 \in \partial \hat{g}(p) \text{ and } \delta > 0.$$

It follows that there are sequences (j_k) and (z_k) such that $j_k > k$ and $z_k \in \partial p_k \subset D_{j_k}$ for all $k \in \mathbb{N}$, and

$$\operatorname{dist}(f_j(z_k), \hat{f}_j(p_j)) \geqslant \delta \quad \text{for all } j \geqslant j_k.$$

By passing to another subsequence if necessary, we may assume that

$$z_k \to z \in p.$$

Fix $k \in \mathbb{N}$ and $j \geqslant j_k$. We construct a suitable path family $\Gamma(j,k)$ and estimate its modulus to arrive at a contradiction. Let $w \in \mathbb{C}$ be the point in $\hat{f}_j(p_j)$ closest to $f_j(z_k)$, and denote

$$I = (w, f_j(z_k)), \quad \ell = \text{ the line containing segment } I.$$

Let V_j be the bounded component of $\mathbb{C}\setminus f_j(\partial U)$, and denote the $f_j(z_k)$- and w-components of $\overline{V}_j \cap (\ell \setminus I)$ by P' and Q', respectively. Moreover, let

$$P = \hat{f}_j^{-1}(\pi_{f_j(D_j)}(P')),\ Q = \hat{f}_j^{-1}(\pi_{f_j(D_j)}(Q')) \subset \hat{D}_j.$$

There are unique points $a, b \in \partial U$ so that $\pi_{D_j}(a) \in P$ and $\pi_{D_j}(b) \in Q$. Let J_1, J_2 be the connected components of $\partial U \setminus \{a, b\}$, and let

$$\Gamma(j,k) = \{\text{paths joining } \pi_{D_j}(J_1) \text{ and } \pi_{D_j}(J_2) \text{ in } \pi_{D_j}(U) \setminus (P \cup Q)\}.$$

Then every $\gamma \in \Gamma(j,k)$ passes through $\pi_j(B(z, |z - z_k|))$, so if we denote $r_k = |z - z_k|$ and choose k large enough so that $S(z, r_k) \subset U$, we have

$$\Gamma(j,k) \subset \Gamma(S(z, r_k) \setminus p_j, \partial U; D_j)$$

(observe that $\pi_{D_j}(p_j) \in Q$). Thus,

$$\operatorname{mod}(\Gamma(j,k)) \leqslant \operatorname{mod}(S(z, r_k) \setminus p_j, \partial U; D_j)$$

so by (13),

$$\lim_{k\to\infty} \limsup_{j\to\infty} \operatorname{mod}(\Gamma(j,k)) = 0. \tag{14}$$

Lemma 2.5. *We have*

$$\operatorname{mod}(\hat{f}_j\Gamma(j,k)) \geqslant M > 0 \quad \textit{for all } k \in \mathbb{N} \textit{ and } j \geqslant j_k, \tag{15}$$

where $\hat{f}_j\Gamma(j,k) = \{\hat{f}_j \circ \gamma : \gamma \in \Gamma(j,k)\}$ *and* M *does not depend on* j *or* k.

Combining (14) with Lemmas 2.3 and 2.5 leads to a contradiction, so once Lemma 2.5 has been proved we know that Proposition 2.1 follows from Proposition 2.4.

Proof of Lemma 2.5. We consider the subfamily Γ of $\hat{f}_j\Gamma(j,k)$ consisting of projections of segments orthogonal to ℓ. More precisely, denote by T the length of I, $T = |w - f_j(z_k)|$, and let $\eta(s) = (1 - \frac{s}{T})w + \frac{s}{T} f_j(z_k)$, $0 < s < T$, be an arc-length parametrization of I. Notice that $T \geqslant \delta$.

Fix $0 < s < T$, and denote by ℓ_s the line orthogonal to ℓ passing through $\eta(s)$. Then there is a component I_s of $\ell_s \cap \overline{V}_j$ with endpoints $m_1 \in f_j(J_1)$ and $m_2 \in f_j(J_2)$ (recall that V_j is the bounded component of $\mathbb{C} \setminus f_j(\partial U)$). Choose a parametrization γ_s of $\pi_{D_j}(I_s)$, and let

$$\Gamma = \{\gamma_s : 0 < s < T\}.$$

Then $\Gamma \subset \hat{f}_j\Gamma(j,k)$, so it suffices to prove (15) with $\hat{f}_j\Gamma(j,k)$ replaced by Γ.

Fix $\rho \in X(\Gamma)$, and denote by $\mathcal{D}_j$ the family of disks $\tau \in \hat{f}_j(\mathcal{C}(D_j))$ satisfying $\tau \subset \pi_{D_j}(V_j)$. Then

$$1 \leqslant \int_{I_s} \rho\, ds + \sum_{q \in \mathcal{D}_j \cap |\gamma_s|} \rho(q) \quad \text{for all } 0 < s < T.$$

Integrating from 0 to T and applying Fubini's theorem and Hölder's inequality yields

$$\begin{aligned} \delta &\leqslant T \leqslant \int_{f_j(U\cap D_j)} \rho\, dA + \sum_{\tau\in\mathcal{D}_j} \operatorname{diam}(\tau)\rho(\tau) \\ &\leqslant |V_j|^{1/2}\Big(\int_{f_j(U\cap D_j)} \rho^2\, dA\Big)^{1/2} + \Big(\sum_{\tau\in\mathcal{D}_j} \operatorname{diam}(\tau)^2\Big)^{1/2}\Big(\sum_{\tau\in\mathcal{D}_j} \rho(\tau)^2\Big)^{1/2} \\ &\leqslant \Big(\frac{4}{\pi}|V_j|\Big)^{1/2}\Big(\Big(\int_{f_j(U\cap D_j)} \rho^2\, dA\Big)^{1/2} + \Big(\sum_{\tau\in\mathcal{D}_j} \rho(\tau)^2\Big)^{1/2}\Big), \end{aligned}$$

where the last inequality follows since the disks τ are disjoint subsets of V_j.

Recall that by our normalization $V_j \subset B(0,1)$ and therefore $|V_j| \leqslant \pi$ for all j. Combining with the estimate above and inequality

$$(a^{1/2} + b^{1/2})^2 \leqslant 3(a+b), \quad a, b > 0,$$

leads to

$$\frac{\delta^2}{12} \leqslant \int_{f_j(U\cap D_j)} \rho^2\, dA + \sum_{\tau\in\mathcal{D}_j} \rho(\tau)^2. \tag{16}$$

Since (16) holds for all $\rho \in X(\Gamma)$, we have $\operatorname{mod}(\Gamma) \geqslant \delta^2/12$. □

2.4. **Proof of Proposition 2.4.** We use the following notation: if $G, V \subset \hat{\mathbb{C}}$ are domains, then

$$\mathcal{C}(G,V) = \{q \in \mathcal{C}(G) :\, q \subset V\}.$$

Lemma 2.6. *Suppose D, Φ', p and U are as in Proposition 2.4. Then Φ' has a refinement $\Phi_p = (D_j(p))$ so that $D_j(p) \setminus U = D'_j \setminus U$ and*

$$\mathcal{C}(D_j(p), U) = \hat{\mathcal{C}}_{e,j} \cup \hat{\mathcal{C}}_{d,j} \cup \{\hat{p}_j\}$$

for all $j \in \mathbb{N}$, where $\hat{p}_j \supset p$ and $\hat{p}_j \notin \hat{\mathcal{C}}_{e,j} \cup \hat{\mathcal{C}}_{d,j}$,

$$\sum_{\hat{q}(j)\in\hat{\mathcal{C}}_{d,j}} \operatorname{diam}(\hat{q}(j)) \leqslant 2^{-j-1}, \tag{17}$$

and for every $\hat{q}(j) \in \hat{\mathcal{C}}_{e,j}$ there is $q = \overline{B}(x,t) \in \mathcal{C}(D,U)$, $t > 0$, such that

$$\overline{B}(x,t) \subset \hat{q}(j) \subset B(x,t+s), \quad s = \min\Big\{\frac{t}{100}, \frac{\operatorname{dist}(\hat{q}(j),p)}{100}\Big\}. \tag{18}$$

Proof. We have $\mathcal{C}(D,U) = \mathcal{C}_e \cup \mathcal{C}_d \cup \{p\}$, $p \notin \mathcal{C}_e \cup \mathcal{C}_d$, where $\mathcal{C}_e$ is a family of disks with positive radius and $\mathcal{C}_d$ a family of point components. We enumerate the elements of $\mathcal{C}_d$:

$$\mathcal{C}_d = \{q_1, q_2, \ldots\}.$$

We define $\Phi_p = (D_j(p))$ as follows: First, let $D_j(p) \setminus U = D'_j \setminus U$, $j \in \mathbb{N}$. To describe the sets $D_j(p) \cap U$, assume that $j = 1$ or $j \geqslant 2$, and $D_k(p)$ has been defined for all $k \leqslant j-1$.

We denote by $\hat{p}_j$ the element of $\mathcal{C}(D'_j, U)$ containing p. We lose no generality by assuming that $\hat{p}_1 \subset U$. Then $\mathcal{C}(D_j, U \setminus \hat{p}_j)$ is non-empty for all $j \geqslant 1$.

Each $q \in \mathcal{C}(D, U)$ is contained in some $\hat{q}(j)$ such that

(i) $\hat{p}(j) = \hat{p}_j$,

(ii) $\hat{q}(j) \in \mathcal{C}(D'_{j'}, U \setminus \hat{p}_j)$ for some $j' \geqslant j$,

(iii) if $j \geqslant 2$ then $\hat{q}(j) \subset \hat{q}(j-1)$ for some $\hat{q}(j-1) \in \mathcal{C}(D_{j-1}(p), U)$,

(iv) if $q = q_m \in \mathcal{C}_d$, then

$$\operatorname{diam}(\hat{q}_m(j)) \leqslant 2^{-j-m-1},$$

(v) if $q = \overline{B}(x,t) \in \mathcal{C}_e$, then $\hat{q}(j)$ satisfies (18).

Denote $\mathcal{Q}_j = \{\hat{q}(j) : q \in \mathcal{C}(D, U)\}$. If $\hat{q}(j), \hat{q}'(j) \in \mathcal{Q}_j$, then either $\hat{q}(j) \cap \hat{q}'(j) = \emptyset$ or one is contained in the other. Thus we can define $D_j(p) \cap U$ as the domain for which $\mathcal{C}(D_j(p), U)$ is the set of maximal elements in $\mathcal{Q}_j$.

Properties (i)–(iii) guarantee that $\{D_j(p)\}$ is a refinement of $\{D'_j\}$. Moreover, every $\hat{q}(j)$ satisfies (iv) or (v). We define

$$\begin{aligned}
\hat{\mathcal{C}}_{d,j} &= \{\hat{q}(j) \in \mathcal{C}(D_j(p), U) \setminus \{\hat{p}_j\} : \hat{q}(j) \text{ satisfies (iv)}\},\\
\hat{\mathcal{C}}_{e,j} &= \{\hat{q}(j) \in \mathcal{C}(D_j(p), U) \setminus \{\hat{p}_j\} : \hat{q}(j) \text{ satisfies (v)}\}.
\end{aligned}$$

□

We complete the proof of Proposition 2.4 by showing that any refinement $\Phi = (D_j)$ of the Φ_p in Lemma 2.6 satisfies the remaining estimate (13), i.e.,

$$\lim_{r \to 0} \limsup_{j \to \infty} \operatorname{mod}(S(z,r) \setminus p_j, \partial U; D_j) = 0 \quad \text{for every } z \in p.$$

Lemma 2.7. *Every refinement $\Phi = (D_j)$ of Φ_p satisfies* (13).

Proof. Fix $z \in p$ and let v be the largest integer such that

$$B(z, e^v) = B(z, R) \subset U.$$

It suffices to show that if j is large enough, then

$$\operatorname{mod}(S(z,r) \setminus p_j, S(z,R); D_j) \leqslant \epsilon(r) \to 0 \quad \text{as } r \to 0, \tag{19}$$

where $\epsilon(r)$ does not depend on j. We will do this by first constructing a suitable sequence of disjoint annuli, and then applying them to find admissible functions.

First, let $v_1 = v$. Then, fix $n \geqslant 1$ and assume that $v_n < v_{n-1} < \cdots < v_1$ have been defined. Denote $R_k = e^{v_k}$ and $A_k = B(z, R_k) \setminus \overline{B}(z, R_k/e)$, and let $v_{n+1} < v_n$ be the largest integer such that

$$\overline{B}(z, R_{n+1}) \cap \overline{B}(x, t+s) = \emptyset \quad \text{for all } q = \overline{B}(x,t) \in \mathcal{C}(D,U),\ q \cap A_n \neq \emptyset,$$

where s is as in (18).

Recall from Lemma 2.6 that

$$\mathcal{C}(D_j(p),U)=\hat{\mathcal{C}}_{e,j}\cup\hat{\mathcal{C}}_{d,j}\cup\{\hat{p}_j\}.$$

We denote $\hat{\mathcal{C}}_{e,j}=\hat{\mathcal{C}}_{b,j}\cup\hat{\mathcal{C}}_{s,j}$, where

$$\begin{aligned}\hat{\mathcal{C}}_{b,j} &= \{\hat{q}(j)\in\hat{\mathcal{C}}_{e,j}:\operatorname{diam}(\hat{q}(j))\geqslant\operatorname{dist}(\hat{q}(j),z)\},\\ \hat{\mathcal{C}}_{s,j} &= \{\hat{q}(j)\in\hat{\mathcal{C}}_{e,j}:\operatorname{diam}(\hat{q}(j))<\operatorname{dist}(\hat{q}(j),z)\}.\end{aligned}$$

Moreover, let $\mathcal{C}_j=\mathcal{C}_{d,j}\cup\mathcal{C}_{b,j}\cup\mathcal{C}_{s,j}$, where

$$\mathcal{C}_{d,j}=\cup_{m\geqslant j}\hat{\mathcal{C}}_{d,m},\quad \mathcal{C}_{b,j}=\cup_{m\geqslant j}\hat{\mathcal{C}}_{b,m},\quad \mathcal{C}_{s,j}=\cup_{m\geqslant j}\hat{\mathcal{C}}_{s,m}.$$

Fix a refinement $\Phi=(D_j)$ of Φ_p, and $u<v-100$. We denote $r=e^u$. Let j be large enough so that $2^{-j+1}<r/e$, and p_j the element of $\mathcal{C}(D_j,U)$ containing p. Since Φ is a refinement of Φ_p, we have $\mathcal{C}(D_j,U\setminus p_j)\subset\mathcal{C}_j$. In particular,

$$\mathcal{C}(D_j,U\setminus p_j)=\mathcal{D}_j\cup\mathcal{B}_j\cup\mathcal{S}_j,\quad \text{where } \mathcal{D}_j\subset\mathcal{C}_{d,j},\ \mathcal{B}_j\subset\mathcal{C}_{b,j},\ \mathcal{S}_j\subset\mathcal{C}_{s,j}.$$

By Lemma 2.6 and the definition of the above sets, the following hold: First,

$$\sum_{q(j)\in\mathcal{D}_j}\operatorname{diam}(q(j))\leqslant 2^{-j}<\frac{r}{2e}. \tag{20}$$

Secondly, denoting $\mathcal{B}_j(n)=\{q(j)\in\mathcal{B}_j:q(j)\cap A_n\neq\emptyset\}$, we have $\mathcal{B}_j(n)\cap\mathcal{B}_j(n')=\emptyset$ if $n\neq n'$. Moreover, since every $q(j)\in\mathcal{B}_j(n)$ contains a disk whose area is comparable to the area of A_n, the cardinality of $\mathcal{B}_j(n)$ has an absolute bound;

$$|\mathcal{B}_j(n)|\leqslant 30\quad\text{for all } n\in\mathbb{N}. \tag{21}$$

Finally, every $q(j)\in\mathcal{S}_j$ satisfies

$$\operatorname{diam}(q(j))^2\leqslant 2\operatorname{Area}(q(j)). \tag{22}$$

Moreover, denoting $\mathcal{S}_j(n)=\{q(j)\in\mathcal{S}_j:q(j)\cap A_n\neq\emptyset\}$, we have $\mathcal{S}_j(n)\cap\mathcal{S}_j(n')=\emptyset$ if $n\neq n'$.

We construct an admissible function

$$\rho\in X(\Gamma(S(z,r)\setminus p_j,S(z,R);D_j)) \tag{23}$$

as follows: let m be the largest integer such that $v_{m+1}\geqslant u$, and $1\leqslant n\leqslant m$. Define $\rho_n:\hat{D}_j\to[0,\infty]$,

$$\rho_n(w)=\begin{cases}\frac{1}{m}, & w\in\mathcal{B}_j(n),\\ \frac{2e\operatorname{diam}(w)}{mR_n} & w\in\mathcal{S}_j(n),\\ \frac{2}{m|w-z|}, & w\in A_n\cap D_j,\end{cases}$$

and $\rho_n(w)=0$ otherwise. We claim that

$$\frac{1}{m}\leqslant\int_\gamma\rho_n\,ds+\sum_{q\in\mathcal{C}_j\cap|\gamma|}\rho_n(q) \tag{24}$$

for all $\gamma \in \Gamma(S(z,r) \setminus p_j, S(z,R); D_j)$. Fix such a γ, and denote

$$\begin{aligned}\Omega_1 &= \{R_n/e < T < R_n : T = |y-z| \text{ for some } y \in |\gamma| \cap D_j\},\\ \Omega_2 &= \{R_n/e < T < R_n : T = |y-z| \text{ for some } y \in w,\ w \in |\gamma| \cap \mathcal{S}_j(n)\},\\ \Omega_3 &= \{R_n/e < T < R_n : T = |y-z| \text{ for some } y \in w,\ w \in |\gamma| \cap \mathcal{D}_j(n)\}.\end{aligned}$$

We may assume that γ does not intersect any $w \in \mathcal{B}_j(n)$, otherwise (24) follows directly from the definition of ρ_n. We then have

$$\int_{\Omega_1} \frac{dT}{T} + \int_{\Omega_2} \frac{dT}{T} + \int_{\Omega_3} \frac{dT}{T} \geqslant 1,$$

which combined with (20) yields

$$\int_{\Omega_1} \frac{dT}{T} + \int_{\Omega_2} \frac{dT}{T} \geqslant \frac{1}{2}.$$

The definition of ρ_n in $A_n \cap D_j$ yields

$$\int_\gamma \rho_n\, ds \geqslant \frac{2}{m} \int_{\Omega_1} \frac{dT}{T}.$$

On the other hand, combining the definition of ρ_n in $\mathcal{S}_j(n)$ with inequality

$$\frac{e(\beta-\alpha)}{R_n} \geqslant \log\beta - \log\alpha, \qquad \frac{e}{R_n} \leqslant \alpha \leqslant \beta,$$

yields

$$\sum_{q \in \mathcal{S}_j(n) \cap |\gamma|} \rho_n(q) \geqslant \frac{2}{m} \int_{\Omega_2} \frac{dT}{T}.$$

Combining the estimates yields (24). In particular, $\rho = \sum_{n=1}^m \rho_n$ satisfies (23), i.e., ρ is admissible for $\Gamma(S(z,r) \setminus p_j, S(z,R); D_j)$.

We prove (19) by estimating the energy

$$\int_{D_j \cap U} \rho^2\, dA + \sum_{w \in \mathcal{C}_j} \rho(w)^2 \tag{25}$$

from above. First, we have

$$\int_{D_j \cap U} \rho^2\, dA \leqslant \frac{4}{m^2} \sum_{n=1}^m \int_{D_j \cap A_n} \frac{dA(w)}{|w-z|^2} \leqslant \frac{8\pi}{m}. \tag{26}$$

In order to estimate the sum in (25), we recall that each $w \in \mathcal{B}_j \cup \mathcal{S}_j$ intersects at most one A_n. By (21),

$$\sum_{w \in \mathcal{B}_j} \rho(w)^2 \leqslant \sum_{n=1}^m \frac{|\mathcal{B}_j(n)|}{m^2} \leqslant \frac{30}{m}. \tag{27}$$

Finally, since every $w \in \mathcal{S}_j(n)$ is a subset of $B(z, 2R_n)$, (22) yields

$$\begin{aligned}\sum_{w \in \mathcal{S}_j} \rho(w)^2 &\leqslant \frac{4e^2}{m^2} \sum_{n=1}^m \sum_{w \in \mathcal{S}_j(n)} \frac{\operatorname{diam}(w)^2}{R_n^2} \\ &\leqslant \frac{8e^2}{m^2} \sum_{n=1}^m \frac{\operatorname{Area}(B(z,2R_n))}{R_n^2} = \frac{32\pi e^2}{m}.\end{aligned} \tag{28}$$

Combining (26), (27) and (28), we conclude

$$\int_{D_j \cap U} \rho^2 \, dA + \sum_{w \in \mathcal{C}_j} \rho(w)^2 \leqslant \frac{1000}{m} \to 0 \quad \text{as } r \to 0,$$

and (19) follows. The proof is complete. □

Remark 2.8. Schramm [19, Theorem 5.1] gives a simplified proof for the Koebe conjecture for countably connected domains, i.e., for the He-Schramm theorem. Although his proof uses transfinite induction, he explains in a remark how the proof can be modified to avoid it. Using his method, it is possible to modify the proof of Theorem 1.3 so that it no longer uses transfinite induction.

Remark 2.9. He and Schramm [12] and Schramm [19, Theorem 5.4] prove the Koebe conjecture for *almost circle domains*, i.e., domains D for which there is a closed and countable $B \subset \mathcal{C}(D)$ so that every $p \in \mathcal{C}(D) \setminus B$ is a circle or a point. Slight modifications to the proof above show that Theorem 1.3 also holds for almost circle domains D.

3. Proof of Theorem 1.1

3.1. **Construction of the domain.** We will construct a countably connected *square domain* $D \subset \hat{\mathbb{C}}$ so that $\{0\} \in \mathcal{C}(D)$, and an exhaustion Φ of D so that $\hat{f}_\Phi(\{0\})$ is non-trivial. The following result, which follows from the modulus estimate in [19, Theorem 6.2], then shows that $f_\Phi(D)$ cannot be a circle domain.

Proposition 3.1. *If f is a conformal map from domain $D \subset \hat{\mathbb{C}}$ with the above properties onto a circle domain, then $\hat{f}(\{0\})$ is a point-component.*

The construction of D is flexible in terms of the shapes of the complementary components. In particular, there are circle domains D satisfying the requirements of Theorem 1.1. We use squares in our construction for convenience of presentation. We start with a sequence of disjoint squares

$$Q_k = [a_k - R_k, a_k + R_k] \times [-R_k, R_k], \quad R_1 = 1, \, R_k < a_k,$$

where $(a_k)_{k=1}^\infty$, $(R_k)_{k=1}^\infty$ are decreasing sequences converging to zero, so that

$$D_k := \operatorname{dist}(Q_k, Q_{k+1}) = a_k - (a_{k+1} + R_k + R_{k+1}) = 2^{-k} R_{k+1}. \tag{29}$$

Each Q_k, $1 \leqslant k \leqslant j$, is surrounded by a sequence $(Q_{k,j})$ of inflated squares

$$Q_{k,j} = [a_k - T_{k,j}, a_k + T_{k,j}] \times [-T_{k,j}, T_{k,j}], \quad T_{k,j} = R_k + 2^{-j-1} D_k.$$

We also denote

$$Q_{0,j} = [-T_j, T_j] \times [-T_j, T_j], \quad T_j = a_{j+1} + R_{j+1} + D_j/2.$$

Then

$$\cup_{k=j+1}^\infty Q_j \subset \operatorname{int}(Q_{0,j}), \quad \cup_{k=1}^j Q_{k,j} \cap Q_{0,j} = \emptyset \quad \text{for every } j \in \mathbb{N}.$$

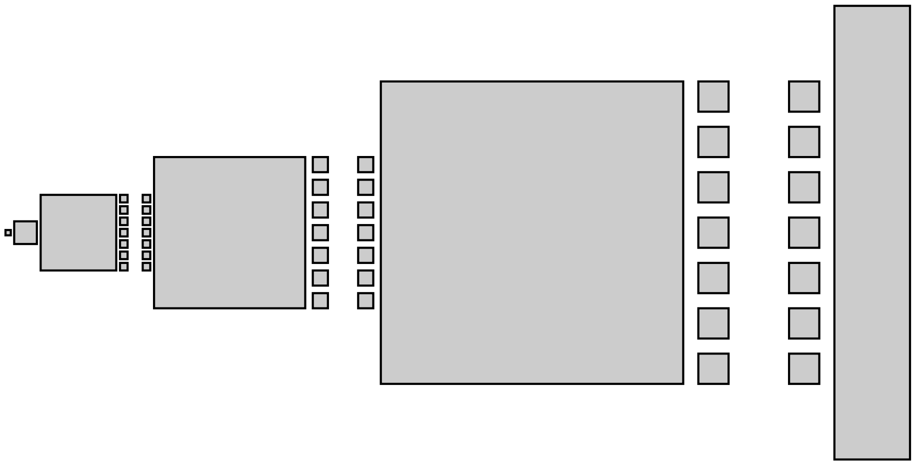

FIGURE 1. Part of the complement of D

Next, for $m \in \mathbb{N}$ and $1 \leqslant \ell \leqslant M_m$ (M_m will be chosen later), let

$$q_{m,\ell} = [(\ell-1)(s_m+d_m), (\ell-1)s_m + \ell d_m] \times [0, d_m], \tag{30}$$

where d_m, s_m are positive numbers so that

$$(M_m - 1)s_m + M_m d_m = 1 \quad \text{and} \quad d_m \geqslant s_m.$$

In particular, $d_m \leqslant M_m^{-1}$. For a fixed $m \in \mathbb{N}$, the sets $q_{m,\ell}$ are evenly spaced squares of sidelength d_m inside the rectangle $[0,1] \times [0, d_m]$.

For each $m \in \mathbb{N}$ and $1 \leqslant k \leqslant m$, let $\phi_{k+1,m}$ be the Möbius transformation so that $\phi_{k+1,m}(\infty) = \infty$,

$$\begin{aligned}
\phi_{k+1,m}(0,0) &= (a_{k+1} + (1-s_m)T_{k+1,m-1}, -R_{k+1}) \quad \text{and} \\
\phi_{k+1,m}(1,0) &= (a_{k+1} + (1-s_m)T_{k+1,m-1}, R_{k+1}).
\end{aligned}$$

We denote

$$t_{k,m} = (a_k - T_{k,m-1}) - (a_{k+1} + (1-2s_m)T_{k+1,m-1}) + d_m, \tag{31}$$

and define

$$\begin{aligned}
q^e_{k+1,m,\ell} &= \phi_{k+1,m}(q_{m,\ell}), \\
q^w_{k,m,\ell} &= \phi_{k+1,m}(q_{m,\ell}) + (t_{k,m}, 0), \quad 1 \leqslant \ell \leqslant M_m.
\end{aligned}$$

The squares $q^e_{k+1,m,\ell}, q^w_{k,m,\ell}$ lie "between" Q_{k+1} and Q_k, and we can choose M_m large enough so that

$$q^e_{m+1,m,\ell} \subset \operatorname{int}(Q_{0,m}),$$

$$q^e_{k+1,m,\ell} \subset \operatorname{int}(Q_{k+1,m-1}) \setminus Q_{k+1,m} \quad \text{for all } 1 \leqslant k \leqslant m-1, \tag{32}$$

$$q^w_{k,m,\ell} \subset \operatorname{int}(Q_{k,m-1}) \setminus Q_{k,m} \quad \text{for all } 1 \leqslant k \leqslant m. \tag{33}$$

We define D by

$$\hat{\mathbb{C}} \setminus D = \{0\} \cup \Big(\bigcup_{k=1}^{\infty} Q_k \cup \Big(\bigcup_{m=1}^{\infty} \bigcup_{\ell=1}^{M_m} \bigcup_{k=1}^{m} (q^e_{k+1,m,\ell} \cup q^w_{k,m,\ell}) \Big) \Big).$$

3.2. **Construction of the exhaustion.** We define exhaustion $\Phi_0 = (D_j)$ of D. First, every $\mathcal{C}(D_j)$ includes

$$Q_{0,j} \quad \text{and} \quad Q_{k,j}, \quad 1 \leqslant k \leqslant j.$$

To describe the rest of the elements of $\mathcal{C}(D_j)$, we first define

$$q^e_{k+1,m,\ell,j} = (1+\epsilon(j))q^e_{k+1,m,\ell} \quad 1 \leqslant k \leqslant j-1, \tag{34}$$

$$q^w_{k,m,\ell,j} = (1+\epsilon(j))q^w_{k,m,\ell} \quad 1 \leqslant k \leqslant j \tag{35}$$

for all $k \leqslant m \leqslant j$ and $1 \leqslant \ell \leqslant M_m$, i.e., squares with same center and $(1+\epsilon(j))$ times the sidelength of $q^e_{k+1,m,\ell}$ and $q^w_{k,m,\ell}$, respectively. Here $(\epsilon(j))$ is a strictly decreasing sequence converging to zero, and $\epsilon(1)$ is small enough such that for any fixed $j \in \mathbb{N}$ we have

(i) none of the squares intersect each other, and
(ii) (32) holds for $q^e_{k+1,m,\ell,j}$ and (33) holds for $q^w_{k,m,\ell,j}$.

We let $\mathcal{C}(D_j)$ include all the squares in (34) and (35) for $k \leqslant m \leqslant j-1$ and $1 \leqslant \ell \leqslant M_m$. Notice that the squares for which $m = j$ are not included.

The remaining elements of $\mathcal{C}(D_j)$ will be components $\overline{U}_{k,j,\ell}$ which "surround" $Q_{k,j}$ and contain both $q^e_{k,j,\ell,j}$ and $q^w_{k,j,\ell,j}$. More precisely, fix $2 \leqslant k \leqslant j$, and let $U_{k,j,\ell}$, $1 \leqslant \ell \leqslant M_j$, be Jordan domains so that

(i) the sets $\overline{U}_{k,j,\ell}$ are pairwise disjoint,
(ii) $\overline{U}_{k,j,\ell} \subset \operatorname{int}(Q_{k,j-1}) \setminus Q_{k,j}$,
(iii) $\overline{U}_{k,j,\ell}$ contains both $q^e_{k,j,\ell,j}$ and $q^w_{k,j,\ell,j}$,
(iv) if $(x,y) \in \partial U_{k,j,\ell}$ has the largest x-coordinate among all points of $\partial U_{k,j,\ell}$, then $(x,y) \in q^e_{k,j,\ell,j}$,
(v) if $(x,y) \in \partial U_{k,j,\ell}$ has the smallest x-coordinate among all points of $\partial U_{k,j,\ell}$, then $(x,y) \in q^w_{k,j,\ell,j}$.

We conclude the definition of $\mathcal{C}(D_j)$ by including $\overline{U}_{1,j,\ell} := q^w_{1,j,\ell,j}$ and

$$\overline{U}_{k,j,\ell} \quad 2 \leqslant k \leqslant j, \; 1 \leqslant \ell \leqslant M_j.$$

Then D_j is the set for which $\hat{\mathbb{C}} \setminus D_j = \cup\{p \in \mathcal{C}(D_j)\}$, and $\Phi_0 = (D_j)$.

Proposition 3.2. *There is $\delta > 0$ such that*

$$\operatorname{mod}(Q_{0,j_0}, Q_{1,j_0}; D_j) \geqslant \delta \quad \textit{for all } j_0 \in \mathbb{N} \textit{ and } j \geqslant j_0.$$

We postpone the proof of Proposition 3.2 and first show how it implies Theorem 1.1. Choose any subsequence $\Phi = (D_{j_n})$ of (D_j) so that (f_{j_n}) converges to f_Φ. By Proposition 3.1 it suffices to show that $\hat{f}_\Phi(\{0\})$ is non-trivial. But this follows directly by combining Proposition 3.2 with Proposition 4.2 below. Here the latter proposition can be applied with $E = Q_{1,1}$ since every $j_0 \geqslant 1$ satisfies

$$\operatorname{mod}(Q_{0,j_0}, Q_{1,j_0}; D_j) \leqslant \operatorname{mod}(Q_{0,j_0}, Q_{1,1}; D_j).$$

Thus, Theorem 1.1 follows once we have proved these propositions.

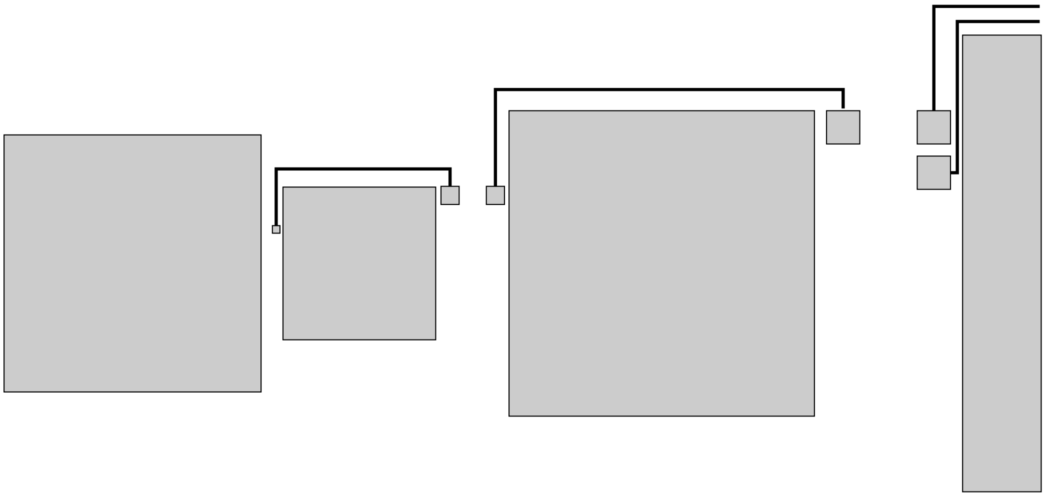

FIGURE 2. Some of the sets $\overline{U}_{k,j,\ell}$

3.3. **Proof of Proposition 3.2.** Fix j_0 and $j \geqslant j_0$, and let F_j be the projection of $\cup_{\ell=1}^{M_j} q_{j,\ell}$ to the real axis, recall the definition in (30). We construct a family of paths Γ parametrized by F_j, so that each $\gamma \in \Gamma$ connects $\pi_{D_j}(Q_{1,j_0})$ and $\pi_{D_j}(Q_{0,j_0})$ in $\hat{D}_j$. We then give a lower bound for $\mathrm{mod}(\Gamma)$.

Fix $\tau \in F_j$, and denote

$$\begin{aligned} z_{k+1}^e(\tau) &= \phi_{k+1,j}((\tau, d_j/2)), \quad 1 \leqslant k \leqslant j-1, \\ z_k^w(\tau) &= \phi_{k+1,j}((\tau, d_j/2)) + (t_{k,j}, 0), \quad 1 \leqslant k \leqslant j, \end{aligned}$$

where $t_{k,j}$ is the number in (31) and $\phi_{k+1,j}$ the Möbius transformation defined before (31). Then

$$z_{k+1}^e(\tau) \in q_{k+1,j,\ell,j}^e \subset \overline{U}_{k+1,j,\ell} \quad \text{and} \quad z_k^w(\tau) \in q_{k,j,\ell,j}^w \subset \overline{U}_{k,j,\ell},$$

where $\ell = \ell(j, \tau)$ is the index for which $(\tau, 0) \in q_{j,\ell}$.

We denote

$$\overline{U}_{k,j,\ell} =: \overline{U}_k(\tau),$$

and let $I_k(\tau)$ be the horizontal line segment in $\mathbb{C}$ which connects Q_{1,j_0} to $z_2^e(\tau)$, $z_k^w(\tau)$ to $z_{k+1}^e(\tau)$ if $2 \leqslant k \leqslant j-1$, and $z_j^w(\tau)$ to $Q_{0,j}$. Then

$$J(\tau) = (\cup_{k=1}^{j} I_k(\tau)) \cup (\cup_{k=2}^{j} \overline{U}_k(\tau))$$

is a continuum connecting Q_{1,j_0} and Q_{0,j_0} in $\mathbb{C}$. Moreover, $\pi_{D_j}(J(\tau))$ is a rectifiable curve in $\hat{D}_j$, with arc-length parametrization γ_τ. We define

$$\Gamma = \{\gamma_\tau : \tau \in F_j\}.$$

We now estimate the modulus of Γ. Let $\rho \in X(\Gamma)$ be an admissible function and $\tau \in F_j$. We denote by $\mathcal{A}_j$ all the sets in $\mathcal{C}(D_j)$ of the form $\overline{U}_{k,j,\ell}$, and by $\mathcal{B}_j$ all the other squares in $\mathcal{C}(D_j)$ of the form (34) or (35).

Then

$$1 \leqslant \int_{\gamma_\tau} \rho\, ds + \sum_{q \in \mathcal{C}(D_j)\cap|\gamma_\tau|} \rho(q) \tag{36}$$
$$= \sum_{k=1}^{j} \int_{I_k(\tau)\cap D_j} \rho\, ds + \sum_{k=1}^{j} \rho(\overline{U}_k(\tau)) + \sum_{q\in\mathcal{B}_j\cap|\gamma_\tau|} \rho(q).$$

Given $1 \leqslant k \leqslant j$, let A_k be the smallest rectangle containing all the segments $I_k(\tau)\cap D_j$, $\tau \in F_j$. Then by (29),

$$\mathrm{Area}(A_k) \leqslant 2D_k R_{k+1} \leqslant 2^{1-k} R_{k+1}^2. \tag{37}$$

To estimate the modulus, we integrate both sides of (36) over τ and apply change of variables and Fubini's theorem to get

$$\ell(F_j) \leqslant \sum_{k=1}^{j} (2R_{k+1})^{-1} \int_{A_k\cap D_j} \rho\, dA + \int_{F_j} \sum_{k=1}^{j} \rho(\overline{U}_k(\tau))\, d\tau \tag{38}$$
$$+ \int_{F_j} \sum_{q\in\mathcal{B}_j\cap|\gamma_\tau|} \rho(q)\, d\tau = S_1 + S_2 + S_3.$$

We apply Hölder's inequality and (37) to estimate S_1 as follows:

$$S_1 \leqslant \sum_{k=1}^{j} (2R_{k+1})^{-1}\, \mathrm{Area}(A_k)^{1/2} \Big(\int_{A_k\cap D_j} \rho^2\, dA \Big)^{1/2} \tag{39}$$
$$\leqslant \Big(\sum_{k=1}^{j} 2^{-1-k} \Big)^{1/2} \Big(\int_{D_j} \rho^2\, dA \Big)^{1/2} \leqslant \Big(\int_{D_j} \rho^2\, dA \Big)^{1/2}.$$

To estimate S_2 and S_3, we choose M_m so that

$$M_m \geqslant m 2^{m+1} \quad \text{for all } m \in \mathbb{N}. \tag{40}$$

We notice that the length of the set of parameters τ for which a given $\overline{U}_{k,j,\ell} \in \mathcal{A}_j$ is $\overline{U}_k(\tau)$ equals d_j. We have $d_j M_j \leqslant 1$ by construction. Thus, Hölder's inequality and (40) yield

$$S_2 = d_j \sum_{k=1}^{j} \sum_{\ell=1}^{M_j} \rho(\overline{U}_{k,j,\ell}) \leqslant d_j (jM_j)^{1/2} \Big(\sum_{k=1}^{j} \sum_{\ell=1}^{M_j} \rho(\overline{U}_{k,j,\ell,j})^2 \Big)^{1/2}$$
$$\leqslant \Big(\sum_{\overline{U}\in\mathcal{A}_j} \rho(\overline{U})^2 \Big)^{1/2}. \tag{41}$$

Next, we notice that the length of the set of parameters τ for which a given $q = q^y_{k,m,\ell,j} \in \mathcal{B}_j \cap |\gamma_\tau|$ is at most M_m^{-1}. Here $y = e$ or w. As before, Hölder's inequality yields

$$S_3 \leqslant \sum_{q\in\mathcal{B}_j} M_m^{-1} \rho(q) \leqslant \Big(\sum_{q\in\mathcal{B}_j} M_m^{-2} \Big)^{1/2} \Big(\sum_{q\in\mathcal{B}_j} \rho(q)^2 \Big)^{1/2}. \tag{42}$$

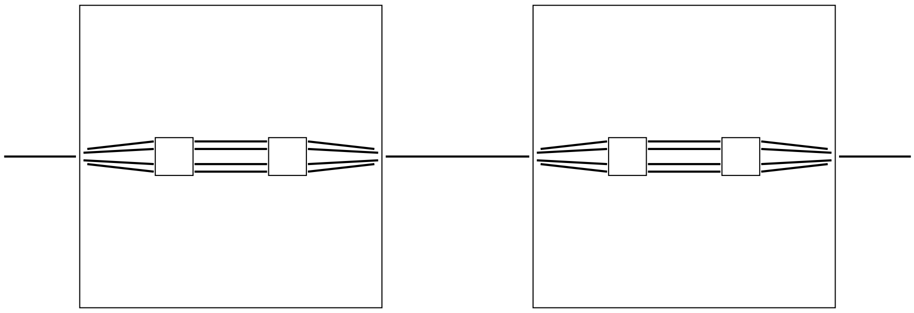

Figure 3. First steps in the construction of D

We estimate the first sum from above by summing over all $q \in \mathcal{B}_j$ and applying (40) to have

$$\sum_{q\in\mathcal{B}_j} M_m^{-2} \leqslant 2\sum_{m=1}^{j}\sum_{k=1}^{m}\sum_{\ell=1}^{M_m} M_m^{-2} = 2\sum_{m=1}^{j} mM_m^{-1} \leqslant \sum_{m=1}^{\infty} 2^{-m} = 1. \tag{43}$$

We have $\ell(F_j) \geqslant \frac{1}{2}$ by construction. Combining with (38), (39), (41), (42), and (43), yields

$$\begin{aligned}\frac{1}{2} &\leqslant \Big(\int_{D_j} \rho^2\, dA\Big)^{1/2} + \Big(\sum_{\overline{U}\in\mathcal{A}_j} \rho(\overline{U})^2\Big)^{1/2} + \Big(\sum_{q\in\mathcal{B}_j} \rho(q)^2\Big)^{1/2}\\ &\leqslant 3\Big(\int_{D_j} \rho^2\, dA + \sum_{q\in\mathcal{C}(D_j)} \rho(q)^2\Big)^{1/2}. \qquad (44)\end{aligned}$$

Since (44) holds for all $\rho \in X(\Gamma)$, we conclude that

$$\operatorname{mod}(Q_{0,j_0}, Q_{1,j_0}; D_j) \geqslant \operatorname{mod}(\Gamma) \geqslant \frac{1}{36}.$$

The proof is complete.

4. Proof of Theorem 1.2

4.1. **Construction of the domain.** The set $\mathcal{C}(D)$ of complementary components of D, which we now describe, consists of countably many segments and a Cantor set. The size of the Cantor set is not relevant for our construction. For instance, the construction can be carried out so that $\hat{\mathbb{C}} \setminus D$ has σ-finite length. Let $\mathcal{W}_0 = \{e\}$, $\mathcal{Y}_0 = \{(e,e)\}$, and for $k = 1, 2, \ldots$, let

$$\begin{aligned}\mathcal{W}_k &= \{w = w_1w_2w_3\cdots w_k : w_\ell \in \{0,1\} \text{ for } 1 \leqslant \ell \leqslant k\},\\ \mathcal{W}_\infty &= \{\bar{w} = w_1w_2w_3\cdots : w_\ell \in \{0,1\} \text{ for } \ell = 1,2,\ldots\}, \quad \text{and}\\ \mathcal{Y}_k &= \{(w,v) : w \in \mathcal{W}_k,\, v = v_1v_2v_3\cdots v_k,\, v_\ell \in \{0,1,2,3\} \text{ for } 1 \leqslant \ell \leqslant k\}.\end{aligned}$$

If $\bar{w} = w_1w_2\cdots \in \mathcal{W}_\infty$ and $k \in \mathbb{N}$, we denote $\bar{w}(k) = w_1\cdots w_k$.

Next, let (R_k) be a sequence of positive real numbers so that $R_{k+1} < R_k/2$ for all $k = 0, 1, 2, \ldots$. Moreover, given such a k let

$$\mathcal{Q}_k = \{Q_w = [x_w - R_k, x_w + R_k] \times [-R_k, R_k] : w \in \mathcal{W}_k\} \tag{45}$$

be a family of disjoint, congruent closed squares in $\mathbb{C}$ with centers on the real axis so that

$$\text{if } w \in \mathcal{W}_k \text{ and } a \in \{0,1\}, \text{ then } Q_{wa} \subset \operatorname{int}(Q_w).$$

The intersection

$$K = \bigcap_{k=0}^{\infty} \Big(\bigcup_{w \in \mathcal{W}_k} Q_w \Big) \tag{46}$$

is a Cantor set on the real axis. It is the Cantor set part of $\mathcal{C}(D)$. Each $p = p_{\bar{w}} \in K$ is uniquely determined by the $\bar{w} \in \mathcal{W}_\infty$ that satisfies

$$\{p_{\bar{w}}\} = \cap_{k=1}^{\infty} Q_{\bar{w}(k)}.$$

We now inductively define the segments in $\mathcal{C}(D)$. The definition involves a sequence (ϵ_k) of positive real numbers converging rapidly to zero. We initially require that $\epsilon_k < R_{k-2}/10$. The segments are of the form

$$I_m(w,v) = [a_m(w,v), b_m(w,v)], \quad m = 1,2,3,$$

where $a_m(w,v), b_m(w,v) \in \mathbb{C}$ and $(w,v) \in \mathcal{Y}_k$ for some $k = 0,1,2,\ldots$. We denote by $\pi_1 : \mathbb{C} \to \mathbb{R}$ the projection to the real axis.

We first choose

$$[a_1,b_1] = I_1 = I_1(e,e),\ [a_2,b_2] = I_2 = I_2(e,e),\ [a_3,b_3] = I_3 = I_3(e,e) \tag{47}$$

of length larger than ϵ_1 in $Q_e \setminus (Q_0 \cup Q_1)$, so that

$$\begin{aligned} \pi_1(a_1) &< \pi_1(b_1) < x_0 - R_1 < \pi_1(b_1) + \epsilon_2/10, \\ \pi_1(a_2) - \epsilon_2/10 &< x_0 + R_1 < \pi_1(a_2), \\ \pi_1(a_2) &< \pi_1(b_2) < x_1 - R_1 < \pi_1(b_2) + \epsilon_2/10, \\ \pi_1(a_3) - \epsilon_2/10 &< x_1 + R_1 < \pi_1(a_3) < \pi_1(b_3). \end{aligned}$$

We can also require the segments to be horizontal, but this is not necessary and such a requirement cannot be made below when $k \geqslant 1$.

Next fix $k \geqslant 1$ and assume that $I_m(w',v')$ and ϵ_ℓ are defined for $(w',v') \in \mathcal{Y}_\ell$, $0 \leqslant \ell \leqslant k-1$, so that

$$\text{if } B_1 \in \mathcal{B}_{\ell_1} \text{ and } B_2 \in \mathcal{B}_{\ell_2},\ B_1 \neq B_2, \text{ then } \overline{B}_1 \cap \overline{B}_2 = \emptyset. \tag{48}$$

Here

$$\mathcal{B}_\ell = \{B(z,\epsilon_\ell) :\ z \text{ endpoint of } I_m(w,v),\ (w,v) \in \mathcal{Y}_\ell,\ m = 1,2,3\}. \tag{49}$$

Let

$$I_1(w,v),\ I_2(w,v),\ I_3(w,v) \subset \operatorname{int}(Q_w) \setminus (Q_{w0} \cup Q_{w1}), \quad (w,v) = (w'\alpha, v'\beta) \in \mathcal{Y}_k,$$

be disjoint segments with the following properties: if we denote $a_m(w,v) = a_m$ and $b_m(w,v) = b_m$, then

$$\begin{aligned} \pi_1(a_1) &< \pi_1(b_1) < x_{w0} - R_{k+1} < \pi_1(b_1) + \epsilon_{k+1}/10, \\ \pi_1(a_2) - \epsilon_{k+1}/10 &< x_{w0} + R_{k+1} < \pi_1(a_2), \\ \pi_1(a_2) &< \pi_1(b_2) < x_{w1} - R_{k+1} < \pi_1(b_2) + \epsilon_{k+1}/10, \\ \pi_1(a_3) - \epsilon_{k+1}/10 &< x_{w1} + R_{k+1} < \pi_1(a_3) < \pi_1(b_3). \end{aligned}$$

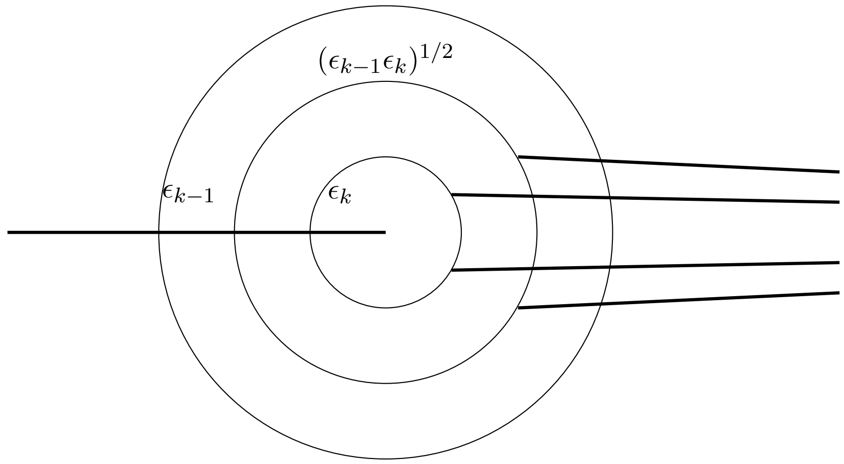

FIGURE 4. Positioning of the segments $I_m(w,v)$

We also require that (See Figure 4) if we denote

$$r_v = \epsilon_k,\ R_v = (\epsilon_{k-1}\epsilon_k)^{1/2} \quad \text{if } \beta = 0 \text{ or } 1,$$
$$r_v = (\epsilon_{k-1}\epsilon_k)^{1/2},\ R_v = \epsilon_{k-1} \quad \text{if } \beta = 2 \text{ or } 3, \tag{50}$$

then for each $r_v < r < R_v$ there are arcs

$$J_1(r,w,v) \subset S(b_m(w',v'),r), \quad m = \alpha+1,$$
$$J_3(r,w,v) \subset S(a_m(w',v'),r), \quad m = \alpha+2,$$

whose relative interiors are disjoint and do not intersect any segment $I_m(\tilde{w},\tilde{v})$, $(\tilde{w},\tilde{v}) \in \mathcal{Y}_\ell$, $0 \leqslant \ell \leqslant k$, so that

(i) the endpoints of $J_1(r,w,v)$ lie in $I_{\alpha+1}(w',v')$ and $I_1(w,v)$.
(ii) the endpoints of $J_3(r,w,v)$ lie in $I_{\alpha+2}(w',v')$ and $I_3(w,v)$.

We are now ready to define D; it is the domain for which

$$\mathcal{C}(D) = K \cup \{I_m(w,v) : m = 1,2,3,\ (w,v) \in \mathcal{Y}_k,\ k = 0,1,2,\ldots\},$$

where K is the Cantor set in (46).

4.2. **Construction of the exhaustion.** We now construct an exhaustion $\Phi_0 = (D_j)$ of D. We fix $k \in \mathbb{N}$ and $(w,v) \in \mathcal{Y}_k$. First, let $U(w,v)$ be a Jordan domain so that if we denote $I(w,v) = I_1(w,v) \cup I_2(w,v) \cup I_3(w,v)$ then

$$I(w,v) \subset U(w,v) \subset \overline{U}(w,v) \subset \operatorname{int}(Q_w) \setminus (Q_{w0} \cup Q_{w1}).$$

We also require that if $B \in \cup_\ell \mathcal{B}_\ell$ where $\mathcal{B}_\ell$ is the family of balls in (49), then either $U(w,v) \cap B = \emptyset$ or $I(w,v) \cap B \neq \emptyset$ and

$$U(w,v) \cap B \subset N_k(I(w,v)) \cap B. \tag{51}$$

Here $N_k(I(w,v))$ is the set of those $x \in \mathbb{C}$ for which

$$\operatorname{dist}(x, I(w,v)) < \frac{\min\{\epsilon_{k+1}, \operatorname{dist}(I(w,v), \mathbb{C} \setminus (D \cup I(w,v))\}}{100}. \tag{52}$$

Next, for $m = 1,2,3$ denote

$$U_m(k+1,w,v) = U(w,v)$$

and let

$$U_m(j,w,v), \quad j=k+2,k+3,\dots$$

be Jordan domains so that

$$\overline{U}_m(k+2,w,v)\cap\overline{U}_{m'}(k+2,w,v)=\emptyset \quad \text{if } m\neq m',$$

and, with $N_j(I_m(w,v))$ defined as in (52),

$$I_m(w,v)\subset U_m(j,w,v)\subset\overline{U}_m(j,w,v)\subset U_m(j-1,w,v)\cap N_j(I_m(w,v)).$$

We denote

$$\mathcal{A}_j=\{\overline{U}_m(j,w,v):\, m=1,2,3,\ (w,v)\in\mathcal{Y}_k,\ 0\leqslant k\leqslant j-1\},$$

and define D_j by

$$\mathcal{C}(D_j)=\mathcal{Q}_j\cup\mathcal{A}_j,$$

where $\mathcal{Q}_j$ is the family of squares in (45). Then $\Phi_0=(D_j)$ is an exhaustion of D. Theorem 1.2 follows by combining the two propositions below and choosing any $\Phi=(D_{j_n})$ so that (f_{j_n}) converges.

Proposition 4.1. *There is $\delta>0$ such that if $p=p_{\bar{w}}\in K$, then*

$$\operatorname{mod}(I_1,Q_{\bar{w}(j_0)};D_j)\geqslant\delta \quad \textit{for all } j_0\in\mathbb{N} \textit{ and } j>j_0. \tag{53}$$

Here I_1 is the segment in (47).

Recall that, given a domain $D\subset\hat{\mathbb{C}}$, $p\in\mathcal{C}(D)$, and an exhaustion $\Phi=(D_j)$ of D, we denote by p_ℓ the component in $\mathcal{C}(D_\ell)$ containing p. With this notation, $Q_{\bar{w}(j_0)}=p_{j_0}$ in (53).

Proposition 4.2. *Suppose $D\subset\hat{\mathbb{C}}$ is a domain with exhaustion $\Phi=(D_j)$. Fix $p\in\mathcal{C}(D)$ and a compact set $E\subset\hat{\mathbb{C}}$ such that $E\cap p=\emptyset$. If*

$$\lim_{\ell\to\infty}\liminf_{j\to\infty}\operatorname{mod}(E,p_\ell;D_j)>0, \tag{54}$$

then $\hat{f}(p)$ is non-trivial for all $f\in\mathcal{F}_\Phi$.

4.3. **Proof of Proposition 4.1.** Fix $p=p_{\overline{w}}\in K$, $j_0\in\mathbb{N}$, and $j>j_0$. Let $\mathcal{V}_0=\{e\}$, and for $k=1,2,\dots$, let

$$\mathcal{V}_k=\{v=v_1v_2\cdots v_k:\, v_\ell=\{0,1,2,3\}\,\text{for all } 1\leqslant\ell\leqslant k\},$$

so that $\mathcal{Y}_k=\mathcal{W}_k\times\mathcal{V}_k$. We consider the family of continua

$$\eta(v,t)\subset\hat{D}_j,\ v\in\mathcal{V}_{j-1},\ 1/4<t<3/4,$$

defined as follows: if $v=v_1v_2\dots v_{j-1}$, let $\eta(v,t)=A_j(v)\cup B_j(v,t)$, where

$$\begin{aligned}A_j(v)&=\cup\{\overline{U}_m(j,w,v_k)\in\mathcal{A}_j:\, m=1,2,3,\ w\in\mathcal{W}_k,\ 0\leqslant k\leqslant j-1\}, \text{ and}\\ B_j(v,t)&=\cup\{J_m(r[t,k],w,v_k):\, m=1,3,\ w\in\mathcal{W}_k,\ 1\leqslant k\leqslant j-1\}.\end{aligned}$$

Here $r[t,k]=R_{v_k}^t r_{v_k}^{1-t}$ and R_{v_k},r_{v_k} are the radii in (50).

Each $\eta(v,t)$ is a continuum joining $\overline{U}_1(j,e,e)$ and $\overline{U}_3(j,e,e)$ in $\hat{D}_j$. Moreover, each $\eta(v,t)$ intersects $Q_{\bar{w}(j_0)}$. By (51), we have $\eta(v,t)\setminus A_j(v)\subset D_j$. It is important to notice that the continua $\eta(v,t)$ do not intersect any of the squares in $\mathcal{Q}_j\subset\mathcal{C}(D_j)$.

Let $\gamma_{v,t}$ be an arc-length parametrization of $\eta(v,t)$, and

$$\Gamma_j = \{\gamma_{v,t} : v \in \mathcal{V}_{j-1},\, 1/4 < t < 3/4\}.$$

In view of the comments above, (53) follows if we can prove a lower bound for $\operatorname{mod}\Gamma_j$ independent of j. Fix $\rho \in X(\Gamma_j)$;

$$1 \leqslant \sum_{q\in A_j(v)} \rho(q) + \sum_{J\in B_j(v,t)} \int_J \rho\, ds.$$

Integrating both sides over $1/4 < t < 3/4$ and summing over $v \in \mathcal{V}_{j-1}$ yields

$$\frac{4^{j-1}}{2} \leqslant \frac{1}{2} \sum_{v\in\mathcal{V}_{j-1}} \sum_{q\in A_j(v)} \rho(q) + \sum_{v\in\mathcal{V}_{j-1}} \int_{1/4}^{3/4} \sum_{J\in B_j(v,t)} \int_J \rho\, ds\, dt = S_1 + S_2.$$

We estimate the sums S_1, S_2 from above. First, changing the order of summation yields

$$2S_1 = \sum_{k=0}^{j-2} 4^{j-1-k} \sum_{\substack{(w,v')\in\mathcal{Y}_k\\ m=1,2,3}} \rho(\overline{U}_m(j,w,v')) + \sum_{(w,v)\in\mathcal{Y}_{j-1}} \rho(\overline{U}(j,w,v)) = \sum_{k=0}^{j-1} S'_k.$$

Hölder's inequality yields

$$S'_k \leqslant 4^{j-k-1}(3\cdot 2^k \cdot 4^k)^{1/2} \Big(\sum_{q\in\mathcal{C}(D_j)} \rho(q)^2\Big)^{1/2} \leqslant 2^{2j-k/2-1} \Big(\sum_{q\in\mathcal{C}(D_j)} \rho(q)^2\Big)^{1/2}$$

for all $0 \leqslant k \leqslant j-1$. Thus, summing over k we have

$$S_1 \leqslant 4^j \sum_{k=0}^{j-1} 2^{-k/2} \Big(\sum_{q\in\mathcal{C}(D_j)} \rho(q)^2\Big)^{1/2} \leqslant 4^{j+1} \Big(\sum_{q\in\mathcal{C}(D_j)} \rho(q)^2\Big)^{1/2}.$$

We now estimate S_2. First, we denote by $\mathcal{Z}_\ell$ the set of centers z in the definition of $\mathcal{B}_\ell$ in (49). Fubini's theorem and (48) yield

$$S_2 \leqslant \sum_{k=1}^{j-1} 4^{j-k-1} \sum_{z\in\mathcal{Z}_k} \int_{1/4}^{3/4} \int_{S(z,r[t,k])} \rho\, ds\, dt = \sum_{k=1}^{j-1} T_k. \tag{55}$$

We apply change of variables to the integral in (55) to conclude that

$$T_k \leqslant 4^{j-k} \Big(\log \frac{\epsilon_{k-1}}{\epsilon_k}\Big)^{-1} \sum_{z\in\mathcal{Z}_k} \int_{B(z,\epsilon_{k-1})\setminus \overline{B}(z,\epsilon_k)} \frac{\rho(x)}{|x|}\, dA(x). \tag{56}$$

Applying Hölder's inequality to the integral in (56) yields

$$T_k \leqslant (2\pi)^{1/2} 4^{j-k} \Big(\log \frac{\epsilon_{k-1}}{\epsilon_k}\Big)^{-1/2} \sum_{z\in\mathcal{Z}_k} \Big(\int_{B(z,\epsilon_{k-1})} \rho(x)^2\, dA(x)\Big)^{1/2}.$$

Since $\operatorname{card}(Z_k) \leqslant 6\cdot 8^{k-1} \leqslant 8^k$ for all $0 \leqslant k \leqslant j-1$, we moreover have

$$T_k \leqslant (2\pi)^{1/2} 4^j \cdot 2^{-k/2} \Big(\log \frac{\epsilon_{k-1}}{\epsilon_k}\Big)^{-1/2} \Big(\int_{D_j} \rho(x)^2\, dA(x)\Big)^{1/2}.$$

Thus, if we require that $\epsilon_k \leqslant \epsilon_{k-1}/e$ for all k, we have

$$S_2 \leqslant (2\pi)^{1/2} 4^j \sum_{k=1}^{j-1} 2^{-k/2} \Big(\int_{D_j} \rho(x)^2 \, dA(x) \Big)^{1/2} \leqslant 4^{j+2} \Big(\int_{D_j} \rho(x)^2 \, dA(x) \Big)^{1/2}.$$

Combining the estimates yields

$$\begin{aligned} 4^{j-2} &\leqslant 4^{j+1} \Big(\sum_{q \in \mathcal{C}(D_j)} \rho(q)^2 \Big)^{1/2} + 4^{j+2} \Big(\int_{D_j} \rho(x)^2 \, dA(x) \Big)^{1/2} \\ &\leqslant 4^{j+3} \Big(\int_{D_j} \rho(x)^2 \, dA(x) + \sum_{q \in \mathcal{C}(D_j)} \rho(q)^2 \Big)^{1/2}. \end{aligned}$$

We conclude that $\mathrm{mod}(\Gamma_j) \geqslant 4^{-10}$. The proof is complete.

4.4. **Proof of Proposition 4.2.** By taking a subsequence of (D_j), we may assume $f_j \to f$. Suppose towards contradiction that $\hat{f}(p)$ is a point component. We lose no generality by assuming $\hat{f}(p) = \{0\}$.

Lemma 4.3. *Suppose $\hat{f}(p) = \{0\}$. For every $R > 0$ there are $r > 0$ and $m \in \mathbb{N}$ so that if $j \geqslant m$ and if $q \in \mathcal{C}(D_j)$ satisfies $\hat{f}_j(q) \cap S(0,R) \neq \emptyset$, then $\hat{f}_j(q) \cap S(0,r) = \emptyset$.*

Proof. Suppose towards contradiction that there is $R > 0$ and a sequence (q_{n_j}), $q_{n_j} \in \mathcal{C}(D_{n_j})$, so that each $\hat{f}_{n_j}(q_{n_j})$ intersects both $S(0,R)$ and $S(0, 2^{-j})$. By passing to a subsequence if necessary, we may assume $n_j = j$.

For each $j \in \mathbb{N}$, fix a point $x_j \in q_j$. Since $\hat{\mathbb{C}} \setminus D$ is compact, (x_j) has a subsequence converging to $x_0 \in q_0$ for some $q_0 \in \mathcal{C}(D)$. We may assume that $x_j \to x_0$. It follows that if $k \in \mathbb{N}$ and if $q_0(k)$ is the element of $\mathcal{C}(D_k)$ containing q_0, then

$$q_j \subset q_0(k) \quad \text{for all } j \geqslant j_k.$$

In particular, since $\hat{f}_j(q_j)$ intersects both $S(0,R)$ and $S(0,2^{-j})$, so does $\hat{f}_j(q_0(k))$. We conclude that $\hat{f}(q_0(k))$ contains both the origin and a point in $S(0,R)$. But this holds for all k, so also $\hat{f}(q_0)$ contains both the origin and a point in $S(0,R)$. This contradicts our assumption, that $\hat{f}(p) = \{0\}$. The proof is complete. □

We use Lemma 4.3 to construct a decreasing sequence (R_n) of positive real numbers and an increasing sequence j_n of indices as follows (compare to the proof of Lemma 2.7): First, choose R_1, j_1 so that $\hat{f}_j(E) \cap B(0, 2R_1) = \emptyset$ for all $j \geqslant j_1$. Here E is the compact set in the statement of the proposition.

Then, assuming that R_n, j_n have been constructed, choose $R_{n+1} < R_n/2$ and $j_{n+1} \geqslant j_n$ such that if $q \in \mathcal{C}(D_j)$, $j \geqslant j_{n+1}$, and $\hat{f}_j(q) \cap S(0,R_n) \neq \emptyset$, then $\hat{f}_j(q) \cap S(0, 2R_{n+1}) = \emptyset$.

Given $k \in \mathbb{N}$, let N be the largest number for which there is $j'_N \geqslant k$ so that $\hat{f}_j(p_k) \subset B(0, R_N)$ for all $j \geqslant j'_N$. We may assume that $j'_N = j_N$. Then

$$\mathrm{mod}(\hat{f}_j(E), \hat{f}_j(p_k); f_j(D_j)) \leqslant \mathrm{mod}(S(0,2R_1), S(0,R_N); f_j(D_j)) \tag{57}$$

for all $j \geqslant j_N$ (here the modulus on the left is over all paths connecting $\hat{f}_j(E)$ and $\hat{f}_j(p_k)$ in $\widehat{f_j(D_j)}$, a slight abuse of earlier terminology). Fix such a j. We construct a test function ρ as follows: First, let $1 \leqslant n \leqslant N$. We denote $A(R,r) = B(0,R) \setminus \overline{B}(0,r)$ and define

$$\rho_n(x) = \begin{cases} \frac{1}{|x|\log 2}, & x \in D_j \cap A(2R_n, R_n) \\ \frac{\operatorname{diam}(x)}{R_n \log 2}, & x \in \mathcal{C}(f_j(D_j)),\, x \cap A(2R_n, R_n) \neq \emptyset,\, \operatorname{diam}(x) \leqslant \operatorname{dist}(x,0), \\ 1, & x \in \mathcal{C}(f_j(D_j)),\, x \cap A(2R_n, R_n) \neq \emptyset,\, \operatorname{diam}(x) > \operatorname{dist}(x,0), \end{cases}$$

and $\rho_n(x) = 0$ otherwise. As in the proof of Lemma 2.7, we have

$$\rho = \frac{1}{N}\sum_{n=1}^{N} \rho_n \in X(S(0,2R_1), S(0,R_N); f_j(D_j)) \quad \text{for all } j \geqslant j_N.$$

For each $q \in \mathcal{C}(D_j)$ there is at most one n such that $\rho_n(q) \neq 0$. Moreover, for every n there are at most 30 elements (disks) $q \in \mathcal{C}(f_j(D_j))$ such that $q \cap A(2R_n, R_n) \neq \emptyset$ and $\operatorname{diam}(q) > \operatorname{dist}(q,0)$. Thus we can estimate

$$\begin{aligned} \int_{f_j(D_j)} \rho_n^2 \, dA + \sum_{q \in \mathcal{C}(f_j(D_j))} \rho_n(q)^2 &\leqslant \frac{1}{(\log 2)^2} \int_{A(2R_n,R_n)} \frac{dA}{|x|^2} \\ + \quad \frac{\operatorname{Area}(B(0,4R_n))}{R_n^2 (\log 2)^2} + 30 &\leqslant \frac{2\pi}{\log 2} + \frac{16\pi}{(\log 2)^2} + 30 \leqslant 1000, \end{aligned}$$

and, since we have chosen j_k so that every $q \in \mathcal{C}(f_j(D_j))$ satisfies $\rho_n(q) \neq 0$ for at most one n,

$$\int_{f_j(D_j)} \rho^2 \, dA + \sum_{q \in \mathcal{C}(f_j(D_j))} \rho(q)^2 \leqslant \frac{1000N}{N^2} = \frac{1000}{N}. \tag{58}$$

Since $N \to \infty$ as $k \to \infty$, combining (58) with (57) yields

$$\lim_{k\to\infty} \liminf_{j\to\infty} \operatorname{mod}(\hat{f}_j(E), \hat{f}_j(p_k); f_j(D_j)) = 0. \tag{59}$$

But (59) and the conformal invariance of the modulus contradict our assumption (54). The proof is complete.

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