MOTIVIC CONGRUENCES AND SHARIFI'S CONJECTURE

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Abstract. Let f be a cuspidal eigenform of weight two and level N, let $p \nmid N$ be a prime at which f is congruent to an Eisenstein series and let V_f denote the p-adic Tate module of f. Beilinson [Bei] constructed a class $\kappa_f \in H^1(\mathbb{Q}, V_f(1))$ arising from the cup product of two Siegel units and proved a striking relationship with the first derivative $L'(f,0)$ at the near central point $s=0$ of the L-series of f, which led him to formulate his celebrated conjecture. In this note we prove two congruence formulae relating the "motivic part" of $L'(f,0) \pmod p$ and $L''(f,0) \pmod p$ with circular units. The proofs make use of delicate Galois properties satisfied by various integral lattices within V_f and exploits Perrin-Riou's, Coleman's and Kato's work on the Euler systems of circular units and Beilinson–Kato elements and, most crucially, the work of Sharifi [Sha], Fukaya–Kato [FK] and Ohta [Oh1].

Contents

1. Introduction

The aim of this paper is to show how the ideas underlying Sharifi's conjecture [Sha] and the work [FK] of Fukaya and Kato can be exploited to study congruences among motivic classes that naturally arise from the Euler systems of Beilinson-Kato elements and circular units.

In order to set the stage, let $\theta : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ be an even, primitive, Dirichlet character of conductor $N \geq 4$ and let $f \in S_2(N, \theta)$ be a normalized cuspidal eigenform of level N, weight 2 and nebentype θ.

Fix a prime $p \nmid 6N\varphi(N)$. Let $T_{f,X}$ and $T_{f,Y}$ denote the integral p-adic Galois representations given as the f-isotypical quotient of $H^1_{\mathrm{et}}(\bar{X}, \mathbb{Z}_p(1))$, resp. $H^1_{\mathrm{et}}(\bar{Y}, \mathbb{Z}_p(1))$, for the closed, resp. open, modular curve X, resp. Y, of level $\Gamma_1(N)$. Cf. §2 for the particular models of these curves we employ in this article and precise definitions. We also denote by $\bar{X}$ and $\bar{Y}$ the corresponding curves over a fixed algebraic closure of $\mathbb{Q}$. Set as usual $V_f = T_{f,X} \otimes \mathbb{Q} = T_{f,Y} \otimes \mathbb{Q}$.

Beilinson [Bei] introduced a motivic element in the K-group $K_2(Y)$ giving rise to a global Galois cohomology class

$$\kappa_f = \kappa_f(\chi_1, \chi_2) \in H^1(\mathbb{Q}, T_{f,Y}(\psi)(1)) \tag{1}$$

2010 *Mathematics Subject Classification.* 11F33 (primary); 11F67, 11F80 (secondary).

that was later the basis of Kato's Euler system in [Ka]. This class depends on auxiliary data (cf. loc. cit. and [BD2], [Han], [KLZ1, §9], [Sch] for several presentations of the subject in the literature). With our normalizations, κ_f depends on the choice of two auxiliary even Dirichlet characters $\chi_1, \chi_2 : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ whose product of conductors N_1 and N_2 is N (see §3), and $\psi = \bar{\theta}\chi_2$. It is straightforward to relate it to other equivalent conventions adopted in loc. cit.

The main motivation for Beilinson's construction of κ_f was providing evidence for his celebrated conjecture on values of L-functions of mixed motives, encompassing Dirichlet's unit theorem and the Birch and Swinnerton-Dyer conjecture for elliptic curves as instances of it. This conjecture was later refined by Bloch-Kato in [BK], and in the case at hand predicts that

$$\operatorname{ord}_{s=0} L(f,\psi,s) \stackrel{?}{=} \dim H^1_{\mathrm{f}}(\mathbb{Q}, V_f(\psi)(1)), \tag{2}$$

where the left-hand side is the L-series associated to f and the character ψ, and the right-hand side denotes the space of *crystalline* classes in $H^1(\mathbb{Q}, V_f(\psi)(1))$. Since $L(f,\psi,0)$ vanishes (for innocent reasons, due to the vanishing of a Γ-factor arising from analytic continuation), (2) suggests that $H^1_{\mathrm{f}}(\mathbb{Q}, V_f(\psi)(1))$ should contain non-trivial classes, and Beilinson proved in loc. cit. that κ_f is indeed crystalline and non-trivial, provided $L'(f,\psi,0) \neq 0$.

Let F be the finite extension of $\mathbb{Q}$ generated by the field of coefficients of f and the values of all Dirichlet characters of conductor N; let $\mathcal{O}$ be its ring of integers and $\mathfrak{p} \subset \mathcal{O}$ a prime ideal above p.

In this note we assume f is congruent to an Eisenstein series modulo $\mathfrak{p}$. Up to replacing f with a twist of it, we may assume without loss of generality that

$$f \equiv E_2(\theta, 1) \bmod \mathfrak{p}^t \tag{3}$$

for some $t \geq 1$, where $E_2(\theta,1)$ is the classical Eisenstein series recalled in (7) below. Note that congruence (3) is equivalent to $f^* \equiv E_2(1,\bar{\theta})$, where $f^* = f \otimes \theta^{-1}$ is the dual form of f, and this in turn implies that $\mathfrak{p}^t$ divides the generalized Bernoulli number $B_{2,\bar{\theta}}$ (or equivalently the L-value $L(\bar{\theta},-1)$). We take t to be the largest power satisfying (3).

As is well-known, $T_{f,X}$ and $T_{f,Y}$ are finitely generated $\mathcal{O}_{\mathfrak{p}}[G_{\mathbb{Q}}]$-modules giving rise to the same Galois representation V_f over $F_{\mathfrak{p}}$. However, the lattices $T_{f,X}$ and $T_{f,Y}$ are not isomorphic as $G_{\mathbb{Q}}$-modules, and in general neither of them are necessarily free as $\mathcal{O}_{\mathfrak{p}}$-modules. Setting $\bar{T} := T \otimes \mathcal{O}/\mathfrak{p}^t$ for any $\mathcal{O}_{\mathfrak{p}}$-module, congruence (3) does imply (cf. §2) that one always has surjective homomorphisms of $G_{\mathbb{Q}}$-modules

$$\bar{T}_{f,Y} \xrightarrow{\bar{\pi}_1} \mathcal{O}/\mathfrak{p}^t(\theta), \quad \bar{T}_{f,X} \xrightarrow{\bar{\pi}_2} \mathcal{O}/\mathfrak{p}^t(1). \tag{4}$$

The maps $\bar{\pi}_1$ and $\bar{\pi}_2$ in (4) are non-canonical, but we exploit the work of Ohta [Oh1], [Oh2], Sharifi [Sha] and Fukaya-Kato [FK] to rigidify them in a canonical way, in the sense that both $\bar{\pi}_1$ and $\bar{\pi}_2$ only depend on canonical periods naturally associated to f; cf. (42) and (52).

Write $\bar{\kappa}_f = \kappa_f \pmod{\mathfrak{p}^t}$ and define

$$\bar{\kappa}_{f,1} = \bar{\pi}_{1*}(\bar{\kappa}_f) \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1)).$$

Motivated by the above discussion of Beilinson's conjecture, $\bar{\kappa}_{f,1}$ may be regarded as a motivic avatar of the first derivative $L'(f,\psi,0) \pmod{\mathfrak{p}^t}$. The first main result of this note, Theorem 1.1 below, is an explicit formula for $\bar{\kappa}_{f,1}$ in terms of algebraic L-values and circular units, which in particular provides a criterion for this class to vanish. In the parlance of [BD1], our Theorem 1.1 may be interpreted as a *Jochnowitz congruence*

$$L'_{\mathrm{alg}}(f,\psi,0) \equiv L'_{\mathrm{alg}}(\theta\psi,0) \pmod{\mathfrak{p}^t}$$

between the "algebraic" or "motivic parts" of the derivative at $s=0$ of the Hasse–Weil L-function $L(f,\psi,s)$ and Dirichlet's L-function $L(\theta\psi,s)$.

We further show that κ_f may be lifted to an element in $H^1(\mathbb{Q}, T_{f,X}(\psi)(1))$ if and only if the $(\bmod\ \mathfrak{p}^t)$ class $\bar{\kappa}_{f,1}$ vanishes. When this happens, such a lift is unique and we thus continue to denote it κ_f by slight abuse of notation; we may then define

$$\bar{\kappa}_{f,2} = \bar{\pi}_{2*}(\bar{\kappa}_f) \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\psi)(2)),$$

which we regard as the motivic counterpart of the second derivative $L''(f,\psi,0) \pmod{\mathfrak{p}^t}$.

The second main result of this note, Theorem 1.2 below, provides an explicit formula for $\bar{\kappa}_{f,2}$ as the cup product of two circular units. We find it interesting that the circle of ideas appearing in [FK] and [Sha] can be applied to the computation of second derivatives, a type of result which appears to be quite novel.

In order to state our results precisely, let

$$\mathbb{Z}[\mu_{N_2}]^\times[\bar\chi_2] = (\mathbb{Z}[\mu_{N_2}]^\times \otimes \mathcal{O}_\mathfrak{p}(\chi_2))^{\mathrm{Gal}\,(\mathbb{Q}(\mu_{N_2})/\mathbb{Q})} \simeq \mathrm{Hom}(\mathcal{O}_\mathfrak{p}(\bar\chi_2), \mathbb{Z}[\mu_{N_2}]^\times \otimes \mathcal{O}_\mathfrak{p}) \tag{5}$$

denote the $\bar\chi_2$-isotypic component of $\mathbb{Z}[\mu_{N_2}]^\times \otimes \mathcal{O}_\mathfrak{p}$ on which $\mathrm{Gal}\,(\mathbb{Q}(\mu_{N_2})/\mathbb{Q})$ acts through $\bar\chi_2$, which may be naturally identified with a $\mathcal{O}_\mathfrak{p}$-submodule of $\mathbb{Z}[\mu_{N_2}]^\times \otimes \mathcal{O}_\mathfrak{p}$ of rank 1 when $\chi_2 \neq 1$ (resp. rank 0 when $\chi_2 = 1$). Kummer theory gives rise to an injective homomorphism

$$\mathbb{Z}[\mu_{N_2}]^\times[\bar\chi_2] \to \mathrm{Hom}(G_{\mathbb{Q}(\mu_{N_2})}, \mathcal{O}_\mathfrak{p}(1))[\bar\chi_2] \to H^1(\mathbb{Q}, \mathcal{O}_\mathfrak{p}(\chi_2)(1)).$$

Fix a primitive N_2-th root of unity ζ_{N_2} and define the circular unit

$$c_{\chi_2} := \prod_{a=1}^{N_2-1} (1-\zeta_{N_2}^a)^{\chi_2(a)} \in \mathbb{Z}[\mu_{N_2}]^\times[\bar\chi_2]. \tag{6}$$

Let

$$c_{\chi_2} \in H^1(\mathbb{Q}, \mathcal{O}_\mathfrak{p}(\chi_2)(1))$$

denote, with the same symbol by a slight abuse of notation, its image under the identification provided by the Kummer map. Write $\bar c_{\chi_2} = c_{\chi_2} \pmod{\mathfrak{p}^t} \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1))$.

Our main theorems are conditional on the following two hypotheses, that we assume throughout this article for simplicity, although some of them can be easily relaxed:

(H1) *Non-trivial zeroes mod $\mathfrak{p}$:*

$$\theta(p)-1,\ \chi_2(p)-1,\ \theta\bar\chi_1(p)-1,\ \chi_1(p)-1 \neq 0 \pmod{\mathfrak{p}}.$$

(H2) Letting Σ_X and Σ_Y denote the torsion submodules of $T_{f,X}$ and $T_{f,Y}$ respectively, the $G_{\mathbb{Q}_p}$-module $\mathcal{O}/\mathfrak{p}^t(\theta)$ does not show up as a quotient of Σ_Y/Σ_X.

Note that (H1) implies that χ_2 and in fact $(\chi_2)_{|\mathbb{Q}_p}$ is non-trivial, even mod $\mathfrak{p}$; in particular, c_{χ_2} is a non-trivial unit. Note that (H2) follows automatically if the localization of the Hecke algebra acting on $M_2(\Gamma_1(N))$ at the Eisenstein ideal is Gorenstein, as this implies that $T_{f,Y}$ is free as $\mathcal{O}_\mathfrak{p}$-module. We wonder whether (H2) might be weaker and more tractable than asking the Hecke algebra to be Gorenstein.

Define the algebraic L-value

$$L^{\mathrm{alg}}(f^*, \theta\bar\chi_1, 1) = L(f^*, \theta\bar\chi_1, 1)/\Omega_f^+ \in \mathcal{O}$$

where Ω_f^+ is Shimura's complex period associated to f^*, chosen in a specific way that we recall in §3.1. Let also

$$\mathfrak{g}(\xi) = \sum_{a=1}^{M-1} \xi(a)\zeta_M^a$$

denote the Gauss sum attached to a Dirichlet character ξ of conductor M, where ζ_M is a primitive M-th root of unity.

Theorem 1.1. *In $H^1(\mathbb{Q}_p, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1))$ we have*

$$\bar\kappa_{f,1} \equiv \frac{B_{2,\bar\theta\chi_2}}{2\mathfrak{g}(\bar\theta\chi_2)} \cdot L^{\mathrm{alg}}(f^*, \theta\bar\chi_1, 1) \cdot \bar c_{\chi_2} \pmod{\mathfrak{p}^t},$$

where $B_{2,\bar\theta\chi_2}$ is the generalized Bernoulli number, defined in Section 2. This equality takes place globally in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1))$ if p is $\bar\chi_2$-regular as specified in (29).

Let us state now our second main result. As we describe in more detail in §3, Kato's class is constructed as

$$\kappa_f = \pi_{f*}(u \cup v) \in H^1(\mathbb{Q}, T_{f,Y}(\psi)(1)),$$

namely the push-forward to the f-isotypic component of the cup product of two modular units

$$u = u_{\chi_1,\chi_2} \quad \text{and} \quad v = u_{\theta\bar\chi_1\bar\chi_2,1}$$

whose logarithmic derivatives are respectively the classical Eisenstein series $E_2(\chi_1,\chi_2)$ and $E_2(\theta\bar\chi_1\bar\chi_2, 1)$ given in (7).

The unit u_{χ_1,χ_2} is determined by its logarithmic derivative only up to a multiplicative constant, and therefore the first non-vanishing coefficient in the Laurent expansion of u_{χ_1,χ_2} at ∞, which we simply denote $u_{\chi_1,\chi_2}(\infty)$ as in [FK, §5], may be chosen arbitrarily. Since $\mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q})$ acts on $E_2(\chi_1,\chi_2)$ via χ_1 [St1, Theorem 1.3.1], it is natural to normalize u_{χ_1,χ_2} likewise, so that $u_{\chi_1,\chi_2}(\infty)$ may be any power of the circular unit c_{χ_1}. In the literature one finds different normalizations, typically either $u_{\chi_1,\chi_2}(\infty)=1$ or c_{χ_1}. In the statement below we have chosen to normalize the modular units above so that

$$u_{\chi_1,\chi_2}(\infty)=c_{\chi_1}, \quad u_{\theta\bar{\chi}_1\bar{\chi}_2,1}(\infty)=c_{\theta\bar{\chi}_1\bar{\chi}_2}$$

but any other choice would be perfectly fine, upon replacing accordingly the two circular units appearing in the cup product below.

Theorem 1.2. *Assume that $\bar{\kappa}_{f,1}=0$. Let $L_p(\bar{\theta},s)$ denote the Kubota–Leopoldt p-adic L-function attached to $\bar{\theta}$ and assume $L_p'(\bar{\theta},-1)$ is a p-adic unit. The following equality holds in $H^1(\mathbb{Q},\mathcal{O}/\mathfrak{p}^t(\psi)(2))$:*

$$\bar{\kappa}_{f,2}\equiv\frac{L_p'(\bar{\theta},-1)}{1-p^{-1}}\cdot\frac{\bar{c}_{\chi_1}\cup\bar{c}_{\theta\bar{\chi}_1\bar{\chi}_2}}{\cup\log_p(\varepsilon_{\mathrm{cyc}})} \pmod{\mathfrak{p}^t}.$$

Here $\varepsilon_{\mathrm{cyc}}$ is the cyclotomic character and $1/\cup\log_p(\varepsilon_{\mathrm{cyc}})$ denotes the inverse of the map

$$H^1(\mathbb{Q},\mathcal{O}/\mathfrak{p}^t(\psi)(2))\to H^2(\mathbb{Q},\mathcal{O}/\mathfrak{p}^t(\psi)(2)), \quad \kappa\mapsto\kappa\cup\log_p(\varepsilon_{\mathrm{cyc}}),$$

which is invertible under our assumptions.

In the above statement note that our running assumptions imply that $L_p(\bar{\theta},-1)\equiv 0 \pmod{\mathfrak{p}^t}$ and it is thus natural that the first derivative of the Kubota–Leopoldt p-adic L-function makes an appearance.

Theorems 1.1 and 1.2 are proved in §3 and §4 respectively. We hope the methods introduced in this note may help to extend Sharifi's conjectures to other scenarios where the theory of Euler systems has experienced exciting progress in recent years (cf. e.g. [Eu], [KLZ1], [LPSZ], [LSZ]).

Acknowledgements. We thank P. Wake for guiding us through the delicate subject of integral lattices in Galois representations. We also thank G. Gras for pointing us to suitable references and providing careful explanations of his work. We are also grateful to H. Darmon and R. Sharifi for their comments. We would like to sincerely thank the anonymous referee for a very careful reading of the manuscript, whose comments notably contributed to improve the exposition and correct inaccuracies in earlier versions of the paper. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 682152). The first author has also received financial support through "la Caixa" Fellowship Grant for Doctoral Studies (grant LCF/BQ/ES17/11600010) and from the Royal Society Newton International Fellowship NIF\R1\202208. The second author is supported by Icrea through an Icrea Academia Grant.

2. Eisenstein series, modular curves and lattices

The aim of this section is recalling well-known facts and setting notations concerning Eisenstein series, models of modular curves and various integral lattices associated to them. We take the chance to prove some elementary relationships among the latter, which are surely well-known to experts but that we include because we failed to find precise references in the literature.

Fix algebraic closures $\bar{\mathbb{Q}}$, $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively, and embeddings of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ and $\mathbb{C}$. Fix a field extension $F/\mathbb{Q}$, and let $\mathcal{O}$ be its ring of integers. The choice of the embedding singles out a prime ideal $\mathfrak{p}$ of $\mathcal{O}$ lying above p, and we let $\mathcal{O}_{\mathfrak{p}}$ denote the completion of $\mathcal{O}$ at $\mathfrak{p}$. We also fix throughout an uniformizer ϖ of $\mathcal{O}_{\mathfrak{p}}$ and an isomorphism $\mathbb{C}_p\simeq\mathbb{C}$.

Given a variety $Y/\mathbb{Q}$, let $Y_F=Y\times F$ denote the base change of Y to F and set $\bar{Y}=Y_{\bar{\mathbb{Q}}}$. We denote by $\mathcal{O}$ the ring of integers of F. Fix an integer $N\geq 4$ and let $Y_1(N)\subset X_1(N)$ denote the canonical models over $\mathbb{Q}$ of the (affine and projective, respectively) modular curves classifying pairs (A,i) where A is a (generalized) elliptic curve and $i:\mu_N\to A$ is an embedding of group schemes. It is important to recall that this is not the model used by Fukaya and Kato, as they consider the one which classifies pairs (A,P), where A is a (generalized) elliptic curve and P is an N-torsion point of it. In any case, the model of [FK] can be obtained from ours, as explained in [FK, §1.4.2].

Let $C_N := X_1(N) \setminus Y_1(N)$ denote the finite scheme of cusps; among them one may distinguish the cusp $\infty \in C_N(\mathbb{Q})$ associated to Tate's elliptic curve over $\mathbb{Z}((q))$, which is rational over $\mathbb{Q}$ in this choice of model (cf. e.g. [St1, §1.3], [St2]). (Again, note that in the model of [FK, §1.3.3], while the cusp 1 is defined over $\mathbb{Q}$, the cusp ∞ is only rational over $\mathbb{Q}(\mu_N)^+$, the maximal totally real subfield of $\mathbb{Q}(\mu_N)$.)

Assume now that F contains the values of all Dirichlet characters of conductor dividing N and continue denoting by $\mathcal{O}$ its ring of integers. Then a basis of $\mathrm{Eis}_2(\Gamma_1(N), F)$ is indexed by triples (χ_1, χ_2, r) where χ_1 and χ_2 are primitive Dirichlet characters of conductors N_1 and N_2 with $N_1 \cdot N_2 \mid N$, $\chi_1(-1) = \chi_2(-1)$, and r is a positive integer with $1 < rN_1N_2 \mid N$, provided by the Eisenstein series (cf. e.g. [DS, Theorem 4.6.2], [St1, Def. 3.4.1]):

$$E_2(\chi_1,\chi_2,r) = a_0 + \sum_{n=1}^{\infty}\Big(\sum_{d|n}\chi_1(n/d)\chi_2(d)d\Big)q^{rn}, \quad a_0 = \begin{cases}\frac{L(\chi_2,-1)}{2} & \text{if } \chi_1 = 1\\ 0 & \text{if } \chi_1 \neq 1\end{cases} \tag{7}$$

unless $\chi_1 = \chi_2 = 1$, in which case $E_2(1,1,r) = \sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^n - r\sum_{n=1}^{\infty}\Big(\sum_{d|n} d\Big)q^{rn}$.

When $r = 1$ we shall simply denote $E_2(\chi_1,\chi_2) := E_2(\chi_1,\chi_2,1)$.

When $\chi_1 = 1$, the constant term may also be recast as a generalized Bernoulli number: setting $B_2(x) = x^2 - x + 1/6$, define

$$B_{2,\chi} = N\sum_{a=1}^{N-1}\chi(a)\cdot B_2(a/N)$$

for any Dirichlet character χ of conductor N. One then has $-2L(\chi,-1) = B_{2,\chi}$.

Define the group of modular units $U(N)$ as the subgroup of rational functions of $X_1(N)_{\mathbb{Q}(\mu_N)}$ with zeroes and poles concentrated at the cusps, that is to say

$$U(N) = \mathcal{O}(Y_1(N)_{\mathbb{Q}(\mu_N)})^{\times}.$$

Similarly to (5), let $U(N)[\chi]$ denote the χ-isotypic component of $U(N)\otimes\mathcal{O}_{\mathfrak{p}}$ on which $\mathrm{Gal}\left(\mathbb{Q}(\mu_N)/\mathbb{Q}\right)$ acts through the character χ. In light of [St1, Theorem 1.3.1], there exists a modular unit $u_{\chi_1,\chi_2} \in U(N)[\chi_1]$ satisfying

$$\mathrm{dlog}(u_{\chi_1,\chi_2}) = E_2(\chi_1,\chi_2)\frac{dq}{q} \quad \text{and} \quad u_{\chi_1,\chi_2}(\infty) = c_{\chi_1}. \tag{8}$$

The q-expansion of the modular units u_{χ_1,χ_2} can be written down explicitly. Given a pair of integers (a,b) between 0 and $N-1$, not both equal to 0, define the Siegel unit

$$u_{a,b;N} = q^w\prod_{n\geq 0}\Big(1 - q^{n+a/N}\zeta_N^b\Big)\prod_{n\geq 1}\Big(1 - q^{n-a/N}\zeta_N^{-b}\Big),$$

where $w = \frac{1}{12} - \frac{a}{N} + \frac{a^2}{2N^2}$. Then the q-expansion of the modular unit $u_{\chi_1,1}$ is given by

$$u_{\chi_1,1} = \frac{-1}{2\mathfrak{g}(\bar{\chi}_1)}\sum_{b=1}^{N-1}\bar{\chi}_1(b)\otimes u_{0,b;N}, \tag{9}$$

where here N stands for the conductor of χ_1. Although we will not use them here in this note, similar expressions can be given for u_{χ_1,χ_2} for arbitrary χ_2 by averaging $u_{a,b;N}$ and choosing an appropriate uniformizer.

Kummer theory induces a morphism

$$\delta : U(N)[\chi] \to H^1_{\mathrm{et}}(Y_1(N), \mathcal{O}_{\mathfrak{p}}(\bar{\chi})(1)). \tag{10}$$

Let $\mathbb{T} \subset \mathrm{End}\, H^1_{\mathrm{et},c}(\overline{Y}_1(N), \mathcal{O}_{\mathfrak{p}}(1))$ denote the Hecke algebra acting on the compactly-supported cohomology of the open modular curve generated by the standard Hecke operators T_ℓ for every (good or bad) prime ℓ and the diamond operators $\langle d\rangle$ for all $d \in \mathbb{Z}_{\geq 1}$ with $(d,N) = 1$. At primes $\ell \mid N$ we may also denote T_ℓ by U_ℓ. As in [FK, §1.2.6], define also $\mathbb{T}^* = \langle T_\ell^*, \langle d\rangle\rangle \subset \mathrm{End}\, H^1_{\mathrm{et}}(\overline{Y}_1(N), \mathcal{O}_{\mathfrak{p}}(1))$ where T_ℓ^* stand for the dual Hecke operators as defined in [Oh1, §3.4], [KLZ1, Def. 2.4.3].

Recall from the introduction the newform $f \in S_2(N,\theta)$ satisfying $f \equiv E_2(\theta,1) \bmod \mathfrak{p}^t$. Let $f(q) = \sum a_n(f)q^n$ denote its q-expansion at the cusp ∞. Enlarge F so that it also contains the eigenvalues $\{a_n(f)\}_{n\geq 1}$. We make the following assumptions:

- $p \nmid N\varphi(N)$;

- θ is a primitive and even character of conductor N;
- $\theta(p) \neq 1 \bmod \mathfrak{p}$.

Note that in [FK] the authors do not assume that $p \nmid N$, but we do need it here at several points in the article.

Let $I_f^* = (T_\ell^* - a_\ell(f)) \subset \mathbb{T}^*$ denote the ideal associated to the system of eigenvalues of f with respect to the dual Hecke operators. Define the $\mathcal{O}_\mathfrak{p}$-modules

$$T_{f,X} = H^1_{\mathrm{et}}(\overline{X}_1(N), \mathcal{O}_\mathfrak{p}(1))/I_f^*, \quad T_{f,Y} = H^1_{\mathrm{et}}(\overline{Y}_1(N), \mathcal{O}_\mathfrak{p}(1))/I_f^*. \tag{11}$$

Note that the two lattices $T_{f,X}$ and $T_{f,Y}$ may give rise to different $\mathcal{O}_\mathfrak{p}[G_\mathbb{Q}]$-modules in spite of the fact that the associated rational Galois representations

$$V_f := T_{f,X} \otimes_{\mathcal{O}_\mathfrak{p}} F_\mathfrak{p} \simeq T_{f,Y} \otimes_{\mathcal{O}_\mathfrak{p}} F_\mathfrak{p}$$

are isomorphic.

Proposition 2.1. *The natural inclusion $X \hookrightarrow Y$ induces by push-forward in cohomology a map $T_{f,X} \to T_{f,Y}$ that sits in an exact sequence of $\mathcal{O}_\mathfrak{p}[G_\mathbb{Q}]$-modules*

$$0 \to T_{f,X} \longrightarrow T_{f,Y} \xrightarrow{\pi_1} \mathcal{O}/\mathfrak{p}^t(\theta) \to 0.$$

Proof. Write for short $H^1(X) = H^1_{\mathrm{et}}(\bar{X}_1(N), \mathcal{O}_\mathfrak{p}(1))$ and $H^1(Y) = H^1_{\mathrm{et}}(\bar{Y}_1(N), \mathcal{O}_\mathfrak{p}(1))$. As shown for instance in [St1, §1.8], there is a short exact sequence of the form

$$0 \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow \mathrm{Div}^0[C_N] \longrightarrow 0, \tag{12}$$

where $\mathrm{Div}^0[C_N]$ is the free $\mathcal{O}_\mathfrak{p}$-module of degree 0 divisors supported on C_N with coefficients in $\mathcal{O}_\mathfrak{p}$. Since the map $H^1(X) \hookrightarrow H^1(Y)$ induces an injection

$$H^1(X)/(I_f^* \cdot H^1(Y) \cap H^1(X)) \hookrightarrow H^1(Y)/I_f^* \cdot H^1(Y),$$

the exactness on the left follows once we show that

$$I_f^* \cdot H^1(Y) \cap H^1(X) = I_f^* \cdot H^1(X). \tag{13}$$

The inclusion $I_f^* H^1(X) \subset I_f^* H^1(Y) \cap H^1(X)$ is clear. As for the opposite one, take an element $t\alpha \in I_f^* H^1(Y) \cap H^1(X)$ with $t \in I_f^*$ and $\alpha \in H^1(Y)$. After inverting p, there is an isomorphism

$$H^1(Y)[1/p] \simeq H^1(X)[1/p] \oplus \mathrm{Div}^0[C_N][1/p],$$

since (12) is split after tensoring with $\mathbb{Q}_p$ by Manin-Drinfeld's theorem. We may thus write $\alpha = \beta + \gamma$, where $\beta \in H^1(X)[1/p]$ and $\gamma \in \mathrm{Div}^0[C_N][1/p]$. Since $t\alpha \in H^1(X)$, we have $t\gamma = 0$, and thus $t\alpha = t\beta$ in $I_f^*(H^1(Y) \cap H^1(X)[1/p])$.

Note that $H^1(Y) \cap H^1(X)[1/p] = H^1(X)$. Indeed, otherwise there would exist an element $y \in H^1(Y) \setminus H^1(X)$ such that $p^u y \in H^1(X)$ for some u. This would imply that $H^1(Y)/H^1(X)$ contains non-trivial torsion, but this quotient is isomorphic to $\mathrm{Div}^0[C_N]$, which is free. It thus follows that $t\alpha$ lies in $I_f^* H^1(X)$, and this proves exactness on the left.

In order to conclude the proof of the proposition, let us now show that $T_{f,Y}/T_{f,X} \simeq \mathcal{O}/\mathfrak{p}^t(\theta)$. As follows from the above, (12) induces an exact sequence

$$0 \longrightarrow T_{f,X} \longrightarrow T_{f,Y} \longrightarrow \mathrm{Div}^0[C_N]/I_f^*\mathrm{Div}^0[C_N] \longrightarrow 0. \tag{14}$$

The Hecke action on $\mathrm{Div}^0[C_N]$ is Eisenstein, and the eigenvalues of T_ℓ^* are $\chi_1(\ell) + \ell \cdot \chi_2(\ell)$, where (χ_1, χ_2) range through pairs of even Dirichlet characters, not both trivial and whose product of conductors is a divisor N (as described e.g. in [St1, §1.3]).

Consider $\mathrm{Div}^0[C_N]/\mathfrak{p}^t := \mathrm{Div}^0[C_N] \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{O}/\mathfrak{p}^t$. Our running assumptions imply that $E_2(\theta, 1)$ occurs with multiplicity one in $\mathrm{Div}^0[C_N]/\mathfrak{p}^t$, i.e., the system of eigenvalues attached to $E_2(\theta, 1)$ appears in $\mathrm{Div}^0[C_N]/\mathfrak{p}^t$ with multiplicity one. Indeed, otherwise there would exist a pair of Dirichlet characters (ξ_1, ξ_2) as above such that $E_2(\theta, 1) \equiv E_2(\xi_1, \xi_2) \pmod{\mathfrak{p}}$. Proceeding as in [Oh3, Lemma 1.4.9] (since our Eisenstein series is p-distinguished), this amounts to saying that $\bar{\theta}\xi_1 \equiv \xi_2 \equiv 1 \pmod{\mathfrak{p}^t}$, and here this is only possible when $(\xi_1, \xi_2) = (\theta, 1)$ because $p \nmid \varphi(N)$.

Since $f \equiv E_2(\theta, 1) \pmod{\mathfrak{p}^t}$, it follows that $T_{f,Y}/T_{f,X} \simeq \mathcal{O}/\mathfrak{p}^t$ as $\mathcal{O}_\mathfrak{p}$-modules. The action of $G_\mathbb{Q}$ on $T_{f,Y}/T_{f,X}$ is given by the character θ by [St1, Theorem 1.3.1]. $\square$

Since f is ordinary at $\mathfrak{p}$, we may let $\alpha_f \in \mathcal{O}_{\mathfrak{p}}^\times$ denote the unit root of the pth Hecke polynomial of f. Let $\psi_f : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_{\mathfrak{p}}^\times$ denote the unramified character characterized by $\psi_f(\mathrm{Fr}_p) = \alpha_f$.

It is well-known (cf. e.g. [DR, §1.5] and [FK, 1.7] for a twisted version that boils down to the one below with our normalizations) that there are exact sequences of finitely generated $\mathcal{O}_{\mathfrak{p}}[G_{\mathbb{Q}_p}]$-modules

$$0 \to T_{f,X}^{\mathrm{sub}} \to T_{f,X} \to T_{f,X}^{\mathrm{quo}} \to 0 \qquad (15)$$

$$0 \to T_{f,Y}^{\mathrm{sub}} \to T_{f,Y} \to T_{f,Y}^{\mathrm{quo}} \to 0$$

such that

(i) $T_{f,X}^{\mathrm{quo}}$ and $T_{f,Y}^{\mathrm{quo}}$ are unramified as $G_{\mathbb{Q}_p}$-modules with

$$V_f^{\mathrm{quo}} := T_{f,X}^{\mathrm{quo}} \otimes F_{\mathfrak{p}} \simeq T_{f,Y}^{\mathrm{quo}} \otimes F_{\mathfrak{p}} \simeq F_{\mathfrak{p}}(\psi_f). \qquad (16)$$

(ii) The map $T_{f,X} \to T_{f,Y}$ induces an isomorphism $T_{f,X}^{\mathrm{sub}} = T_{f,Y}^{\mathrm{sub}}$ of free $\mathcal{O}_{\mathfrak{p}}$-modules of rank 1 on which $G_{\mathbb{Q}_p}$ acts through $\theta\psi_f^{-1}\varepsilon_{\mathrm{cyc}}$.

In general $T_{f,X}^{\mathrm{quo}}$ and $T_{f,Y}^{\mathrm{quo}}$ are finitely generated $\mathcal{O}_{\mathfrak{p}}$-modules that are not necessarily free, and we let Σ_X and Σ_Y, respectively, denote their torsion submodules.

Given a $\mathcal{O}_{\mathfrak{p}}$-module T, set

$$\bar{T} = T \otimes \mathcal{O}/\mathfrak{p}^t.$$

For a $G_{\mathbb{Q}}$-module T, we let $T^{\pm}$ denote the submodule on which complex conjugation acts as ± 1. As shown by Sharifi in [Sha, Theorem 4.3] and by Fukaya-Kato in [FK, §6.3.1, §7.1.11], there is an exact sequence of $\mathcal{O}/\mathfrak{p}^t[G_{\mathbb{Q}}]$-modules

$$0 \longrightarrow \bar{T}_{f,X}^{+} \longrightarrow \bar{T}_{f,X} \longrightarrow \bar{T}_{f,X}^{-} \longrightarrow 0. \qquad (17)$$

Note the switch of signs between (17) and [FK, §6.3.1, §7.1.11], which is due to the different Tate twist adopted in the definition of $T_{f,X}$. As explained e.g. in [FKS, §2.5.5] there are isomorphisms of $\mathcal{O}/\mathfrak{p}^t[G_{\mathbb{Q}_p}]$-modules

$$\bar{T}_{f,X}^{+} \simeq \bar{T}_{f,X}^{\mathrm{quo}}, \quad \bar{T}_{f,X}^{\mathrm{sub}} \simeq \bar{T}_{f,X}^{-}. \qquad (18)$$

The first isomorphism is given by the composition of the inclusion in (17) with the projection map in (15) mod $\mathfrak{p}^t$; the second one is given by the analogous composition obtained by switching the roles of (17) and (15). In particular $\bar{T}_{f,X}^{-}$ is free of rank 1 over $\mathcal{O}/\mathfrak{p}^t$.

As for the lattice associated to the open modular curve, let

$$T_{f,Y,\circ}^{\mathrm{quo}} := T_{f,Y}^{\mathrm{quo}}/\Sigma_Y \simeq \mathcal{O}_{\mathfrak{p}}(\psi_f) \qquad (19)$$

denote the free quotient. The latter isomorphism follows from (16).

Since $f \equiv E_2(\theta, 1) \bmod \mathfrak{p}^t$, we have

$$T_{f,Y,\circ}^{\mathrm{quo}} \otimes \mathcal{O}/\mathfrak{p}^t \simeq \mathcal{O}/\mathfrak{p}^t(\theta), \qquad (20)$$

which amounts to the congruence

$$\psi_f \equiv \theta \pmod{\mathfrak{p}^t} \qquad (21)$$

as unramified characters of $G_{\mathbb{Q}_p}$.

Note that π_1 factors through $T_{f,Y}^{\mathrm{quo}}$ because the latter is the maximal unramified $G_{\mathbb{Q}_p}$-quotient of $T_{f,Y}$. Assuming also hypothesis (H2) from the introduction, it follows that π_1 factors further through the natural projection $T_{f,Y} \longrightarrow T_{f,Y,\circ}^{\mathrm{quo}}$ and may thus be written as the following composition of $\mathcal{O}_{\mathfrak{p}}[G_{\mathbb{Q}_p}]$-modules

$$\pi_1 : \quad T_{f,Y} \longrightarrow T_{f,Y,\circ}^{\mathrm{quo}} \longrightarrow \bar{T}_{f,Y,\circ}^{\mathrm{quo}} \xrightarrow{\sim} \mathcal{O}/\mathfrak{p}^t(\theta). \qquad (22)$$

While the first and second arrows are canonical projection maps, the latter isomorphism is non-canonical and depends on the choice of a unit of $\mathcal{O}/\mathfrak{p}^t$.

3. First congruence relation

Keep the notations and assumptions fixed in the introduction concerning the first congruence relation. Here we shall mainly work with the integral Galois representation associated to the eigenform $f \in S_2(N,\theta)$ with respect to the *open* modular curve, and hence throughout this section we abbreviate

$$T_f := T_{f,Y}.$$

We begin by recalling more precisely the definition of Kato classes. Choose auxiliary even Dirichlet characters $\chi_1,\,\chi_2 : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ as in the introduction and set $\xi_1 = \theta\bar\chi_1\bar\chi_2$, $\xi_2 = 1$. By [Nek, (1.2)] (see also [KLZ1, §9.2, 3.3]), together with the fact that $H^j_{\mathrm{et}}(\bar V, \mathcal{O}_{\mathfrak p})$ vanishes for any smooth affine variety V of dimension d and any $j > d$, the Hochschild–Serre spectral sequence yields a homomorphism

$$H^2_{\mathrm{et}}(Y_1(N), \mathcal{O}_{\mathfrak p}(\psi)(2)) \longrightarrow H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\overline{Y_1(N)}, \mathcal{O}_{\mathfrak p}(\psi)(2))), \tag{23}$$

where recall we set $\psi = \bar\theta\chi_2$.

In view of (11) there is a $G_{\mathbb{Q}}$-equivariant projection

$$\pi_f : H^1_{\mathrm{et}}(\overline{Y}_1(N), \mathcal{O}_{\mathfrak p}(1)) \to T_f,$$

which in turn gives rise to a homomorphism

$$\pi_{f*} : H^2_{\mathrm{et}}(Y_1(N), \mathcal{O}_{\mathfrak p}(\psi)(2)) \longrightarrow H^1(\mathbb{Q}, T_f(\psi)(1)). \tag{24}$$

It thus makes sense to define

$$\kappa_f := \pi_{f*}(\delta(u_{\chi_1,\chi_2}) \cup \delta(u_{\xi_1,\xi_2})) \in H^1(\mathbb{Q}, T_f(\psi)(1)).$$

Note that f is ordinary at p because of (3). Label the roots of the pth Hecke polynomial of f as α_f, β_f so that $\mathrm{ord}_p(\alpha_f) = 0$ and $\mathrm{ord}_p(\beta_f) = 1$. Note that $\alpha_f\beta_f = \theta(p)p$.

We are interested in describing p-adic L-functions from the point of view of rigid geometry, as in [Eu, §2.1]. Let $\mathcal{W}$ denote the weight space defined as the formal spectrum of the Iwasawa algebra $\Lambda = \mathcal{O}_{\mathfrak p}[[\mathbb{Z}_p^\times]]$. The set $\mathcal{W}^{\mathrm{cl}}$ of classical points in $\mathcal{W}$ is given by characters of the form $\nu_{s,\xi}(z) = \xi(z)z^{s-1}$ where ξ is a Dirichlet character of p-power conductor, $\varepsilon_{\mathrm{cyc}}$ is the cyclotomic character, and s is an integer; this forms a dense subset in $\mathcal{W}$ for the Zariski topology. Let $\mathcal{W}^\circ$ further denote the set of those points with $\xi = 1$; we shall often write s in place of $\nu_s = \nu_{s,1}$. Let $\mathcal{W}^\pm \subset \mathcal{W}$ denote the topological closure of the set of points $\xi\varepsilon_{\mathrm{cyc}}^{s-1}$ with $(-1)^{s-1}\xi(-1) = \pm 1$. We have $\mathcal{W} = \mathcal{W}^+ \sqcup \mathcal{W}^-$ and we write $\Lambda = \Lambda^+ \oplus \Lambda^-$ for the corresponding decomposition of the Iwasawa algebra.

Let χ be an even Dirichlet character and let $L_p(f^*,\chi)$ denote the Mazur–Tate–Teitelbaum p-adic L-function associated to (f^*,χ). As discussed in [FK, §4.5], $L_p(f^*,\chi)$ depends on the choice of two complex periods (Ω_f^+, Ω_f^-), which in turn are determined by the choice of generators $\delta^\pm$ of $V_f^\pm = T_{f,X}^\pm \otimes \mathbb{Q}$ as

$$\Omega_f^\pm = \int_{\delta^\pm} \omega_f, \tag{25}$$

where ω_f is the canonical differential attached to f. The p-adic L-function $L_p(f^*,\chi)$ (cf. [Bel2]) is a rigid-analytic function on $\mathcal{W}$ characterized by the following formula interpolating classical L-values: let $\xi : (\mathbb{Z}/p^n\mathbb{Z})^\times \longrightarrow \bar{\mathbb{Q}}^\times$ be a homomorphism which does not factor through $(\mathbb{Z}/p^{n-1}\mathbb{Z})^\times$. Then

$$L_p(f^*,\chi)(\xi\varepsilon_{\mathrm{cyc}}) = \begin{cases} \mathfrak{g}(\bar\chi)(1-\bar\chi(p)\beta_f p^{-1})(1-\bar\theta\chi(p)\beta_f p^{-1})\frac{L(f^*,\chi,1)}{\Omega_f^+} & \text{if } \xi = 1 \\ (\theta/\alpha_f)^n \mathfrak{g}(\bar\chi\bar\xi)\frac{L(f^*,\chi\xi,1)}{\Omega_f^\pm} \text{ with } \pm = \xi(-1) & \text{if } n \geq 1, \end{cases} \tag{26}$$

where $\mathfrak{g}(\cdot)$ stands for the Gauss sum of a character. The set of characters of the form $\xi\varepsilon_{\mathrm{cyc}}$ as ξ ranges through all Dirichlet characters of arbitrary p-power conductor is dense within $\mathcal{W}$ and hence $L_p(f^*,\chi)$ is uniquely determined by (26). More classically, one can also view $L_p(f^*,\chi)$ as a one-variable p-adic L-function by setting $L_p(f^*,\chi,s) = L_p(f^*,\chi)(\varepsilon_{\mathrm{cyc}}^s)$.

Similarly, for an even, primitive, and non-trivial Dirichlet character χ, we may define the Kubota–Leopoldt p-adic L-function $L_p(\chi)$. It is a rigid analytic function on $\mathcal{W}$ characterized by the following formula interpolating classical L-values (cf. [PR]):

$$L_p(\chi,j) := L_p(\chi)(\varepsilon_{\mathrm{cyc}}^j) = \begin{cases} (1-p^{j-1}\bar\chi(p))\frac{2N^j(j-1)!}{(-2\pi i)^j\mathfrak{g}(\chi)}L(\chi,j) & \text{for } j \geq 2 \text{ even}, \\ (1-p^{-j}\chi(p))L(\chi,j) & \text{for } j \leq -1 \text{ odd}. \end{cases} \tag{27}$$

As usual we may regard as a one-variable function by setting $L_p(\chi, s) = L_p(\chi)(\varepsilon_{\rm cyc}^s)$.

Define the Euler-like factor

$$\mathcal{E}_f = (1 - \bar{\theta}\chi_2(p)\alpha_f)(1 - \bar{\theta}\chi_2(p)\beta_f)(1 - \bar{\chi}_1(p)\beta_f p^{-1})(1 - \bar{\theta}\chi_1(p)\beta_f p^{-1}). \tag{28}$$

Define also the "p-adic L-value"

$$\ell = -\mathfrak{g}(\bar{\theta}\chi_1)^{-1}\mathfrak{g}(\bar{\theta}\chi_2)^{-1} \cdot (1 - \chi_2(p)) \cdot L_p(\bar{\theta}\chi_2, -1) \cdot L_p(f^*, \theta\bar{\chi}_1, 1).$$

Thanks to Proposition 2.1, we can define as in the introduction the class

$$\bar{\kappa}_{f,1} := \pi_{1*}(\kappa_f) = \bar{\pi}_{1*}(\bar{\kappa}_f) \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1)),$$

where π_1 is as in (22).

In order to state more precisely the first congruence relation, we need to introduce some notation. Let $k = \mathbb{Q}(\mu_{N_2})^+ = \mathbb{Q}(\zeta_{N_2} + \zeta_{N_2}^{-1})$ denote the maximal totally real subfield of $\mathbb{Q}(\mu_{N_2})$ and set $d_{N_2} = [k : \mathbb{Q}]$. Let $\mathrm{Cl}(k)$ denote its class group. As in (5) let $\mathrm{Cl}(k)[\bar{\chi}_2]$ denote its $\bar{\chi}_2$-eigencomponent. It follows from the work of G. Gras [Gr, Théorème I2] that

$$\mathrm{rank}_{\mathbb{Z}/p\mathbb{Z}}\, \mathrm{Cl}(k)[\bar{\chi}_2] \otimes \mathbb{Z}/p\mathbb{Z} \leq \mathrm{rank}_{\mathbb{Z}/p\mathbb{Z}}\, \mathrm{Cl}(k(\mu_p))[\chi_2\omega] \otimes \mathbb{Z}/p\mathbb{Z}, \tag{29}$$

where $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ is the Teichmüller character. This inequality may be regarded as an instance of Leopoldt's Spiegelungssatz.

Remark 3.1. [Gr, Théorème I2] applies because Leopoldt's conjecture is known for (k, p) by the work of Brumer, primes in k above p are totally ramified in $k(\mu_p)$ and therefore the ω-component of the $\mathrm{Gal}\,(k(\mu_p)/k)$-submodule of $\mathrm{Cl}(k(\mu_p))$ generated by ideals above p is trivial. [Gr, Théorème I2] thus asserts that $\mathrm{rank}_{\mathbb{Z}/p\mathbb{Z}}\, \mathrm{Cl}(k(\mu_p))[\chi_2\omega] \otimes \mathbb{Z}/p\mathbb{Z}$ is equal to the rank of the $\bar{\chi}_2$-component of the p-torsion of the Galois group $\mathrm{Gal}\,(H_p/k)$ of the maximal p-abelian extension of k unramified away from p, as in fact $(\mathrm{Gal}\,(H_p/k)[\bar{\chi}_2] \otimes \mathbb{Z}/p\mathbb{Z})^\vee \simeq \mathrm{Cl}(k(\mu_p))[\chi_2\omega] \otimes \mathbb{Z}/p\mathbb{Z}$. Hence (Gras) follows because the Hilbert class field H/k is contained in H_p and $\mathrm{Gal}\,(H/k) = \mathrm{Cl}(k)$.

Definition 3.2. We say that p is $\bar{\chi}_2$-regular if (29) is an equality.

To place in context this condition, let $R_p(k)$ denote the p-adic regulator of k. As explained in e.g. [Gr2, Def. 2.3], one always has $\mathrm{ord}_p R_p(k) \geq d_{N_2} - 1$. It is shown in loc. cit. that (29) is an equality for all non-trivial even Dirichlet characters of conductor N_2 if and only if $\mathrm{ord}_p R_p(k) = d_{N_2} - 1$. We refer to [Gr2, §7.3] for conjectures predicting that such an equality is expected to hold for all primes p away from a set of density 0.

For the following result, recall the circular unit c_{χ_2} defined in the introduction, and also its reduction modulo p, $\bar{c}_{\chi_2}$.

Theorem 3.3. *(First congruence relation) Assuming (H1)-(H2) we have*

$$\mathcal{E}_f \cdot \bar{\kappa}_{f,1} \equiv \ell \cdot \bar{c}_{\chi_2} \quad \text{in } H^1(\mathbb{Q}_p, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1)).$$

If p is $\bar{\chi}_2$-regular, this equality takes place in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1))$.

Theorem 1.1 in the introduction readily follows from the above statement. Indeed, it follows from (26) that

$$L_p(f^*, \theta\bar{\chi}_1, 1) = (1 - \bar{\theta}\chi_1(p)\beta_f p^{-1})(1 - \bar{\chi}_1(p)\beta_f p^{-1}) \cdot \mathfrak{g}(\bar{\theta}\chi_1) \cdot \frac{L(f^*, \theta\bar{\chi}_1, 1)}{\Omega_f^+}.$$

Since $L_p(\bar{\theta}\chi_2, -1) = (1 - \bar{\theta}\chi_2(p)p) \cdot L(\bar{\theta}\chi_2, -1) = (p\bar{\theta}\chi_2(p) - 1) \cdot \frac{B_{2,\bar{\theta}\chi_2}}{2}$, the Euler factors in the above theorem cancel out in light of (H1) and the congruence $(\alpha_f, \beta_f) \equiv (\theta(p), p) \pmod{\mathfrak{p}^t}$. Then, Theorem 1.1 follows.

The remainder of this section is devoted to the proof of Theorem 3.3.

3.1. **Mazur's factorization formula.** The aim of this section is to recall a mod $\mathfrak{p}$ factorization formula for $L_p(f^*,\chi)$ in terms of two Kubota–Leopoldt p-adic L-functions. The first result in this direction was proved by Mazur in [Maz], and further generalizations were provided by Stevens [St1] and Greenberg–Vatsal [GV, Theorem 3.12]. As explained before, this formula actually depends on the choice of periods $\Omega_f^{\pm}$, and we make this choice by invoking the work of Sharifi [Sha] and Fukaya-Kato [FK], which gives rise to a rather precise formula that turns out to be crucial for our purposes.

Note firstly that with the conventions adopted above, character $\xi\varepsilon_{\mathrm{cyc}}$ lies in $\mathcal{W}^+$ if and only if ξ is odd, and in that case the period appearing in (26) is Ω_f^-. The main result we aim to prove in this section is concerned with the value of $L_p(f^*,\chi,s)$ at $s=2$ (and $\xi=1$), which again lies in $\mathcal{W}^+$. For this reason it will suffice to work with the restriction of $L_p(f^*,\chi)$ to $\mathcal{W}^+$, and accordingly we only need to choose the period Ω_f^-.

The work of Sharifi [Sha] and Fukaya-Kato [FK] allows us to take a canonical choice of Ω_f^-. Namely, recall from (18) that $T_{f,X}^-$ is a free $\mathcal{O}_{\mathfrak{p}}$-module of rank 1. Take any generator δ^- of $T_{f,X}^-$ such that δ^- (mod $\mathfrak{p}^t$) is the basis of $\bar{T}_{f,X}^-$ specified in [FK, 6.3.18, 7.1.11, 8.2.4], which in turn builds on the work of Sharifi [Sha, §4] and Ohta [Oh1], [Oh2]. Define Ω_f^- from δ^- as in (25).

It is readily verified that this choice of Ω_f^- satisfies the defining properties imposed by Vatsal in [Va]. Beware however that in loc. cit. these periods are only well-defined up to p-adic units and this is not enough for our aims in this paper.

Recall the uniformizer ϖ fixed at the outset in §2. Let $C_{f^*}=(\varpi^r)\subseteq O_{\mathfrak{p}}$ denote the congruence ideal attached to f^* as defined e.g. in [Oh3]. In Hida's terminology, ϖ^r, is sometimes called a congruence divisor.

With our canonical definition of Ω_f^- at hand, we can now define Ω_f^+ as in [Va2, Remark 2.7], namely the one satisfying

$$\Omega_f^+\Omega_f^- = \frac{4\pi^2 i}{\varpi^r}\langle f,f\rangle. \tag{30}$$

The above equation determines Ω_f^+ in terms of the remaining quantities, which are all canonical except for the choice of ϖ: see the discussion after (40) for more details on this.

Proposition 3.4. *The following congruence relation holds for any even integer s:*

$$L_p(f^*,\chi,s)\equiv 2\cdot L_p(\bar{\chi},1-s)\cdot L_p(\bar{\theta}\chi,s-1) \pmod{\mathfrak{p}^t}. \tag{31}$$

Proof. This follows from [FK, Proposition 8.2.4]. □

Recall we have set $\psi=\bar{\theta}\chi_2$. In particular, since the character ψ is even, it holds that

$$L_p(f^*,\psi,2)=2\cdot L_p(\bar{\psi},-1)\cdot L_p(\bar{\theta}\psi,1)\equiv -B_{2,\bar{\psi}}\cdot L_p(\bar{\theta}\psi,1) \pmod{\mathfrak{p}^t}. \tag{32}$$

3.2. **Dieudonné modules and congruences among Ohta's periods.** Given a p-adic de Rham representation V of $G_{\mathbb{Q}_p}$ with coefficients in $F_{\mathfrak{p}}$, let $D_{\mathrm{dR}}(V)=(V\otimes_{\mathbb{Q}_p}B_{\mathrm{dR}})^{G_{\mathbb{Q}_p}}$ denote its de Rham Dieudonné module and let $\log_{\mathrm{BK}}$ (resp. $\exp^*_{\mathrm{BK}}$) stand for the Bloch–Kato logarithm (resp. dual exponential map) attached to V as defined in [BK], [Bel].

In this note we shall work with the Dieudonné modules associated to the following two basic types of representations. Firstly, if $\chi: G_{\mathbb{Q}_p}\longrightarrow \mathcal{O}_{\mathfrak{p}}^\times$ is a finite order character and s is an integer, let $F_{\mathfrak{p}}(\chi)(s)$ denote the 1-dimensional representation on which $G_{\mathbb{Q}_p}$ acts via $\chi\varepsilon_{\mathrm{cyc}}^s$. Then $D_{\mathrm{dR}}(F_{\mathfrak{p}}(\chi)(s))$ is again 1-dimensional and a canonical generator of $D_{\mathrm{dR}}(F_{\mathfrak{p}}(\bar{\chi})(-s))$ is given by $t^s\mathfrak{g}(\chi)^{-1}$, where t is Fontaine's p-adic analogue of $2\pi i$. Moreover, unravelling the definitions, there is s a perfect pairing

$$\langle\,,\rangle\,:\,D_{\mathrm{dR}}(F_{\mathfrak{p}}(\chi)(s))\times D_{\mathrm{dR}}(F_{\mathfrak{p}}(\bar{\chi})(-s))\longrightarrow F_{\mathfrak{p}}$$

which gives rise to the isomorphism

$$D_{\mathrm{dR}}(F_{\mathfrak{p}}(\chi)(s))\to F_{\mathfrak{p}}\quad c\mapsto \Big\langle c,\frac{t^s}{\mathfrak{g}(\chi)}\Big\rangle. \tag{33}$$

If $s\geq 1$ and $\chi\neq 1$ when $s=1$, Bloch-Kato's logarithm gives rise to an isomorphism

$$\log_{\mathrm{BK}}: H^1(\mathbb{Q}_p,F_{\mathfrak{p}}(\chi)(s))\to D_{\mathrm{dR}}(F_{\mathfrak{p}}(\chi)(s)). \tag{34}$$

Secondly, the de Rham Dieudonné module $D_{\rm dR}(V_f)$ associated to the eigenform f is a $F_{\mathfrak p}$-filtered vector space of dimension 2. As discussed in e.g. [KLZ1, §2.8], Poincaré duality yields a perfect pairing

$$\langle\, ,\, \rangle : D_{\rm dR}(V_f(-1)) \times D_{\rm dR}(V_{f^*}) \to F_{\mathfrak p}$$

and (15) induces an exact sequence of Dieudonné modules

$$0 \to D_{\rm dR}(V_f^{\rm sub}) \to D_{\rm dR}(V_f) \to D_{\rm dR}(V_f^{\rm quo}) \to 0, \tag{35}$$

where $D_{\rm dR}(V_f^{\rm sub})$ and $D_{\rm dR}(V_f^{\rm quo})$ have both dimension 1.

As discussed e.g. in [KLZ2, §2], the space $D_{\rm dR}(V_f)$ is endowed with a filtration, yielding a canonical subspace ${\rm Fil}(D_{\rm dR}(V_f)) \subset D_{\rm dR}(V_f)$. Faltings' theorem associates to f a regular differential form $\omega_f \in {\rm Fil}(D_{\rm dR}(V_f))$, which gives rise to an element in $D_{\rm dR}(V_f^{\rm quo})$ via the rightmost map in (35) and in turn induces a linear form

$$\omega_f \;:\; D_{\rm dR}(V_{f^*}^{\rm sub}(-1)) \to F_{\mathfrak p}, \quad \eta \mapsto \langle \eta, \omega_f\rangle \tag{36}$$

that we continue to denote with same symbol by a slight abuse of notation.

There is also a differential η_f, characterized by the property that it spans the line $D_{\rm dR}(V_f^{\rm sub}(-1))$ and is normalized so that

$$\langle \eta_f, \omega_{f^*}\rangle = 1. \tag{37}$$

Again, it induces a linear form

$$\eta_f \;:\; D_{\rm dR}(V_{f^*}^{\rm quo}) \to F_{\mathfrak p}, \quad \omega \mapsto \langle \eta_f, \omega\rangle \tag{38}$$

We turn now to the more delicate p-adic Hodge theory of *integral* Galois representations. Let T be an unramified $\mathcal{O}_{\mathfrak p}[{\rm Gal}\,(\bar{\mathbb Q}_p/\mathbb{Q}_p)]$-module and set $V = T \otimes F_{\mathfrak p}$. Let $\hat{\mathbb Z}_p^{\rm ur}$ denote the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$, and define as in e.g. [Oh2, Theorem 2.1.11] the integral Dieudonné module

$$D(T) := (T \,\hat\otimes_{\mathbb{Z}_p} \hat{\mathbb Z}_p^{\rm ur})^{{\rm Fr}_p = 1}.$$

As shown in loc. cit. we have $D_{\rm dR}(V) = D(T) \otimes F_{\mathfrak p}$.

As explained e.g. in [FK, Prop. 1.7.6], there is a functorial isomorphism of $\mathcal{O}_{\mathfrak p}$-modules (forgetting the Galois structure) given by

$$T \xrightarrow{\sim} D(T). \tag{39}$$

This map is not canonical as it depends on a choice of root of unity; for $T = \mathcal{O}_{\mathfrak p}(\chi)$ we take it to be given by the rule $1 \mapsto \mathfrak{g}(\chi)$.

Recall from (19) the free $\mathcal{O}_{\mathfrak p}$-quotient

$$T_{f,\circ}^{\rm quo} \simeq \mathcal{O}_{\mathfrak p}(\psi_f);$$

note in particular that $T_{f,\circ}^{\rm quo}$ is unramified.

Taking into account the characterization of the congruence ideal C_{f^*} given in [KLZ1, Not. 7.1.1], together with the construction of η_{f^*} described in the proof of Prop. 10.1.1(2)] in loc. cit., we have that the image of $\eta_{f^*|D(T_f^{\rm quo})}$ is precisely $C_{f^*}^{-1}$. Hence, there is an isomorphism

$$\eta_{f^*} \;:\; D(T_{f,\circ}^{\rm quo}) \longrightarrow C_{f^*}^{-1}, \quad \omega \mapsto \langle \eta_{f^*}, \omega\rangle.$$

Writing $C_f = (\varpi^r)$ as in §3.1 and setting $\tilde\eta_{f^*} := \varpi^r \cdot \eta_{f^*}$, the above map gives rise to an isomorphism of $O_{\mathfrak p}$-modules

$$\tilde\eta_{f^*} \;:\; D(T_{f,\circ}^{\rm quo}) \longrightarrow \mathcal{O}_{\mathfrak p}, \quad \omega \mapsto \langle \tilde\eta_{f^*}, \omega\rangle. \tag{40}$$

While the choice of uniformizer ϖ fixed at the outset is non-canonical, the ambiguity caused by this choice in (40) is cancelled with the prescription of Ω_f^+ in (30).

The map π_1 appearing in Proposition 2.1

$$\pi_1 \;:\; T_f \longrightarrow \mathcal{O}/\mathfrak{p}^t(\theta)$$

is only well-defined up to units in $\mathcal{O}/\mathfrak{p}^t$. We rigidify it by invoking diagram (22), which tells us that π_1 is fixed once we take a choice of an isomorphism of $G_{\mathbb{Q}_p}$-modules

$$\bar\iota : \bar T_{f,\circ}^{\rm quo} \xrightarrow{\sim} \mathcal{O}/\mathfrak{p}^t(\theta). \tag{41}$$

Fixing such a map amounts to choosing the class $(\mathrm{mod}\,\mathfrak{p}^t)$ of an isomorphism of local modules $\iota : T_{f,\circ}^{\mathrm{quo}} \simeq \mathcal{O}_{\mathfrak{p}}(\psi_f)$. In light of the functoriality provided by (39) this determines and is determined by the class $(\mathrm{mod}\,\mathfrak{p}^t)$ of an isomorphism $D(\iota) : D(T_{f,\circ}^{\mathrm{quo}}) \simeq D(\mathcal{O}_{\mathfrak{p}}(\psi_f))$.

We choose ι as the single isomorphism making the following diagram commutative:

(42)
$$\begin{array}{ccc} D(T_{f,\circ}^{\mathrm{quo}}) & \xrightarrow{\tilde{\eta}_{f^*}} & \mathcal{O}_{\mathfrak{p}} \\ \Big\downarrow{\scriptstyle D(\iota)} & \nearrow{\scriptstyle \cdot 1/\mathfrak{g}(\theta)} & \\ D(\mathcal{O}_{\mathfrak{p}}(\psi_f)) & & \end{array}$$

Indeed, since both $\tilde{\eta}_{f^*}$ and $\cdot 1/\mathfrak{g}(\theta)$ are isomorphisms, it follows that such a map $D(\iota)$ exists and is unique, and this in turn pins down ι and $\bar{\iota}$ in light of (39).

3.3. **Coleman's power series and the Kubota-Leopoldt p-adic L-function.** Let $\varepsilon_r : G_{\mathbb{Q}} \to (\mathbb{Z}/p^r\mathbb{Z})^\times$ denote the cyclotomic character associated to the Galois extension $\mathbb{Q}(\mu_{p^r})/\mathbb{Q}$ and let

(43)
$$\underline{\varepsilon}_{\mathrm{cyc}} = \varprojlim_{r\geq 0} \varepsilon_r : G_{\mathbb{Q}} \to \Lambda^\times = \mathcal{O}_{\mathfrak{p}}[[\mathbb{Z}_p^\times]]^\times$$

denote the Λ-adic cyclotomic character sending a Galois element σ to the group-like element $[\varepsilon_{\mathrm{cyc}}(\sigma)]$. It interpolates the powers of the $\mathbb{Z}_p$-cyclotomic character, in the sense that for any classical point $\nu_{s,\xi} \in \mathcal{W}^{\mathrm{cl}}$ as in §3, we have

(44)
$$\nu_{s,\xi} \circ \underline{\varepsilon}_{\mathrm{cyc}} = \xi \cdot \varepsilon_{\mathrm{cyc}}^{s-1}.$$

The following result follows from the general theory of Perrin-Riou maps (see for instance [KLZ1, §8]), considering the Λ-adic representation $\mathcal{O}_{\mathfrak{p}}(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})$. Below, recall from §3.2 Bloch-Kato's logarithm and dual exponential maps with values in Dieudonné modules and the pairings on the latter.

Proposition 3.5. *There exists a morphism of Λ-modules, called the Perrin-Riou regulator,*
$$\mathcal{L}_\chi : H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}})) \longrightarrow \Lambda$$
satisfying that for all integers r, the specialization of $\mathcal{L}_\chi$ at $s \in \mathcal{W}^\circ$ is the homomorphism
$$\mathcal{L}_{\chi,s} : H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi)(s)) \longrightarrow \mathcal{O}_{\mathfrak{p}}$$
given by
$$\mathcal{L}_{\chi,s} = \frac{1-\bar{\chi}(p)p^{-s}}{1-\chi(p)p^{s-1}} \cdot \begin{cases} \frac{(-1)^s}{(s-1)!} \cdot \langle \log_{\mathrm{BK}}, \frac{t^s}{\mathfrak{g}(\chi)} \rangle & \text{if } s \geq 1 \\ (-s)! \cdot \langle \exp^*_{\mathrm{BK}}, \frac{t^s}{\mathfrak{g}(\chi)} \rangle & \text{if } s < 1, \end{cases}$$

As a piece of notation, and for any p-adic representation V, we write $H^1_{\mathrm{f}}(\mathbb{Q}, V)$ for the *finite* Bloch–Kato Selmer group, which is the subspace of $H^1(\mathbb{Q}, V)$ which consists on those classes which are crystalline at p and unramified at $\ell \neq p$.

The following result is a reformulation of Coleman and Perrin-Riou's reciprocity law ([Co], [PR]), with the normalizations used for instance in [Eu, §1.1].

Proposition 3.6. *There exists a Λ-adic cohomology class*
$$\kappa_{\chi,\infty} \in H^1(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}(\chi) \otimes \Lambda(\varepsilon_{\mathrm{cyc}}\underline{\varepsilon}_{\mathrm{cyc}}))$$
such that:

(a) *Its image under restriction at p followed by the Perrin–Riou regulator gives the Kubota–Leopoldt p-adic L-function:*
$$\mathcal{L}_\chi(\mathrm{res}_p(\kappa_{\chi,\infty})) = L_p(\bar{\chi}).$$

(b) *The bottom layer $\kappa_\chi(1) := \nu_1(\kappa_{\chi,\infty})$ lies in $H^1_{\mathrm{f}}(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}(\chi)(1))$ and satisfies*
$$\kappa_\chi(1) = (1-\chi(p)) \cdot c_\chi.$$

3.4. Kato's explicit reciprocity law.

Proposition 3.7. *There exists a homomorphism of Λ-modules*

$$\mathcal{L}^-_{f\otimes\psi} : H^1(\mathbb{Q}_p, T^{\rm quo}_{f,\circ} \otimes \Lambda(\psi)(\varepsilon_{\rm cyc}\underline{\varepsilon}_{\rm cyc})) \to \Lambda$$

satisfying the following interpolation property: for $s \in \mathcal{W}^\circ$, the specialization of $\mathcal{L}^-_{f\otimes\psi}$ at s is the homomorphism

$$\mathcal{L}^-_{f\otimes\psi,s} : H^1(\mathbb{Q}_p, T^{\rm quo}_{f,\circ}(\psi)(s)) \longrightarrow \mathcal{O}_{\mathfrak{p}}$$

given by

$$\mathcal{L}^-_{f\otimes\psi,s} = \frac{1-\bar\theta\bar\psi(p)\beta_f p^{-s-1}}{1-\theta\psi(p)\beta_f^{-1}p^s} \times \begin{cases} \frac{(-1)^s}{(s-1)!} \times \langle \log_{\rm BK}, t^s\tilde\eta_{f^*\otimes\bar\psi}\rangle & \text{if } s\geq 1 \\ (-s)! \times \langle \exp^*_{\rm BK}, t^s\mathfrak{g}(\psi)^{-1}\tilde\eta_{f^*\otimes\bar\psi}\rangle & \text{if } s<1, \end{cases}$$

where $\log_{\rm BK}$ is the Bloch–Kato logarithm and $\exp^*_{\rm BK}$ is the dual exponential map.

Proof. This follows from Coleman and Perrin-Riou's theory of Λ-adic logarithm maps as extended by Loeffler and Zerbes in [LZ]. This is recalled for instance in [KLZ1, §8,9]. More precisely, [KLZ1, Theorem 8.2.3] and, more particularly, the second displayed equation in [KLZ1, p. 82] yield an injective map

$$H^1(\mathbb{Q}_p, T^{\rm quo}_{f,\circ} \otimes \Lambda(\psi)(\varepsilon_{\rm cyc}\underline{\varepsilon}_{\rm cyc})) \longrightarrow D(T^{\rm quo}_f)\otimes\Lambda,$$

since $H^0(\mathbb{Q}_p, T^{\rm quo}_{f,\circ}(\psi)(1)) = 0$ because of the assumption that $\alpha_f\psi \equiv \theta\psi(p) \not\equiv 1$ modulo $\mathfrak{p}$.

This map is characterized by the interpolation property formulated in [LZ, Appendix B]. Next we apply the pairing of (40) and the result follows. □

Theorem 3.8. *There exists a Λ-adic cohomology class*

$$\kappa_{f,\infty} \in H^1(\mathbb{Q}, T_f \otimes \Lambda^-(\psi)(\varepsilon_{\rm cyc}\underline{\varepsilon}_{\rm cyc}))$$

such that:

(a) *There is an explicit reciprocity law*

$$\mathcal{L}^-_{f\otimes\psi}(\mathrm{res}_p(\kappa_{f,\infty})^-) = \frac{-L_p(f^*,\theta\bar\chi_1,1)}{2\mathfrak{g}(\bar\theta\chi_1)\mathfrak{g}(\bar\theta\chi_2)} \times L_p(f^*,\bar\psi,1+s),$$

where res_p stands for the map corresponding to localization at p and $\mathrm{res}_p(\kappa_{f,\infty})^-$ is the map induced in cohomology from the projection map $T_f \to T^{\rm quo}_{f,\circ}$ of (15). (The s on the right-hand side is a variable.)

(b) *The bottom layer $\kappa_f(1)$ lies in $H^1_{\rm f}(\mathbb{Q}, T_f(\psi)(1))$ and satisfies*

$$\kappa_f(1) = \mathcal{E}_f \cdot \kappa_f,$$

where $\mathcal{E}_f$ is the Euler factor introduced in (28).

Proof. This is due to Kato [Ka] and has been reported in many other places in the literature. See [Och] and, more specifically, [BD2, Theorems 4.4 and 5.1] combined with Besser's results [Bes, Proposition 9.11 and Corollary 9.10] showing that the p-adic regulator can be recast as the composition of the p-adic étale regulator followed by the Bloch–Kato logarithm. In particular, these results express the left hand side as a Hida–Rankin p-adic L-function, that factors as the product of two Mazur–Kitagawa p-adic L-functions.

Note however that the normalizations adopted in loc. cit. are slightly different from ours, thus affecting the scaling factors. More precisely, in [BD2] the authors employ the functional η_f instead of $\tilde\eta_f$, but here this discrepancy is compensated by our choice of periods Ω_f^+ and Ω_f^-, which we have normalized according to (30). Taking this into account, the theorem holds as stated. □

Recall our running assumption that $f \equiv E_2(\theta,1) \pmod{\mathfrak{p}^t}$.

Corollary 3.9. *The following equality holds in $\Lambda^-/\mathfrak{p}^t\Lambda^-$:*

$$\mathcal{L}^-_{f\otimes\psi}(\mathrm{res}_p\,\kappa^-_{f,\infty}) \equiv \frac{-L_p(f^*,\theta\bar\chi_1,1)}{\mathfrak{g}(\bar\theta\chi_1)\mathfrak{g}(\bar\theta\chi_2)} L_p(\bar\theta\chi_2,-1)\cdot\mathcal{L}_\theta(\mathrm{res}_p\,\kappa_{\chi_2,\infty}) \pmod{\mathfrak{p}^t}.$$

Proof. By Proposition 3.4,

$$L_p(f^*, \theta\bar{\chi}_2, 1+s) \equiv 2 \cdot L_p(\bar{\theta}\chi_2, -s) \cdot L_p(\bar{\chi}_2, s) \pmod{\mathfrak{p}^t}.$$

Applying now part (a) of Theorem 3.8 and Proposition 3.6 to the left and right hand sides respectively, the result follows. □

3.5. **Proof of Theorem 3.3.** We now prove Theorem 3.3. Note that since $(\chi_2)_{|G_{\mathbb{Q}_p}}$ is a non-trivial unramified character, it follows from e.g. [Bel, §2.2, Ex. 2.16] (see also [BK, Ex. 3.9]) that

$$H^1_{\mathrm{f}}(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi_2)(1)) = H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi_2)(1)),$$

where H^1_{f} stands for the subgroup of unramified classes in H^1.

We have seen in Proposition 3.5 that there is a homomorphism

$$\mathcal{L}_{\chi_2,1} : H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi_2)(1)) \to \mathcal{O}_{\mathfrak{p}}. \tag{45}$$

Lemma 3.10. *The map $\mathcal{L}_{\chi_2,1}$ is an isomorphism.*

Proof. According to Proposition 3.5, the map (45) is given by

$$\mathcal{L}_{\chi_2,1} = \frac{\bar{\chi}_2(p)p^{-1}-1}{1-\chi_2(p)} \cdot \Big\langle \log_{\mathrm{BK}}, \frac{t}{\mathfrak{g}(\chi_2)} \Big\rangle.$$

Given a place v of $\mathbb{Q}(\mu_N)$ above p, let $\mathbb{Z}[\mu_N]_v$ denote the completion of $\mathbb{Z}[\mu_N]$ at v. Define the module of local units $U_p(N) = \prod_{v|p} \mathbb{Z}[\mu_N]_v^\times$, where $v = v_1, \ldots, v_r$ ranges over all places of $\mathbb{Q}(\mu_N)$ above p. Note that $G = \mathrm{Gal}\,(\mathbb{Q}(\mu_N)/\mathbb{Q})$ acts on $U_p(N)$ by permuting the places v, and hence it makes sense to pick the eigen-component of $U_p(N)$ with respect to a character of G. In particular, we have

$$U_p(N)[\bar{\chi}_2] := (U_p(N) \otimes \mathcal{O}_{\mathfrak{p}}(\chi_2))^G.$$

Kummer theory identifies $H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi_2)(1))$ with $U_p(N)[\bar{\chi}_2]$, which is a $\mathcal{O}_{\mathfrak{p}}$-module of rank one. In a similar way, let $U_p^1(N_2) = \prod_{v|p} (\mathbb{Z}[\mu_{N_2}]_v^+)^1$, where $(\mathbb{Z}[\mu_{N_2}]_v^+)^1$ stands for the set of local units in $\mathbb{Z}_p[\mu_{N_2}]_v^+$ which are congruent to 1 modulo v. For our further use, let $U^+(N_2)$ denote the closure of the set of units of $\mathbb{Z}[\mu_{N_2}]^+$ congruent to 1 modulo each place above p, diagonally embedded in $U_p^1(N_2)$.

Since $\mathbb{Q}(\mu_N)_v$ is an unramified extension of $\mathbb{Q}_p$, the maximal ideal of $\mathbb{Z}[\mu_N]_v$ is $p\mathbb{Z}[\mu_N]_v$ and the logarithm defines an isomorphism, as recalled for instance in [Con, §8]

$$\log_v : \mathbb{Z}[\mu_N]_v^\times \otimes \mathcal{O}_{\mathfrak{p}} \longrightarrow p\mathbb{Z}[\mu_N]_v \otimes \mathcal{O}_{\mathfrak{p}}. \tag{46}$$

Note that $\prod_v \mathbb{Z}[\mu_N]_v$ is naturally a G-module isomorphic to the regular representation and hence $(\prod_v \mathbb{Z}[\mu_N]_v)[\bar{\chi}_2]$ is again a free module of rank 1 over $\mathcal{O}_{\mathfrak{p}}$. Define

$$\log_{\bar{\chi}_2} := \sum_{\sigma \in G} \chi_2(\sigma) \log_{\sigma(v_1)} : U_p(N)[\bar{\chi}_2] \longrightarrow p\Big(\prod_v \mathbb{Z}[\mu_N]_v\Big)[\bar{\chi}_2].$$

A generator of the target of the previous map may be taken to be the Gauss sum $\mathfrak{g}(\chi_2)$ diagonally embedded in $\prod_v \mathbb{Z}[\mu_N]_v$ and this yields an identification $\frac{1}{\mathfrak{g}(\chi_2)}(\prod_v \mathbb{Z}[\mu_N]_v)[\bar{\chi}_2] = \mathcal{O}_{\mathfrak{p}}$. Under these identifications, Bloch-Kato's logarithm may be recast classically as

$$\Big\langle \log_{\mathrm{BK}}, \frac{t}{\mathfrak{g}(\chi_2)} \Big\rangle = \frac{1}{\mathfrak{g}(\chi_2)} \log_{\bar{\chi}_2} : H^1(\mathbb{Q}_p, \mathcal{O}_{\mathfrak{p}}(\chi_2)(1)) = U_p(N)[\bar{\chi}_2] \longrightarrow \mathcal{O}_{\mathfrak{p}},$$

and we already argued that this yields an isomorphism onto $p\mathcal{O}_{\mathfrak{p}}$.

Since $\mathrm{ord}_p(\frac{\bar{\chi}_2(p)p^{-1}-1}{1-\chi_2(p)}) = -1$, it follows that $\mathcal{L}_{\chi_2,1}$ is an isomorphism onto $\mathcal{O}_{\mathfrak{p}}$, as claimed. □

Recall from Proposition 3.7 the map

$$\mathcal{L}^-_{f\otimes\psi,1} : H^1(\mathbb{Q}_p, T^{\mathrm{quo}}_{f,\circ}(\psi)(1)) \to D(T^{\mathrm{quo}}_{f,\circ}(\psi)(1)) \stackrel{\cdot t\tilde{\eta}_{f^*}}{\to} \mathcal{O}_{\mathfrak{p}}.$$

Recall the isomorphism $T^{\mathrm{quo}}_{f,\circ} \otimes \mathcal{O}/\mathfrak{p}^t \simeq \mathcal{O}/\mathfrak{p}^t(\chi_2)$ of (41) fixed as in (42) above and use it to identify the source of $\mathcal{L}^-_{f\otimes\psi,1} \otimes \mathcal{O}/\mathfrak{p}^t$ with $H^1(\mathbb{Q}_p, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1))$.

Lemma 3.11. *As homomorphisms $H^1(\mathbb{Q}_p, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1)) \longrightarrow \mathcal{O}/\mathfrak{p}^t$ we have the congruence*

$$\mathcal{L}^-_{f\otimes\psi,1} \equiv \mathcal{L}_{\chi_2,1} \pmod{\mathfrak{p}^t}.$$

Proof. This follows by comparing the maps $\mathcal{L}_{\chi_2,1}$ and $\mathcal{L}^-_{f\otimes\psi,1}$ described respectively in Proposition 3.5 and 3.7. Note firstly that the Euler factors involved in the latter agree modulo $\mathfrak{p}^t$ with those of the former (recall that $\alpha_f\beta_f = \theta(p)p$).

Next, observe that in Proposition 3.5 the pairing takes place against $t\mathfrak{g}(\chi_2)^{-1}$, while in Proposition 3.7 this pairing is with $t\tilde{\eta}_{f^*\otimes\bar{\psi}}$. The lemma follows from the commutativity of the diagram (42). □

We are finally in position to provide the *proof of Theorem 3.3:* After specializing Corollary 3.9 at $s=1$ we obtain

$$\mathcal{L}^-_{f\otimes\psi,1}(\mathrm{res}_p\,\kappa_f(1)^-) \equiv \frac{-L_p(f^*,\theta\bar{\chi}_1,1)}{\mathfrak{g}(\bar{\theta}\chi_1)\mathfrak{g}(\bar{\theta}\chi_2)} L_p(\bar{\theta}\chi_2,-1)\cdot\mathcal{L}_{\chi_2,1}(\mathrm{res}_p\,\kappa_{\chi_2}(1)) \pmod{\mathfrak{p}^t}$$

Recall that Proposition 3.6 and Theorem 3.8 assert that

$$\kappa_{\chi_2}(1) = (1-\chi_2(p))\cdot c_{\chi_2}, \quad \kappa_f(1) = \mathcal{E}_f\cdot\kappa_f$$

and hence

$$\mathcal{E}_f\mathcal{L}^-_{f\otimes\psi,1}(\mathrm{res}_p\,\kappa_f^-) \equiv \frac{-L_p(f^*,\theta\bar{\chi}_1,1)}{\mathfrak{g}(\bar{\theta}\chi_1)\mathfrak{g}(\bar{\theta}\chi_2)} L_p(\bar{\theta}\chi_2,-1)(1-\chi_2(p))\cdot\mathcal{L}_{\chi_2,1}(\mathrm{res}_p\,c_{\chi_2}) \pmod{\mathfrak{p}^t}$$

Recall we have set

$$\ell = \frac{-L_p(f^*,\theta\bar{\chi}_1,1)}{\mathfrak{g}(\bar{\theta}\chi_1)\mathfrak{g}(\bar{\theta}\chi_2)}\cdot(1-\chi_2(p))\cdot L_p(\theta\bar{\chi}_2,-1).$$

Using Lemma 3.11 together with Lemma 3.10, we deduce the equality of local classes

$$\mathcal{E}_f\cdot\mathrm{res}_p\,\kappa_f^- \equiv \ell\cdot\mathrm{res}_p\,c_{\chi_2} \pmod{\mathfrak{p}^t} \tag{47}$$

in $H^1(\mathbb{Q}_p,\mathcal{O}/\mathfrak{p}^t(\chi_2)(1))$. Observe that $\mathrm{res}_p(\kappa_f)^-$ is the local class obtained in cohomology by push-forward under the map induced by the projection $\bar{T}_f \to \bar{T}^{\mathrm{quo}}_{f,\circ}$ of (15), as already introduced in Theorem 3.8. This corresponds, modulo $\mathfrak{p}^t$, to what we have called $\bar{\kappa}_{f,1}$. The first (local) part of Theorem 3.3 follows.

The next lemma is conditional on the $\bar{\chi}_2$-regularity of p (cf. Definition 3.2), and is needed to derive the second part of the theorem.

Lemma 3.12. *Assuming p is* $\bar{\chi}_2$-regular*, the global-to-local restriction map*

$$H^1_{\mathrm{f}}(\mathbb{Q},\mathcal{O}_{\mathfrak{p}}(\chi_2)(1)) \to H^1(\mathbb{Q}_p,\mathcal{O}_{\mathfrak{p}}(\chi_2)(1))$$

is an isomorphism.

Proof. Recall from the proof of Lemma 3.10 the definition of the group $U_p(N_2)$ of local units. Consider the following commutative diagram, where vertical arrows are isomorphisms induced from Kummer theory and the upper horizontal arrow stands for the map corresponding to localization at p:

$$\begin{array}{ccc} H^1_{\mathrm{f}}(\mathbb{Q},\mathcal{O}_{\mathfrak{p}}(\chi_2)(1)) & \longrightarrow & H^1(\mathbb{Q}_p,\mathcal{O}_{\mathfrak{p}}(\chi_2)(1)) \\ \uparrow & & \uparrow \\ \mathbb{Z}[\mu_{N_2}]^\times[\bar{\chi}_2] & \longrightarrow & U_p(N_2)[\bar{\chi}_2] \end{array}$$

The bottom horizontal arrow is injective because it is induced by the natural inclusion $\mathbb{Z}[\mu_{N_2}]^\times \hookrightarrow U_p(N_2)$, and so it follows by known cases (due to Brumer) of Leopoldt's conjecture. Moreover, since χ_2 is even and nontrivial, both $\mathbb{Z}[\mu_{N_2}]^\times[\bar{\chi}_2]$ and $U_p(N_2)[\bar{\chi}_2]$ are $\mathcal{O}_{\mathfrak{p}}$-modules of rank 1. The cokernel

$$Q[\bar{\chi}_2] = U_p(N_2)[\bar{\chi}_2]/\mathbb{Z}[\mu_{N_2}]^\times[\bar{\chi}_2]$$

is thus a finite group.

In order to prove the lemma it thus suffices to show that $Q[\bar{\chi}_2]$ is trivial. Write $k=\mathbb{Q}(\mu_{N_2})^+$ (resp. $\mathbb{Z}[\mu_{N_2}]^+$) for the maximal totally real subfield of $\mathbb{Q}(\mu_{N_2})$ (resp. its ring of integers), and note that $Q[\bar{\chi}_2] = U^1_p(N_2)/U^+(N_2)[\bar{\chi}_2]$.

According to [Neu, Chapter 4, Theorem 7.8], $Q[\bar{\chi}_2] \simeq \mathrm{Gal}\,(H_p/H)[\bar{\chi}_2]$, where H_p (resp. H) is the maximal p-abelian extension of k unramified away from primes above p (resp. everywhere unramified). Here the $\bar{\chi}_2$-eigencomponent on the Galois group is taken with respect to the natural action of Gal $(k/\mathbb{Q})$ by conjugation on Gal (H_p/H). The lemma hence follows from the running hypothesis, since $(\mathrm{Gal}\,(H_p/k)[\bar{\chi}_2] \otimes \mathbb{Z}/p\mathbb{Z})^\vee \simeq \mathrm{Cl}(k(\mu_p))[\chi_2\omega] \otimes \mathbb{Z}/p\mathbb{Z}$. □

Assuming $\bar{\chi}_2$-*regularity*, Lemma 3.12 allows us to upgrade (47) to an equality of global classes in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\chi_2)(1))$, namely

$$\mathcal{E}_f \cdot \kappa_{f,1} \equiv \ell \cdot c_{\chi_2} \pmod{\mathfrak{p}^t}.$$

Theorem 3.3 follows.

4. Second congruence relation

As in the introduction, let $\theta : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ be an even, primitive, Dirichlet character of conductor $N \geq 4$. Let $f \in S_2(N, \theta)$ a normalized cuspidal eigenform of level N, weight 2 and nebentype θ. Fix a prime $p \nmid 6N\varphi(N)$ and assume as in (3) that $f \equiv E_2(\theta, 1) \bmod \mathfrak{p}^t$ for some $t \geq 1$. This implies that $L_p(\bar{\theta}, -1) \equiv 0 \pmod{\mathfrak{p}^t}$.

We keep the notations introduced along the introduction, §2 and §3. In particular $\chi_1, \chi_2 : (\mathbb{Z}/N\mathbb{Z})^\times \to \bar{\mathbb{Q}}^\times$ are even Dirichlet characters of conductor N_1 and N_2 with $N_1 \cdot N_2 = N$, and $\xi_1 = \theta\bar{\chi}_1\bar{\chi}_2$, $\xi_2 = 1$, $\psi = \bar{\theta}\chi_2$. Recall that the value of the modular unit u_{χ_1,χ_2} at ∞ is some power of the circular unit c_{χ_1}, and likewise for u_{ξ_1,ξ_2}. For the sake of concreteness, in the statement below we normalize them so that $u_{\chi_1,\chi_2}(\infty) = c_{\chi_1}$ and $u_{\xi_1,\xi_2}(\infty) = c_{\xi_1}$ although any other normalization would work.

Proposition 4.1. *The Beilinson–Kato class $\kappa_f \in H^1(\mathbb{Q}, T_{f,Y}(\psi)(1))$ may be lifted to $H^1(\mathbb{Q}, T_{f,X}(\psi)(1))$ if and only if $\bar{\kappa}_{f,1} = 0$.*

Proof. Recall from Proposition 2.1 the short exact sequence of $G_\mathbb{Q}$-modules

$$0 \longrightarrow T_{f,X} \longrightarrow T_{f,Y} \longrightarrow \mathcal{O}/\mathfrak{p}^t(\theta) \longrightarrow 0. \tag{48}$$

After taking a Tate twist, it gives rise to the long exact sequence in cohomology

$$0 \longrightarrow H^1(\mathbb{Z}[1/Np], T_{f,X}(\psi)(1)) \longrightarrow H^1(\mathbb{Z}[1/Np], T_{f,Y}(\psi)(1)) \longrightarrow H^1(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(\theta\psi)(1)) \tag{49}$$

The last map sends the class κ_f to $\bar{\kappa}_{f,1}$. Hence, the latter vanishes if and only if κ_f belongs to $H^1(\mathbb{Z}[1/Np], T_{f,X}(\psi)(1))$. □

Assume for the remainder of this section that $\bar{\kappa}_{f,1} = 0$ and hence $\kappa_f \in H^1(\mathbb{Q}, T_{f,Y}(\psi)(1))$ lifts to a class in $H^1(\mathbb{Q}, T_{f,X}(\psi)(1))$, that by a slight abuse of notation we continue to denote with the same symbol. As explained in the introduction, this allows us to define the global class

$$\bar{\kappa}_{f,2} = \bar{\pi}_{2*}(\bar{\kappa}_f) \in H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\psi)(2)).$$

Theorem 4.2. *(Second congruence relation) Assume that $L_p'(\bar{\theta}, -1) \not\equiv 0 \pmod{\mathfrak{p}}$. Then the following equality holds in $H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\psi)(2))$:*

$$\bar{\kappa}_{f,2} = \frac{L_p'(\bar{\theta}, -1)}{1 - p^{-1}} \cdot \frac{\bar{c}_{\chi_1} \cup \bar{c}_{\xi_1}}{\cup \log_p(\varepsilon_{\mathrm{cyc}})} \pmod{\mathfrak{p}^t}.$$

Here, $1/\cup \log_p(\varepsilon_{\mathrm{cyc}})$ denotes the inverse of the map

$$H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\psi)(2)) \to H^2(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\psi)(2)), \quad \kappa \mapsto \kappa \cup \log_p(\varepsilon_{\mathrm{cyc}}),$$

which is invertible under our running assumptions.

The remainder of this section is devoted to the proof of this theorem.

4.1. Cohomology and Eisenstein quotients.

For any $r \geq 0$ and $j \in \mathbb{Z}$ let

$$H_r(j) = H^1_{\mathrm{et}}(\bar{X}_1(Np^r), \mathcal{O}_\mathfrak{p}(\psi)(j))^{\mathrm{ord}}$$

denote the ordinary component of the étale cohomology group $H^1_{\mathrm{et}}(\bar{X}_1(Np^r), \mathcal{O}_\mathfrak{p}(j))$ with respect to the Hecke operator U_p^*. This is naturally an $\mathcal{O}_\mathfrak{p}[G_\mathbb{Q}]$-module. As in §2, let $\mathfrak{h}_r$ be the subring of $\mathrm{End}_{\mathcal{O}_\mathfrak{p}}(H_r)$ spanned over $\mathcal{O}_\mathfrak{p}$ by the diamond and Hecke operators T_n^*. As in [FK, §1.9.1], the Eisenstein ideal $I_r^* = I_{\mathrm{Eis},r}^* \subset \mathfrak{h}_r$ is the $\mathcal{O}_\mathfrak{p}$-submodule of $\mathfrak{h}_r$ generated by $U_\ell^* - 1$ and $T_\ell^* - \ell\langle \ell^{-1}\rangle - 1$ for primes ℓ dividing (resp. not dividing) the level.

Passing to the projective limit we may define:

$$H(j) := \varprojlim_{r\geq 0} H_r(j), \quad \mathfrak{h} = \varprojlim_{r \geq 0} \mathfrak{h}_r, \quad I^* = \varprojlim_{r\geq 0} I_r^* \subset \mathfrak{h}.$$

We may simply denote the above modules H_r and H when the Galois action is understood or irrelevant.

The ideal I^* is a height one ideal contained in the maximal ideal $\mathfrak{M} = (I^*, \mathfrak{p})$; for any $t \geq 1$ we shall denote $\mathfrak{M}^{(t)} = (I^*, \mathfrak{p}^t)$, so that $\mathfrak{M} = \mathfrak{M}^{(1)}$. The ideal I^* is the intersection of a finite number of height one prime ideals $P \subset \mathfrak{M}$, each of which corresponds to a weight two eigenform that is congruent to an Eisenstein series mod $\mathfrak{p}$, like the modular form f of the introduction.

Let $\Lambda_N := \lim_{\leftarrow} \mathcal{O}_\mathfrak{p}[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$ denote the Iwasawa algebra of tame level N. Any Dirichlet character $\xi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}_\mathfrak{p}^\times$ may be extended by linearity to yield a homomorphism

$$\xi : \Lambda_N \longrightarrow \mathcal{O}_\mathfrak{p}[(\mathbb{Z}/N\mathbb{Z})^\times] \longrightarrow \mathcal{O}_\mathfrak{p}$$

that we continue to denote with the same symbol.

For any Λ_N-module V let $V_\xi = V \otimes_{\Lambda_N,\xi} \mathcal{O}_\mathfrak{p}$ stand for the associated ξ-isotypical component. Note that $\Lambda_N = \oplus \Lambda_{N,\xi}$ where ξ ranges over all characters of $(\mathbb{Z}/N\mathbb{Z})^\times$ and $\Lambda_{N,\xi} \simeq \Lambda = \mathcal{O}_\mathfrak{p}[[\mathbb{Z}_p^\times]]$.

We begin by rephrasing the results of [FK, §9] on Sharifi's conjecture in a convenient way for our purposes.

Recall the cyclotomic character ε_r corresponding to $\mathrm{Gal}(\mathbb{Q}(\mu_{p^r})/\mathbb{Q})$ as in (43). There is a commutative diagram of $\Lambda_{N,\bar\theta}$-modules

$$\begin{array}{ccc} \lim_{\leftarrow_{r\geq 0}} H^2_{\mathrm{et}}(X_1(Np^r), \mathcal{O}_\mathfrak{p}(2)(\varepsilon_r\psi))^{\mathrm{ord}}_{\bar\theta} & \longrightarrow & H^2_{\mathrm{et}}(X_1(N), \mathcal{O}_\mathfrak{p}(2)(\psi))^{\mathrm{ord}}_{\bar\theta} \\ \downarrow & & \downarrow \\ H^1(\mathbb{Q}, H_{\bar\theta}(2)(\varepsilon_{\mathrm{cyc}})) & \longrightarrow & H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\bar X_1(N), \mathcal{O}_\mathfrak{p}(2)(\psi))^{\mathrm{ord}}_{\bar\theta}) \end{array} \tag{50}$$

where the vertical arrows arise from the Hochschild–Serre spectral sequence in étale cohomology as in the discussion before equation (23) and the horizontal arrows are specialization at $r = 0$.

The module H is endowed with an action of complex conjugation, yielding the decomposition $H = H^+ \oplus H^-$; in the sequel we shall employ a similar notation for any $\mathcal{O}_\mathfrak{p}$-module acted on by complex conjugation. Proceeding as in [FK, Prop. 6.3.2], we see that the quotient

$$(H/\mathfrak{M}^{(t)})^+_{\bar\theta} \tag{51}$$

is still endowed with a compatible action of $G_\mathbb{Q}$; in fact, $G_\mathbb{Q}$ acts on this module through the character ψ, as it follows from [FK, Remark 6.3.3].

Our running assumptions imply that the (mod $\mathfrak{p}^t$) Galois representation $\bar T_{f,X}(\psi)(1) = T_{f,X}(\psi)(1) \otimes \mathcal{O}/\mathfrak{p}^t$ arises as a quotient of the specialization at $r = 0$ of $(H/\mathfrak{M}^{(t)})(2)(\varepsilon_{\mathrm{cyc}})$. It follows from [FK, (7.1.11)], together with [FK, (1.4.3)] for the translation between the models of $X_1(N)$ chosen here and in loc. cit., that $\bar T_{f,X}(\psi)(1)$ belongs to the $\bar\theta$-isotypical component of the latter.

Recall also from (15) that there is a short exact sequence for $\bar T_{f,X}$, which according to (17) and (18) is compatible with the action of complex conjugation. In particular, in light of (21) we have

$$(\bar T^-_{f,X})(\psi)(1) = (\bar T_{f,X}(\psi)(1))^+ \simeq \mathcal{O}/\mathfrak{p}^t(\psi)(2)$$

and this is naturally a quotient of (51). Henceforth we fix the canonical isomorphism provided by [FK, 6.3.18 and 7.1.11] in order to identify

$$\bar T_{f,X}(\psi)(1)^+ = \mathcal{O}/\mathfrak{p}^t(\psi)(2) \tag{52}$$

as $\mathcal{O}/\mathfrak{p}^t[G_\mathbb{Q}]$-modules.

Summing up there is a commutative diagram of $G_\mathbb{Q}$-modules, where the horizontal arrows arise from specializing to $r = 0$, i.e. level N:

$$
\begin{array}{ccc}
H^1(\mathbb{Q}, H_{\bar\theta}(2)(\varepsilon_{\mathrm{cyc}})) & \longrightarrow & H^1(\mathbb{Q}, H^1_{\mathrm{et}}(\bar X_1(N), \mathcal{O}_{\mathfrak{p}}(\psi)(2))^{\mathrm{ord}}_{\bar\theta}) \\
\downarrow & & \downarrow \\
H^1(\mathbb{Q}, (H/\mathfrak{M}^{(t)})^+_{\bar\theta}(2)(\varepsilon_{\mathrm{cyc}})) & \longrightarrow & H^1(\mathbb{Q}, (H^1_{\mathrm{et}}(\bar X_1(N), \mathcal{O}_{\mathfrak{p}}(\psi)(2))^{\mathrm{ord}}/(I_0^*, \mathfrak{p}^t))^+_{\bar\theta}) \\
& \searrow & \downarrow \\
& & H^1(\mathbb{Q}, \bar T_{f,X}(\psi)(1)^+) = H^1(\mathbb{Q}, \mathcal{O}/\mathfrak{p}^t(\psi)(2))
\end{array}
\tag{53}
$$

4.2. **Fukaya–Kato maps.** Define the module $\mathcal{Q}$ similarly as in [FK, §6.3.1], namely

$$\mathcal{Q} := (H/I^*H)^+_{\bar\theta}(2)(\varepsilon_{\mathrm{cyc}}).$$

Recall from (27) in §3 the Kubota–Leopoldt p-adic L-function $L_p(\bar\theta, s-1) \in \Lambda_{N,\theta}$ attached to the Dirichlet character $\bar\theta$. This is what is denoted ${}^{-1}\xi$ in $\Lambda_{N,\theta}$ in [FK, 6.1.6], taking into account the normalizations adopted in [FK, 4.1.1] versus ours.

Proposition 4.3. *There are isomorphisms of Galois modules*

$$\mathcal{Q} \xrightarrow{\simeq} \Lambda_{N,\theta}/(L_p(\bar\theta, s-1))(2)(\varepsilon_{\mathrm{cyc}}\psi) \simeq (\mathfrak{h}/I^*\mathfrak{h})_{\bar\theta}(2)(\varepsilon_{\mathrm{cyc}}\psi). \tag{54}$$

Proof. Since we already remarked that the Galois action on $(H/I^*H)^+_{\bar\theta}$ is given by ψ, the first identification amounts to the equality $(H/I^*H)^+_{\bar\theta} = \Lambda_{N,\theta}(\psi)/(L_p(\bar\theta, s-1))$, which follows from [FK, §6.3.8 – 6.3.18]. Note once again that the results of loc. cit. are stated for a different choice of model for the modular curve, and our formulation is translated from loc. cit. by means of the isomorphism v_N described in [FK, 1.4.5(1)]. This map, whose precise definition is given in [FK, 1.4.2], preserves the actions of Hecke operators, dual Hecke operators and diamond operators, but changes the action of $G_{\mathbb{Q}}$ by an appropriate twist introduced in [FK, 1.2.9].

The second isomorphism is a consequence of the proof of the Iwasawa main conjecture by Mazur and Wiles (see also [FK, §6.1.7]). □

Recall that $H^i(\mathbb{Z}[1/Np], V) \subset H^i(\mathbb{Q}, V)$ stands for the set of classes which are unramified at primes dividing Np. Shapiro's lemma gives an isomorphism

$$\varprojlim H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathcal{O}_{\mathfrak{p}}(\psi)(2)) \simeq H^2(\mathbb{Z}[1/Np], \Lambda_N(\varepsilon_{\mathrm{cyc}}\psi)(2)).$$

As a piece of notation, and following the definition of [FK, §5.2.6], set

$$\mathcal{S} = \varprojlim H^2(\mathbb{Z}[1/Np, \zeta_{Np^r}], \mathcal{O}_{\mathfrak{p}}(\psi)(2))^+ \simeq H^2(\mathbb{Z}[1/Np], \Lambda_N(\varepsilon_{\mathrm{cyc}}\psi)(2))^+.$$

In [FK, §9.1] Fukaya and Kato established the existence of isomorphisms

$$\mathrm{FK}_1 : H^1(\mathbb{Z}[1/Np], \mathcal{Q}) \simeq \mathcal{S}_{\bar\theta}, \quad \mathrm{FK}_2 : \mathcal{S}_{\bar\theta} \simeq H^2(\mathbb{Z}[1/Np], \mathcal{Q})$$

arising from the long exact sequence in cohomology induced by the short exact sequence

$$0 \to \Lambda_{N,\theta}(\varepsilon_{\mathrm{cyc}}\psi)(2) \xrightarrow{\cdot L_p(\bar\theta, s-1)} \Lambda_{N,\theta}(\varepsilon_{\mathrm{cyc}}\psi)(2) \to \mathcal{Q} \to 0.$$

stemming from (54).

In particular, the map we have denoted as FK_2 is just the $+$-component of the homomorphism

$$H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}(\varepsilon_{\mathrm{cyc}}\psi)(2)) \longrightarrow H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}/(L_p(\bar\theta, s-1))(\varepsilon_{\mathrm{cyc}}\psi)(2)) \tag{55}$$

induced by (a twist of) the natural projection $\Lambda_{N,\theta} \longrightarrow \Lambda_{N,\theta}/L_p(\bar\theta, s-1)$. For an explicit description of FK_1, see [FK, §9.1].

The main result of §9.2 of loc. cit. asserts that the map

$$\mathrm{ev}_\infty : H^2_{\mathrm{et}}(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(\varepsilon_{\mathrm{cyc}}\psi)(2))_{\bar\theta} \longrightarrow \mathcal{S}_{\bar\theta} \tag{56}$$

induced by evaluation at the cusp ∞ factors through the Eisenstein quotient, as stated below.

Proposition 4.4 (Fukaya–Kato)**.** *The map* ev_∞ *of* (56) *agrees with the composition*

$$H^2_{\mathrm{et}}(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(\varepsilon_{\mathrm{cyc}}\psi)(2))_{\bar\theta} \to H^1(\mathbb{Z}[1/Np], H_{\bar\theta}(\varepsilon_{\mathrm{cyc}})(2)) \to H^1(\mathbb{Z}[1/Np], \mathcal{Q}) \simeq \mathcal{S}_{\bar\theta}$$

where:

- *the first map is the composition of*
$$H^2(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(\varepsilon_{\mathrm{cyc}}\psi)(2))_{\bar\theta} \to H^2(X_1(Np^\infty), \mathcal{O}_{\mathfrak{p}}(\varepsilon_{\mathrm{cyc}}\psi)(2))^{\mathrm{ord}}_{\bar\theta}$$
and the left vertical arrow in (50)*, both restricted to the subspace of classes unramified at the primes dividing* Np*;*
- *the second map is induced by the projection* $H_{\bar\theta}(\varepsilon_{\mathrm{cyc}})(2) \to \mathcal{Q}$*;*
- *the last isomorphism is* FK_1*.*

In [FK, §9.3] Fukaya and Kato further introduced two distinguished morphisms

$$a, b : H^1(\mathbb{Z}[1/Np], \mathcal{Q}) \to H^2(\mathbb{Z}[1/Np], \mathcal{Q}) \tag{57}$$
$$a = \mathrm{FK}_2 \circ \mathrm{FK}_1,$$
$$b = \cup\, (1-p^{-1}) \log_p(\varepsilon_{\mathrm{cyc}})$$

where $\varepsilon_{\mathrm{cyc}} \in H^1(\mathbb{Q}, \mathcal{O}_{\mathfrak{p}}^\times)$ stands for the cyclotomic character. Note that $(1-p^{-1})\log_p$ takes values in $\mathbb{Z}_p$ and hence b is indeed well-defined. Under these conditions, they show the following.

Proposition 4.5 (Fukaya–Kato)**.** *Let* $L_p'(\bar\theta) \in \Lambda_{N,\theta}$ *denote the derivative of* $L_p(\bar\theta, s-1)$*. Then*

$$b = L_p'(\bar\theta) \cdot a. \tag{58}$$

Proof. This is proved in [FK, Proposition 9.3.1]. □

Corollary 4.6. *The map*

$$H^1(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(\psi)(2)) \longrightarrow H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(\psi)(2)), \quad \kappa \mapsto \kappa \cup (1-p^{-1})\log_p(\varepsilon_{\mathrm{cyc}})$$

is invertible.

Proof. Observe that FK_1 and FK_2 are Λ-adic isomorphisms, as it has been proved in [FK, §9.1]. Hence, once we consider the specialization map at the trivial character, we still have isomorphisms of $\mathcal{O}_{\mathfrak{p}}$-modules. The same must be true for their composition multiplied by the p-adic unit $L_p'(\bar\theta, -1)$, and according to Proposition 4.5 and the definitions provided in [FK, §4.1.3], this is precisely the above map. □

After applying the Fukaya–Kato map FK_1 to the bottom row of diagram (53), restricting to the subspace of unramified classes at primes dividing Np, we reach the commutative diagram
(59)

$$\begin{array}{ccc} H^1(\mathbb{Z}[1/Np], (H/\mathfrak{M}^{(t)})^+_{\bar\theta}(\varepsilon_{\mathrm{cyc}})(2)) & \longrightarrow & H^1(\mathbb{Z}[1/Np], \bar T_{f,X}(\psi)(1)^+) = H^1(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(\psi)(2)) \\ \downarrow \bar{\mathrm{FK}}_1 & & \downarrow \bar{\mathrm{FK}}_1(r=0) \\ H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}/\mathfrak{p}^t(\varepsilon_{\mathrm{cyc}}\psi)(2)) & \longrightarrow & H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(\psi)(2)), \end{array}$$

where the left most vertical map is $\bar{\mathrm{FK}}_1 = \mathrm{FK}_1 \pmod{\mathfrak{p}^t}$. As in (53), the horizontal arrows are specialization at $r=0$, and the right-most vertical arrow is accordingly the specialization of $\bar{\mathrm{FK}}_1$ at $r=0$.

We may further apply now Fukaya–Kato's map $\bar{\mathrm{FK}}_2 = \mathrm{FK}_2 \pmod{\mathfrak{p}^t}$ to the above diagram and obtain the following one:

$$\begin{array}{ccc} H^2(\mathbb{Z}[1/Np], \Lambda_{N,\theta}/\mathfrak{p}^t(\varepsilon_{\mathrm{cyc}}\psi)(2)) & \longrightarrow & H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(\psi)(2)) \\ \downarrow \bar{\mathrm{FK}}_2 & & \downarrow \| \\ H^2(\mathbb{Z}[1/Np], (H/\mathfrak{M}^{(t)})^+_{\bar\theta}(\varepsilon_{\mathrm{cyc}})(2)) & \longrightarrow & H^2(\mathbb{Z}[1/Np], \mathcal{O}/\mathfrak{p}^t(\psi)(2)) \end{array} \tag{60}$$

Again the horizontal maps are specialization in level N at $r=0$ and the right-most vertical map is the specialization of $\bar{\mathrm{FK}}_2$ at $r=0$. In view of (55) the latter may be identified with the identity map: according to the definitions provided in [FK, §6.1.6, 4.1.3] and with our current conventions, the specialization of $L_p(\bar\theta)$ at $r=0$ is

$$L_p(\bar\theta,-1)=(1-\bar\theta(p)p)\cdot L(\bar\theta,-1)=-(1-\bar\theta(p)p)\cdot\frac{B_2(\bar\theta)}{2},$$

which vanishes $\pmod{\mathfrak{p}^t}$ in light of our assumptions.

4.3. **Proof of Theorem 4.2.** We can finally prove Theorem 4.2. With a slight abuse of notation, we identify global units with their image in cohomology under the Kummer map.

Note that the image of the Kato class under the composition of maps described in Proposition 4.4 is $\mathrm{FK}_1(\bar\kappa_{f,2})$. Then, we have

$$\mathrm{FK}_1(\bar\kappa_{f,2})=\mathrm{ev}_\infty(u_{\chi_1,\chi_2}\cup u_{\xi_1,\xi_2})=\bar c_{\chi_1}\cup\bar c_{\xi_1}\pmod{\mathfrak{p}^t},\tag{61}$$

where the circular units involved in the cup product are those resulting from the evaluation at infinity of the modular units u_{χ_1,χ_2} and u_{ξ_1,ξ_2}, respectively. In particular, the last equality directly follows from the construction of the map corresponding to evaluation at infinity. Recall we are assuming that $\xi_1=\theta\bar\chi_1\bar\chi_2$, and it was proved in Theorem 3.8 that κ_f is unramified everywhere.

Next, we apply FK_2 to both sides of (61). Proposition 4.5 together with the commutativity of (60) allow us to establish that

$$\bar\kappa_{f,2}\cup(1-p^{-1})\log_p(\varepsilon_{\mathrm{cyc}})=L_p'(\bar\theta,-1)\cdot(\bar c_{\chi_1}\cup\bar c_{\xi_1}).$$

Theorem 4.2 finally follows from Corollary 4.6.

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