SPHERICAL MAXIMAL FUNCTIONS AND FRACTAL DIMENSIONS OF DILATION SETS

JORIS ROOS ANDREAS SEEGER

ABSTRACT. For the spherical mean operators $\mathcal{A}_t$ in $\mathbb{R}^d$, $d \geq 2$, we consider the maximal functions $M_E f = \sup_{t\in E} |\mathcal{A}_t f|$, with dilation sets $E \subset [1,2]$. In this paper we give a surprising characterization of the closed convex sets which can occur as closure of the sharp L^p improving region of M_E for some E. This region depends on the Minkowski dimension of E, but also other properties of the fractal geometry such the Assouad spectrum of E and subsets of E. A key ingredient is an essentially sharp result on M_E for a class of sets called (quasi-)Assouad regular which is new in two dimensions.

1. Introduction

For a locally integrable function f on $\mathbb{R}^d$ with $d \geq 2$ let

$$\mathcal{A}_t f(x) = \int f(x - ty) d\sigma(y),$$

where $t > 0$ and σ denotes the normalized surface measure on the unit sphere in $\mathbb{R}^d$. Given a set $E \subset (0, \infty)$ consider the maximal function

$$M_E f(x) = \sup_{t\in E} |\mathcal{A}_t f(x)|,$$

which is well–defined at least on continuous functions f. In this paper we study sharp L^p *improving properties* of M_E. By scaling considerations it is natural to restrict attention to sets $E \subset [1, 2]$. We define the *type set* $\mathcal{T}_E$ associated with M_E by

$$\mathcal{T}_E = \{(\tfrac{1}{p}, \tfrac{1}{q}) \in [0,1]^2 \,:\, M_E \text{ is bounded } L^p \to L^q\}.$$

We are interested in determining for a given set E the type set $\mathcal{T}_E$ up to the boundary, i.e. we will focus mainly on the closure of this set. Note that since $\mathcal{T}_E$ is by interpolation convex, the interior of $\mathcal{T}_E$ is determined by $\overline{\mathcal{T}_E}$.

2010 *Mathematics Subject Classification.* 42B25, 28A80.

Key words and phrases. Spherical maximal functions, L^p improving estimates, Minkowski dimension, (quasi-)Assouad dimension, Assouad spectrum, Assouad regular sets.

Research supported in part by the National Science Foundation (DMS 1764295) and a Simons Fellowship.

We consider two natural problems. First, for each given $E \subset [1,2]$ the goal is to determine $\overline{\mathcal{T}_E}$. Second, we ask which closed convex subsets of $[0,1]^2$ arise as $\overline{\mathcal{T}_E}$ for some $E \subset [1,2]$.

In this paper we give a complete solution to the second problem: we will determine exactly which closed convex sets arise as $\overline{\mathcal{T}_E}$. Moreover, we also give a satisfactory answer to the first problem for a large class of sets E that covers all examples previously considered in the literature.

The case of a single average, $E = \{\text{point}\}$, is covered by a classical result of Littman [16]. Sharp results for the case $E = [1,2]$ are due to Schlag [20], Schlag and Sogge [21] and S. Lee [15]. The case $p = q$ for the full spherical maximal operator goes back to Stein [25] in the case $d \geq 3$ and to Bourgain [5] in the case $d = 2$ (see also [18]). For some early results in special cases of dilation sets see [7], [8] and [28, p. 92]. A satisfactory answer for general E in the case $p = q$, depending on the Minkowski dimension of E, was given in [23]; see also [24], [22] for refinements and related results. We remark that, while the question of sharp L^p improving bounds is interesting in its own right, it is also motivated by problems on sparse domination and weighted estimates for global maximal functions $\sup_{t\in E}\sup_{k\in\mathbb{Z}} |\mathcal{A}_{2^k t} f|$, *cf.* [3], [14].

In a recent joint paper [1] with T. Anderson and K. Hughes we addressed the $L^p \to L^q$ problem for M_E when $q > p$ in dimensions $d \geq 3$, with some partial results for $d = 2$. It turned out that satisfactory results cannot just depend on the (upper) Minkowski dimension of E alone and other notions of fractal dimension are needed, in particular the Assouad dimension.

Let us recall some definitions. Let $E \subset [1,2]$. For $\delta > 0$ let $N(E,\delta)$ the δ-covering number, *i.e.* the minimal number of intervals of length δ required to cover E. The *(upper) Minkowski dimension* $\dim_{\mathrm{M}} E$ of E is

$$\dim_{\mathrm{M}} E = \inf \big\{ a > 0 \,:\, \exists\, c > 0 \text{ s.t.} \,\forall\, \delta \in (0,1),\, N(E,\delta) \leq c\,\delta^{-a} \,\big\}.$$

The *Assouad dimension* $\dim_{\mathrm{A}} E$ ([2]) is defined by

$$\dim_{\mathrm{A}} E = \inf \{ a > 0 \,:\, \exists\, c > 0 \text{ s.t.} \,\forall\, I,\, \delta \in (0, |I|), \quad N(E \cap I, \delta) \leq c\,\delta^{-a} |I|^a \,\} \,;$$

here I runs over subintervals of $[1,2]$. Note that $0 \leq \dim_{\mathrm{M}} E \leq \dim_{\mathrm{A}} E \leq 1$. For $0 \leq \beta \leq \gamma \leq 1$ let

$$Q_1 = (0,0),\ Q_{2,\beta} = \left(\tfrac{d-1}{d-1+\beta}, \tfrac{d-1}{d-1+\beta}\right), \quad Q_{3,\beta} = \left(\tfrac{d-\beta}{d-\beta+1}, \tfrac{1}{d-\beta+1}\right),\ Q_{4,\gamma} = \left(\tfrac{d(d-1)}{d^2+2\gamma-1}, \tfrac{d-1}{d^2+2\gamma-1}\right). \tag{1.1}$$

Moreover, let $\mathcal{Q}(\beta,\gamma)$ denote the closed convex hull of the points Q_1, $Q_{2,\beta}$, $Q_{3,\beta}$, $Q_{4,\gamma}$, see Figure 1 below. Let $\mathcal{R}(\beta,\gamma)$ denote the union of the interior of $\mathcal{Q}(\beta,\gamma)$ with the line segment connecting Q_1 and $Q_{2,\beta}$, including Q_1, but excluding $Q_{2,\beta}$. The paper [1] gives a sufficient condition for M_E to be $L^p \to L^q$ bounded, in dimension $d \geq 3$, namely if $\beta = \dim_{\mathrm{M}} E$, $\gamma_* = \dim_{\mathrm{A}} E$ then

$$\mathcal{R}(\beta,\gamma_*) \subset \mathcal{T}_E. \tag{1.2}$$

This inclusion was also obtained for $\gamma_* \leq 1/2$ in two dimension, but the more difficult case $\gamma_* > 1/2$ was left open. Our first main result is that (1.2) remains true for $d = 2$, $\gamma_* > 1/2$.

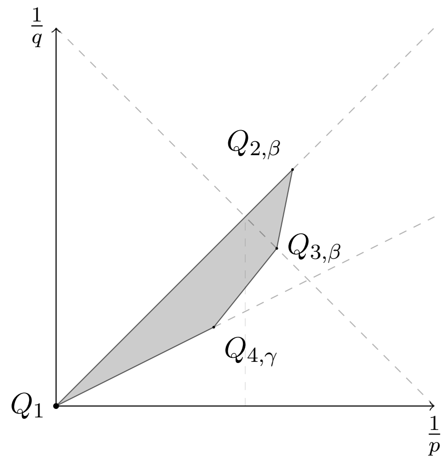

FIGURE 1. The quadrangle $\mathcal{Q}(\beta, \gamma)$ for $d = 2$, $\beta = 0.6$, $\gamma = 0.9$.

We thereby get a rather satisfactory upper bound for M_E, which happens to be essentially sharp for so-called classes of Assouad regular sets discussed below. However, there is a slight shortcoming of this formulation which we will discuss now. Given E the closure of the type set does not change if one replaces E by its union with a set of zero Minkowski dimension; however such unions may change the Assouad dimension (see §6.3) and thus the set $\mathcal{Q}(\dim_{\mathrm{M}} E, \dim_{\mathrm{A}} E)$. To address this issue we replace the notion of Assouad dimension with *quasi-Assouad dimension* introduced by Lü and Xi in [17] (see also [12]).

The definition involves certain intermediate fractal dimensions used in [12], namely the *upper Assouad spectrum* $\theta \mapsto \overline{\dim}_{\mathrm{A},\theta} E$ which for given $\theta \in [0, 1]$ is defined by

$$\overline{\dim}_{\mathrm{A},\theta} E = \inf \{ a > 0 \,:\, \exists\, c > 0 \text{ s.t.} \,\forall\, \delta \in (0,1),\, |I| \geq \delta^{\theta}, \\ N(E \cap I, \delta) \leq c\, \delta^{-a} |I|^{a} \,\};$$

here I runs over subintervals of $[1, 2]$. The upper Assouad spectrum is a variant of the Assouad spectrum, where the condition $|I| \geq \delta^{\theta}$ is replaced by $|I| = \delta^{\theta}$. This was introduced by J. Fraser and H. Yu in [10] (and used in [1] in the discussion of spherical maximal functions). The upper Assouad spectrum has the benefit that it is by definition nondecreasing in θ. One defines the quasi-Assouad dimension as the limit

$$\dim_{\mathrm{qA}} E = \lim_{\theta \to 1} \overline{\dim}_{\mathrm{A},\theta} E. \tag{1.3}$$

We remark that always $\dim_{\mathrm{qA}} E \le \dim_{\mathrm{A}} E$ and the inequality may be strict, see §6.3 for examples. With (1.3) and $\mathcal{R}(\beta,\gamma)$ defined following (1.1) we can now formulate

Theorem 1.1. *Let $d \ge 2$ and $E \subset [1,2]$, $\beta = \dim_{\mathrm{M}} E$, $\gamma = \dim_{\mathrm{qA}} E$. Then $\mathcal{R}(\beta,\gamma) \subset \mathcal{T}_E$.*

The most difficult case is $d = 2$, $\gamma > \frac{1}{2}$, and we will present the complete proof. In the cases $d \ge 3$ and $d = 2$, $\gamma \le 1/2$ the result was essentially established in [1], *cf.* §2 below for further review.

We shall now discuss the second problem mentioned above. Modifications of well-known examples from [20], [21], [23] (see [1, §4] for details) show the lower bound

$$\overline{\mathcal{T}_E} \subset \mathcal{Q}(\beta,\beta) \tag{1.4}$$

if $\beta = \dim_{\mathrm{M}} E$. Theorem 1.1 and (1.4) show that the set $\overline{\mathcal{T}_E}$ is a closed convex set satisfying the relation $\mathcal{Q}(\beta,\gamma) \subset \overline{\mathcal{T}_E} \subset \mathcal{Q}(\beta,\beta)$ for $\gamma = \dim_{\mathrm{qA}} E$. Surprisingly, this necessary condition on $\overline{\mathcal{T}_E}$ is also sufficient:

Theorem 1.2. *Let $\mathcal{W} \subset [0,1]^2$. Then*

(i) $\mathcal{W} = \overline{\mathcal{T}_E}$ holds for some $E \subset [1,2]$ if and only if $\mathcal{W}$ is a closed convex set and

$$\mathcal{Q}(\beta,\gamma) \subset \mathcal{W} \subset \mathcal{Q}(\beta,\beta) \text{ for some } 0 \le \beta \le \gamma \le 1. \tag{1.5}$$

(ii) For $\mathcal{W} = \overline{\mathcal{T}_E}$ in (1.5) one necessarily has $\dim_{\mathrm{M}} E = \beta$ and if in addition γ is chosen minimally, then $\dim_{\mathrm{qA}} E = \gamma$.

Remark 1.3. In the situation of (ii), for every $\gamma_* \in [\gamma, 1]$ the set E can be chosen such that $\dim_{\mathrm{A}} E = \gamma_*$, *cf.* §7.

Figure 2 shows a more detailed look into the critical triangle spanned by the points $Q_{4,\gamma}$, $Q_{4,\beta}$, $Q_{3,\beta}$ and illustrates in particular that the boundary of $\mathcal{T}_E$ may follow an arbitrary convex curve in this triangle.

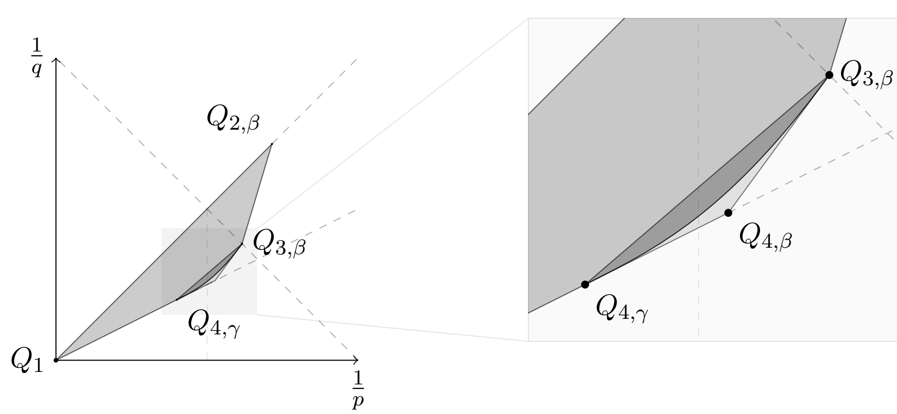

FIGURE 2.

The basic idea of the proof of Theorem 1.2 is to write $\mathcal{W}$ as an at most countable intersection $\cap_n \mathcal{Q}(\beta_n,\gamma_n)$ and to construct the set E in a suitable way as a disjoint union of sets E_n with the property $\overline{\mathcal{T}_{E_n}} = \mathcal{Q}(\beta_n,\gamma_n)$. In order to implement this idea one needs to understand concrete cases in which Theorem 1.1 is sharp.

If $\dim_{\mathrm{qA}} E = \dim_{\mathrm{M}} E = \beta$, then by Theorem 1.1 and (1.4) we have $\overline{\mathcal{T}_E} = \mathcal{Q}(\beta,\beta)$. This happens for example if E is a self-similar Cantor set of dimension β. In particular, Theorem 1.1 is sharp up to endpoints for such E. The theorem is also sharp for a class of sets E with $\dim_{\mathrm{M}} E < \dim_{\mathrm{qA}} E$. Say that a set $E \subset [1,2]$ is (β,γ)-*regular* if either $\gamma = 0$, or

$$\dim_{\mathrm{M}} E = \beta,\ \dim_{\mathrm{qA}} E = \gamma,\ \overline{\dim}_{\mathrm{A},\theta} E = \dim_{\mathrm{qA}} E \text{ for all } 1 > \theta > 1 - \beta/\gamma.$$

A set is *quasi-Assouad regular* if it is (β,γ)-regular for some (β,γ). In [1] we have used a slightly more restrictive definition: a set is called (β,γ)-*Assouad regular* if the above condition holds with $\dim_{\mathrm{qA}}$ replaced by $\dim_A$ and *Assouad regular* if it is (β,γ)-Assouad regular for some (β,γ). Assouad regular sets are also quasi-Assouad regular: for every Assouad regular set of positive Minkowski dimension we have $\dim_{\mathrm{qA}} E = \dim_{\mathrm{A}} E$. Moreover, all sets with $\dim_{\mathrm{M}} E = 0$ are quasi-Assouad regular since the condition is voidly satisfied when $\beta = 0$. When $\beta = \dim_{\mathrm{M}} E = \dim_{\mathrm{qA}} E$ the upper Assouad spectrum is constant, so E is (β,β)-regular.

A convex sequence E which has Minkowski dimension β is $(\beta,1)$-regular. Other examples of (quasi-)Assouad regular sets can be found in [1, §5], see also §6 below for a refinement needed in the proof of Theorem 1.2. The inclusion $\overline{\mathcal{T}}_E \subset \mathcal{Q}(\beta,\gamma)$ for (β,γ)-Assouad regular sets was proved in [1, §4]. Here the maximal operator is tested on characteristic function of δ-neighborhoods of spherical caps which have diameter $\approx \sqrt{\delta^{\beta/\gamma}}$; when $\beta=\gamma$ this reduces to a standard Knapp type example. We refer to §5 for a more general result. In this context we also note that for all E the type set of M_E when restricted to radial functions is strictly larger than the type set of M_E on general functions (*cf.* [19]).

From the necessary conditions and Theorem 1.1 we have

$$\overline{\mathcal{T}_E} = \mathcal{Q}(\beta,\gamma), \qquad \text{for } (\beta,\gamma)\text{-regular } E, \tag{1.6}$$

in all dimensions $d \geq 2$. It turns out that an essentially sharp result can be obtained for a much larger class, namely arbitrary finite unions of quasi-Assouad regular sets in which case the closure of the type set is a closed convex polygon.

Theorem 1.4. *Let $d \geq 2$ and $E = \cup_{j=1}^m E_j$ where E_j is (β_j,γ_j)-regular. Then $\overline{\mathcal{T}_E} = \cap_{j=1}^m \mathcal{Q}(\beta_j,\gamma_j)$.*

This is actually a simple consequence of Theorem 1.1 and the lower bounds, see §5. Nevertheless, Theorem 1.4 is an essential step towards the proof of Theorem 1.2. Moreover, Theorem 1.4 can be used to obtain certain sparse domination results on global spherical maximal functions, see [1, §6].

It would be interesting to extend Theorem 1.4 to a wider class of sets. Moreover, it is also worthwhile to investigate several endpoint results, *cf.* §2.5 below.

Summary of the paper.

- In §2 we recall some previous results from [23], [5], [1] reducing the proof of Theorem 1.1 to Theorem 2.1 concerning the two-dimensional case with $\gamma \geq 1/2$. We also state a key ingredient, Corollary 2.2, for the proof of Theorem 1.2. In §2.5 we discuss some known and some open questions on endpoint estimates.
- In §3 and §4 we prove Theorem 2.1. We use the general strategy from [21]. Our main innovation here appears in §4 and consists of the use of almost orthogonality arguments in conjunction with arguments based on the fractal geometry of the set E.
- In §5 we discuss a relevant necessary condition and prove Theorem 1.4.
- In §6 we present certain uniform constructions of (quasi-)Assouad regular sets. This is a refinement of [1, §5].
- In §7 we prove Theorem 1.2. This uses Theorem 1.1 (in the form of Corollary 2.2), (1.6) and the construction in §6.

Notation. For a sublinear operator T acting on functions on $\mathbb{R}^d$ we denote the $L^p \to L^q$ operator norm by $\|T\|_{p\to q} = \sup\{\|Tf\|_q : \|f\|_p = 1\}$. Fourier transforms will be denoted by $\widehat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)dx$. Weighted L^p spaces are denoted by $L^p(w)$ with $\|f\|_{L^p(w)} = \big(\int_{\mathbb{R}^d} |f(x)|^p w(x)dx\big)^{1/p}$. We will use c to denote a positive constant that may change throughout the text and may depend on various quantities, which are either made explicit or clear from context. We write $A \lesssim B$ to denote existence of a constant c such that $A \leq cB$ and $A \approx B$ to denote $A \lesssim B$ and $B \lesssim A$.

Acknowledgement. We are grateful to the referee for a thorough reading of the paper and helpful suggestions.

2. Setup and preliminary reductions

In this section we review and collect known facts about spherical averages from the literature. This will reduce the proof of Theorem 1.1 to the most difficult case, when $d = 2$, $\gamma > \frac{1}{2}$ and $(\frac{1}{p}, \frac{1}{q})$ near $Q_{4,\gamma}$ (see Theorem 2.1 below). At the same time, this review will pave the way for the proof of Theorem 1.2, which requires a certain uniformity of various constants with respect to the set E. Below we always assume $t \in [1, 2]$.

2.1. *Dyadic decomposition.* Let χ be a smooth radial function on $\mathbb{R}^d$ supported in $\{1/2 \leq |\xi| \leq 2\}$ such that $0 \leq \chi \leq 1$ and $\sum_{j\in\mathbb{Z}} \chi(2^{-j}\xi) = 1$ for every $\xi \neq 0$. Set

$$\chi_0(\xi) = 1 - \sum_{j\geq 1} \chi(2^{-j}\xi), \text{ and } \chi_j(\xi) = \chi(2^{-j}\xi) \quad \text{for } j \geq 1.$$

Next define $\mathcal{A}_t^j f$, $\sigma_{j,t}$ for $j \ge 0$ with

$$\widehat{\mathcal{A}_t^j f}(\xi) = \chi_j(\xi)\widehat{\sigma}(t\xi)\widehat{f}(\xi) = \widehat{\sigma_{j,t}}(\xi)\widehat{f}(\xi). \tag{2.1}$$

Then $A_t = \sum_{j\ge 0} \mathcal{A}_t^j$.

The symbol class $\mathfrak{S}^m$ is defined as the class of functions a on $\mathbb{R}^d$ for which

$$\|a\|_{\mathfrak{S}^m} = \max_{|\alpha|\le 10d} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|)^{-(m-|\alpha|)}|\partial^\alpha a(\xi)| \tag{2.2}$$

is finite. Here $|\alpha| = \sum_{i=1} \alpha_i$ denotes the length of the multindex $\alpha \in \mathbb{N}_0^d$. It is well–known (see [26, Ch. VIII]) that

$$\widehat{\sigma}(\xi) = \sum_{\pm} a_\pm(\xi)e^{\pm i|\xi|} \tag{2.3}$$

with $a_\pm \in \mathfrak{S}^{-(d-1)/2}$. Now (2.3) and Plancherel's theorem imply

$$\|\mathcal{A}_t^j f\|_2 \lesssim 2^{-j\frac{d-1}{2}}\|f\|_2. \tag{2.4}$$

The L^1 functions $\sigma_{j,t}$ satisfy the standard pointwise inequality

$$|\sigma_{j,t}(x)| \le C_N 2^j (1+2^j|\,|x|-t\,|)^{-N}$$

for $1 \le t \le 2$ and thus we get

$$\|\mathcal{A}_t^j\|_{1\to 1} = \|\mathcal{A}_t^j\|_{\infty\to\infty} \lesssim 1 \tag{2.5}$$

and

$$\|\mathcal{A}_t^j\|_{1\to\infty} \lesssim 2^j. \tag{2.6}$$

Appropriate interpolation among (2.4), (2.5), (2.6) yields sharp estimates for the $L^p \to L^q$ operator norm of A_t^j for each *fixed* t.

2.2. *Results near $Q_1, Q_{2,\beta}, Q_{3,\beta}$.* We now turn our attention to the maximal operator associated with each $\mathcal{A}_t^j$. The uncertainty principle suggests that $|\mathcal{A}_t^j f(x)|$ is "roughly constant" as t changes across an interval of length $\lesssim 2^{-j}$. Keeping in mind (2.4), (2.5), this suggests that for all $1 \le p \le \infty$,

$$\|\sup_{t\in E}|\mathcal{A}_t^j f|\|_p \le cN(E,2^{-j})^{\frac{1}{p}} 2^{-j(d-1)\min(\frac{1}{p},\frac{1}{p'})}\|f\|_p \tag{2.7}$$

with c only depending on d. This was proven in [23] (also see [1, Lemma 2.2]). Observe that summing these estimates over $j \ge 0$ already gives $L^p \to L^p$ estimates for M_E in the sharp range $p > 1+\frac{\beta}{d-1}$ unless $d=2$ and $\beta = 1$ (but this case is covered by Bourgain's circular maximal theorem [5]). In view of (2.6), the same argument (see [1, Lemma 2.3]) also yields for $2 \le q \le \infty$,

$$\|\sup_{t\in E}|\mathcal{A}_t^j f|\|_q \le cN(E,2^{-j})^{\frac{1}{q}} 2^{-j(1-\frac{d+1}{q})}\|f\|_{q'} \tag{2.8}$$

with c only depending on d. Appropriate interpolation of (2.7), (2.8) shows that M_E is bounded $L^p \to L^q$ for every $(\frac{1}{p},\frac{1}{q})$ contained in the interior of the triangle with vertices Q_1, $Q_{2,\beta}$, $Q_{3,\beta}$ (see Figure 1).

2.3. *Minkowski and Assouad characteristics.* It is convenient to recast estimates involving $N(E,\delta)$ and $N(E\cap I,\delta)$ in terms of the following functions defined for $0<\delta<1$.

Definition. (i) The function $\chi^E_{\mathrm{M},\beta}:(0,1]\to[0,\infty]$ defined by

$$\chi^E_{\mathrm{M},\beta}(\delta)=\delta^\beta N(E,\delta) \tag{2.9}$$

is called the β-*Minkowski characteristic* of E.

(ii) The function $\chi^E_{\mathrm{A},\gamma}:(0,1]\to[0,\infty]$ defined by

$$\chi^E_{\mathrm{A},\gamma}(\delta)=\sup_{|I|\ge\delta}\big(\tfrac{\delta}{|I|}\big)^\gamma N(E\cap I,\delta) \tag{2.10}$$

is called the γ-*Assouad characteristic* of E.

The estimates (2.7), (2.8) can be rewritten as

$$\|\sup_{t\in E}|\mathcal{A}^j_t f|\|_p\le c[\chi^E_{\mathrm{M},\beta}(2^{-j})]^{\frac1p}\times\begin{cases}2^{-j(\frac{d-1}{p'}-\frac{\beta}{p})}\|f\|_p, & \text{if } 1\le p\le 2,\\ 2^{-j(\frac{d-1-\beta}{p})}\|f\|_p, & \text{if } 2\le p\le\infty,\end{cases} \tag{2.11}$$

$$\|\sup_{t\in E}|\mathcal{A}^j_t f|\|_q\le c[\chi^E_{\mathrm{M},\beta}(2^{-j})]^{\frac1q}2^{-j(1-\frac{d-\beta+1}{q})}\|f\|_{q'},\quad 2\le q\le\infty. \tag{2.12}$$

2.4. *Results near $Q_{4,\gamma}$.* This is the heart of the matter and here the Assouad characteristic enters. The cases $d\ge3$ and $d=2$, $\gamma\le\frac12$ were already handled in [1, §3]. The analysis there is based on TT^* arguments. Rewritten using (2.10), it gives

$$\|\sup_{t\in E}|\mathcal{A}^j_t f|\|_{L^{q_\gamma,\infty}}\le c[\chi^E_{\mathrm{A},\gamma}(2^{-j})]^{\frac{1}{q_\gamma}}2^{-j\frac{(d-1)^2-2\gamma}{2(d-1+2\gamma)}}\|f\|_2,\quad q_\gamma=\frac{2(d-1+2\gamma)}{d-1}. \tag{2.13}$$

In the cases $d\ge3$, and $d=2$, $\gamma<\frac12$ we have $\frac{(d-1)^2-2\gamma}{2(d-1+2\gamma)}>0$ and by interpolation of (2.13) with (2.6) one obtains

$$\|\sup_{t\in E}|\mathcal{A}^j_t f|\|_{q_4}\le c\,[\chi^E_{\mathrm{A},\gamma}(2^{-j})]^{1/q_4}\|f\|_{p_4},\quad d\ge3 \text{ or } d=2,\ \gamma<\tfrac12, \tag{2.14}$$

where c is a positive constants only depending on d and

$$Q_{4,\gamma}=(\tfrac{1}{p_4},\tfrac{1}{q_4})$$

as in (1.1). The remaining case is one of our main results in this paper.

Theorem 2.1. *Let $d=2$ and $\gamma\ge1/2$. Then we have for every $j\ge0$,*

$$\Big\|\sup_{t\in E}|\mathcal{A}^j_t f|\Big\|_{q_4}\le c\,\min\big(\tfrac{j^{1/2}}{2\gamma-1},j\big)^{1/p_4}\big[\chi^E_{\mathrm{A},\gamma}(2^{-j})\big]^{1/q_4}\|f\|_{p_4}, \tag{2.15}$$

where $c>0$ is an absolute constant.

The proof of this theorem is contained in §3 and §4. Interpolation arguments yield the following consequence, that implies Theorem 1.1.

Corollary 2.2. *Let $d \geq 2$, $\beta = \dim_{\mathrm{M}} E$, $\gamma = \dim_{\mathrm{qA}} E$. Then for every $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{Q}(\beta,\gamma)$ there exists a nonnegative $\varepsilon = \varepsilon(\frac{1}{p}, \frac{1}{q}, \beta, \gamma, d)$ depending continuously on $(1/p, 1/q)$ such that $\varepsilon > 0$ for $(\frac{1}{p}, \frac{1}{q})$ in the interior of $\mathcal{Q}(\beta,\gamma)$ and on the open line segment between Q_1 and $Q_{2,\beta}$, and*

$$\| \sup_{t\in E} |\mathcal{A}_t^j f| \|_q \leq c\, [\chi^E_{\mathrm{A},\gamma}(2^{-j})]^{b_1} [\chi^E_{\mathrm{M},\beta}(2^{-j})]^{b_2} (1+j)^{b_3} 2^{-\varepsilon j} \|f\|_p. \tag{2.16}$$

Here $c > 0$ depends only on d, and the nonnegative constants b_1, b_2, b_3 satisfy $b_1 + b_2 = \frac{1}{q}$ and $b_3 \leq \frac{2}{q}$.

Proof. This follows by interpolation arguments using several extreme cases stated above (specifically, using (2.11), (2.12), (2.14), (2.15)). For the $L^\infty \to L^\infty$ estimate (corresponding to the pair $(0,0) = Q_1$) we have (2.16) with $b_1 = b_2 = b_3 = 0$ and $\varepsilon = 0$. For the pair $(p_2^{-1}, q_2^{-1}) = Q_2(\beta)$ (here $p_2 = q_2$) we have (2.16) with $b_1 = 0$, $b_2 = 1/q_2$, $b_3 = 0$ and $\varepsilon = 0$. For the pair $(p_3^{-1}, q_3^{-1}) = Q_3(\beta)$ we have (2.16) with $b_1 = 0$, $b_2 = 1/q_3$, $b_3 = 0$ and $\varepsilon = 0$. Finally we consider $(p_4^{-1}, q_4^{-1}) = Q_4(\gamma)$. Now, in the case $d = 3$ or $d = 2$, $\gamma < 1/2$, we have (2.16) with $b_1 = 1/q_4$, $b_2 = 0$, $b_3 = 0$ and $\varepsilon = 0$, and in the case $d = 2$, $\gamma \geq 1/2$ we have (2.16) with $b_1 = 1/q_4$, $b_2 = 0$, $b_3 = 2/q_4$ and $\varepsilon = 0$. We interpolate these estimates with the L^2 bound in (2.11), corresponding to $b_1 = 0$, $b_2 = 1/2$, $b_3 = 0$, $\epsilon = (d-1-\beta)/2$, except in the case $\beta = 1$ and $d = 2$ when we use Bourgain's result [5] in the form of [18] for $p > 2$. □

Proof of Theorem 1.1 . We have $N(E \cap I, \delta) \leq N(I,\delta) \leq 2\delta^{-\varepsilon}$ if $\delta \leq |I| \leq \delta^{1-\varepsilon}$ and $0 < \delta \leq 1$. Using the assumption on $\dim_{\mathrm{qA}} E$ we see that for any $\varepsilon > 0$ and any $0 < \varepsilon_1 < 1$ there are constants $C(\varepsilon, \varepsilon_1) < \infty$ such that for all intervals I with $\delta \leq |I| \leq 1$

$$N(E\cap I, \delta) \leq \begin{cases} C(\varepsilon,\varepsilon_1)(\delta/|I|)^{-\gamma-\varepsilon_1} & \text{if } \delta^{1-\varepsilon} \leq |I| \leq 1 \\ 2\delta^{-\varepsilon} & \text{if } \delta \leq |I| \leq \delta^{1-\varepsilon}. \end{cases}$$

Thus for the γ-Assouad characteristic (*cf.* §2.3) we get the estimate

$$\chi^E_{\mathrm{A},\gamma}(\delta) \leq 2\delta^{-\varepsilon} + C(\varepsilon,\varepsilon_1)\delta^{-\varepsilon_1}, \quad 0 < \delta \leq 1,$$

and we can conclude by applying Corollary 2.2. □

Remark. The interpolation argument above can be used to compute the exact exponent ε in Corollary 2.2 in terms of $\frac{1}{p}, \frac{1}{q}, \beta, \gamma, d$, but the exact dependence will not matter for us. Moreover, the estimate (2.15) is somewhat stronger than required: to prove the results stated in the introduction, it would suffice to show that for every $\epsilon_1 > 0$,

$$\big\| \sup_{t\in E} |\mathcal{A}_t^j f| \big\|_{q_4} \lesssim_{\epsilon_1} 2^{j\epsilon_1} \big[\chi^E_{\mathrm{A},\gamma}(2^{-j})\big]^{1/q_4} \|f\|_{p_4}.$$

This weaker result together with interpolation arguments as above already implies Theorem 1.1. The reason for stating (2.16) with the indicated degree of precision regarding dependence of the constant on the various parameters will become apparent in the proof of Theorem 1.2, see (7.7) below.

2.5. *Endpoint results and problems.* For the proof of Theorems 1.1 and 1.2 we do not have to consider endpoint questions. Nevertheless such endpoint bounds under the assumptions of bounded β-Minkowski characteristic or bounded γ-Assouad characteristic are very interesting, and some challenging problems are open.

We first consider the case $p = q$. Under the assumption $0 < \beta < 1$ it was noted in [22, Prop. 1.4] that for the pair $Q_{2,\beta}$ the operator M_E is of restricted weak type (q_2, q_2), under the assumption of bounded β-Minkowski characteristic. This is proved by a version of Bourgain's interpolation argument in [4]. It is conjectured (and suggested by the behavior on radial functions [24]) that the restricted weak type estimate can be upgraded to a strong type (q_2, q_2) estimate; however this is known only for special types of sets E such as convex sequences [22], and is open for example for certain Cantor sets.

Now consider the case $p < q$. For the full spherical maximal operator Lee [15] proved a restricted weak type endpoint result for the exponent pairs $Q_{3,1}$ and $Q_{4,1}$, using the above mentioned Bourgain interpolation trick. This yields $L^p \to L^q$ estimates on the open edges $(Q_1, Q_{4,1})$ and $(Q_{3,1}, Q_{4,1})$, moreover $L^{p,1} \to L^q$ bounds on the vertical half-open edge $(Q_{3,1}, Q_{2,1}]$. The endpoint result for $Q_{4,1}$ is especially deep in two dimensions as it relies on Tao's difficult endpoint version [27] of Wolff's bilinear adjoint restriction theorem for the cone [29].

The restricted weak type inequality at $Q_{3,\beta}$, under the assumption of bounded β-Minkowski characteristic, was proved in [1]. Under the assumption of bounded γ-Assouad characteristic, if $d \geq 3$ or $d = 2$, $\gamma < 1/2$ the restricted weak type estimates for $Q_{4,\gamma}$ was also proved in [1]. We remark that for these known restricted weak type endpoint estimates at $Q_{3,\beta}$ and $Q_{4,\gamma}$ it is open whether they can be upgraded to strong type estimates.

Endpoint bounds at $Q_{4,\gamma}$ with bounded Assouad characteristic are open in two dimensions when $1/2 \leq \gamma < 1$. We conjecture that the term $(1+j)^{b_3}$ in Corollary 2.2 can be dropped; moreover that a restricted weak type estimate holds at $Q_{4,\gamma}$.

3. Proof of Theorem 2.1: Fractional integration

In this section we prove Theorem 2.1. For $j = 0$ there is nothing to prove, so we assume $j \geq 1$ from here on. For $a \in \mathfrak{S}^0, t \in [1, 2]$ define

$$T_t^{j,\pm}[a, f](x) = \int_{\mathbb{R}^2} e^{i\langle x,\xi\rangle \pm it|\xi|} \chi_j(\xi) a(t\xi) \widehat{f}(\xi) d\xi \quad (x \in \mathbb{R}^2). \tag{3.1}$$

From (2.1) and (2.3) we see that there exist symbols $a_\pm$ with $\|a_\pm\|_{\mathfrak{S}^0} \leq C(d)$ such that

$$\mathcal{A}_t^j f = 2^{-j/2} \sum_{\pm} T_t^{j,\pm}[a_\pm, f].$$

In the following let us assume without loss of generality that

$$\|a\|_{\mathfrak{S}^0} \le 1 \quad \text{and} \quad \operatorname{supp} a \subset \{\xi : 0 < \xi_1 < 2^{-10}\xi_2\}. \tag{3.2}$$

A first observation is that the effect from oscillation of the factor $e^{\pm it|\xi|}$ in (3.1) is negligible if t varies within an interval of length $\ll 2^{-j}$. This suggests the following standard argument. Define

$$I_{n,j} = [n2^{-j}, (n+1)2^{-j}] \quad (n \in \mathbb{Z}),$$

$$\mathcal{E}_j = \{n2^{-j} : I_{n,j} \cap E \neq \emptyset,\, n \in \mathbb{Z}\} \subset [1,2]. \tag{3.3}$$

We estimate pointwise for each $x \in \mathbb{R}^2$,

$$\sup_{t\in E} |T_t^{j,\pm}[a,f]|(x) \le \Big(\sum_{n\in 2^j\mathcal{E}_j} [\sup_{t\in I_{n,j}} |T_t^{j,\pm}[a,f]|(x)]^{q_4}\Big)^{1/q_4}.$$

For every $n \in 2^j\mathcal{E}_j$ and $t \in I_{n,j}$ we use the fundamental theorem of calculus to estimate

$$|T_t^{j,\pm}[a,f](x)| \le |T_{n2^{-j}}^{j,\pm}[a,f](x)| + \int_0^{2^{-j}} |T_{n2^{-j}+s}^{j,\pm}[\widetilde{a},f](x)|ds,$$

where $\widetilde{a} \in \mathfrak{S}^1$, more precisely $\widetilde{a}(\xi) = \pm i|\xi|a(\xi) + \langle \xi, \nabla a(\xi)\rangle$, and we have used that $t \ge 1$. From the previous two displays,

$$\|\sup_{t\in E} |T_t^{j,\pm}[a,f]|\|_q \le \Big(\sum_{t\in\mathcal{E}_j} \|T_t^{j,\pm}[a,f]\|_q^q\Big)^{\frac{1}{q}} + \int_0^{2^{-j}} \Big(\sum_{t\in\mathcal{E}_j+s} \|T_t^{j,\pm}[\widetilde{a},f]\|_q^q\Big)^{\frac{1}{q}} ds, \tag{3.4}$$

where $q = q_4 = 3 + 2\gamma \ge 4$. The first term on the right hand side and the integrand of the second term will be treated in the same way. Bilinearizing, we write

$$\Big(\sum_{t\in\mathcal{E}} \|T_t^{j,\pm}[a,f]\|_q^q\Big)^{1/q} = \|\mathcal{T}_j(f\otimes f)\|_{L^{q/2}(\mathbb{R}^2\times\mathcal{E})}^{1/2}, \tag{3.5}$$

where $\mathcal{E} \subset [1,2]$ is a finite set, $\mathbb{R}^2 \times \mathcal{E}$ is equipped with the product of the Lebesgue measure and the counting measure, $(f\otimes f)(x,y) = f(x)f(y)$ and

$$\mathcal{T}_j F(x,t) = \int_{\mathbb{R}^4} e^{i\langle x,\xi+\zeta\rangle \pm it(|\xi|+|\zeta|)} \chi_j(\xi)a(t\xi)\chi_j(\zeta)a(t\zeta)\widehat{F}(\xi,\zeta)\, d(\xi,\zeta).$$

Definition. Let $\delta \in (0,1)$. A finite set $\mathcal{E} \subset (0,\infty)$ will be called *uniformly δ–separated* if

$$\mathcal{E} - \mathcal{E} \subset \delta\mathbb{Z}. \tag{3.6}$$

In other words, $\mathcal{E}$ is a subset of an arithmetic progression with common difference of δ.

The sets $\mathcal{E}_j$ and $\mathcal{E}_j + s$ appearing in (3.4) are uniformly 2^{-j}–separated. We have a crucial L^2 estimate with the following weight that blows up near the diagonal:

$$w_\gamma(y,z) = |y-z|^{-(2\gamma-1)} \quad \text{for } y,z \in \mathbb{R}^2,\ y \neq z. \tag{3.7}$$

Proposition 3.1. *Let $\gamma \geq \frac{1}{2}$ and let $a \in \mathfrak{S}^0$ satisfy (3.2). Assume that $j \geq 1$ and that $\mathcal{E} \subset [1,2]$ is uniformly 2^{-j}–separated. Then*

$$\|\mathcal{T}_j F\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \leq c \min\big(\tfrac{j^{1/2}}{2\gamma-1}, j\big)\, 2^j [\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12} \|F\|_{L^2(w_\gamma)}\,, \tag{3.8}$$

where c is an absolute constant.

The proof of this estimate forms the heart of the matter and is contained in Section 4.

Proof of Theorem 2.1 given Proposition 3.1. We assume that $\|a\|_{\mathfrak{S}^0} \leq 1$. Since $q_4 = 2p_4 = 3+2\gamma$, we need to show, by (3.4) and (3.5), that

$$\|\mathcal{T}_j(f\otimes f)\|_{L^{p_4}(\mathbb{R}^2\times\mathcal{E})} \leq c \min\big(\tfrac{j^{1/2}}{2\gamma-1}, j\big)^{2/p_4} 2^j [\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{1/p_4} \|f\|^2_{p_4}\,. \tag{3.9}$$

The case $\gamma = 1/2$ follows immediately from Proposition 3.1. We assume $1/2 < \gamma \leq 1$ and argue as in the paper by Schlag and Sogge [21]. The sectorial localization in (3.2) allows us to estimate

$$\sup_{(x,t)\in\mathbb{R}^2\times[1,2]} |\mathcal{T}_j F(x,t)| \lesssim 2^j \int_{\mathbb{R}^2} \sup_{(y_2,z_2)\in\mathbb{R}^2} |F(y,z)| d(y_1,z_1). \tag{3.10}$$

Indeed, we have the pointwise bound, for $1 \leq t \leq 2$,

$$|\mathcal{T}_j F(x,t)| \lesssim \int_{\mathbb{R}^4} K_{j,t}(x-y,x-z)|F(y,z)|\, d(y,z) \tag{3.11}$$

$$\leq \Big(\sup_{(y_1,z_1)\in\mathbb{R}^2} \int_{\mathbb{R}^2} |K_{j,t}(y,z)| d(y_2,z_2)\Big) \cdot \Big(\int_{\mathbb{R}^2} \sup_{(y_2,z_2)\in\mathbb{R}^2} |F(y,z)| d(y_1,z_1)\Big)$$

where, with $h(s) = \sqrt{1-s^2}$, $K_{j,t}(y,z)$ is a linear combination of five terms

$$\frac{2^j}{(1+t^{-1}(|y|+|z|))^{10}}, \quad 2^j \mathbb{1}_{[-1/2,1/2]^2}(y_1,z_1)\times$$
$$\frac{2^j}{(1+2^j|t^{-1}y_2 \pm h(t^{-1}y_1)|)^{10}} \frac{2^j}{(1+2^j|t^{-1}z_2 \pm h(t^{-1}z_1)|)^{10}}.$$

In (3.11) we integrate in (y_2,z_2) first to obtain (3.10). In (3.8) we may replace $|y-z|^{-\frac{2\gamma-1}{2}}$ with $|y_1-z_1|^{-\frac{2\gamma-1}{2}}$ and then analytically interpolate the resulting inequality with (3.10). This yields for $2 \leq p < \infty$,

$$\|\mathcal{T}_j F\|_{L^p(\mathbb{R}^2\times\mathcal{E}_j)} \lesssim 2^j \min\big(\tfrac{j^{1/2}}{2\gamma-1}, j\big)^{2/p} [\chi^{E}_{\mathrm{A},\gamma}(2^{-j})]^{1/p} \times \tag{3.12}$$
$$\Big(\int_{\mathbb{R}^2} \Big[|y_1-z_1|^{(1-2\gamma)/p} \Big(\int_{\mathbb{R}^2} |F(y,z)|^p d(y_2,z_2)\Big)^{1/p}\Big]^{p'} d(y_1,z_1)\Big)^{1/p'}.$$

We specialize to $F(y,z) = f(y)g(z)$ and apply Hölder's inequality in y_1 with exponents $p/p' \in [1,\infty]$ and $(p/p')'$ to obtain

$$\Big(\int_{\mathbb{R}^2}\Big[|y_1-z_1|^{(1-2\gamma)/p}\Big(\int_{\mathbb{R}^2}|f(y)g(z)|^p d(y_2,z_2)\Big)^{1/p}\Big]^{p'} d(y_1,z_1)\Big)^{1/p'}$$

$$= \Big(\int_{\mathbb{R}^2}\big[|y_1-z_1|^{\frac{1-2\gamma}{p}}\|f(y_1,\cdot)\|_p\|g(z_1,\cdot)\|_p\big]^{p'} d(y_1,z_1)\Big)^{1/p'}$$

$$\le \|f\|_p\Big(\int_{\mathbb{R}}\Big[\int_{\mathbb{R}}|y_1-z_1|^{-(2\gamma-1)\frac{p'}{p}}\|g(z_1,\cdot)\|_p^{p'}dz_1\Big]^{(\frac{p}{p'})'}dy_1\Big)^{\frac{1}{(p/p')'}\frac{1}{p'}}. \tag{3.13}$$

The standard fractional integral theorem says that for $0 < \mathrm{Re}(a) \le 1$ the convolution operator with Schwartz kernel $|s-t|^{a-1}$ maps $L^{\mathfrak{p}}(\mathbb{R})$ to $L^{\mathfrak{q}}(\mathbb{R})$ for $\mathfrak{p}^{-1} - \mathfrak{q}^{-1} = \mathrm{Re}(a)$, and using analytic interpolation with the trivial $L^1 \to L^\infty$ estimate when $\mathrm{Re}(a) = 1$ one notes that the operator norm is bounded as $\mathrm{Re}(a) \to 1$. For $p > 2$ and $1/2 < \gamma \le 1$ the expression $1-(2\gamma-1)p'/p$ belongs to $(0, 1-p'/p)$. Thus, if $p > 2$ and $1 < r < (p/p')'$ is defined by

$$\tfrac{1}{r} - \big(1-\tfrac{p'}{p}\big) = 1-(2\gamma-1)\tfrac{p'}{p} \ =: a \tag{3.14}$$

then we see that (3.13) is bounded by

$$C\|f\|_p\Big(\int_{\mathbb{R}}\|g(z_1,\cdot)\|_p^{p'r}dz_1\Big)^{\frac{1}{rp'}}.$$

where the constant is independent of $\gamma \in (1/2, 1]$. For the special case $r = p/p'$ the relation (3.14) gives $p = (3+2\gamma)/2 = p_4$ (which is > 2 since $\gamma > 1/2$), and we get (3.9) by setting $f = g$. □

4. Proof of Proposition 3.1: An L^2 estimate

In this section we prove the crucial L^2 estimate, Proposition 3.1. As in Schlag–Sogge [21], the key to this estimate will be a second dyadic decomposition in the angle $\measuredangle(\xi,\eta) \in [0,\pi]$ between certain frequency variables ξ and η. For the estimates in [21] the authors relied on space time estimates due to Klainerman and Machedon [13] which are not applicable in our setting. Instead we have to establish an orthogonality property between contributions from different values of t which also depends on the fractal geometry of E.

We find it convenient to introduce the notation

$$S[F,b](\xi,t) = \int_{\mathbb{R}^2} e^{\pm it(|\xi-\eta|+|\eta|)} b(t,\xi,\eta)\widehat{F}(\xi-\eta,\eta)d\eta, \tag{4.1}$$

acting on a function $F : \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{C}$ and a symbol $b : (0,\infty)\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{C}$. For positive integers j, m we set $a_j(t,\xi) = a(t\xi)\chi_j(\xi)$ and

$$b_j(t,\xi,\eta) = a_j(t,\xi-\eta)a_j(t,\eta),$$

$$b_{j,m}(t,\xi,\eta) = b_j(t,\xi,\eta)\chi(2^{m-2j}\det(\xi,\eta)). \tag{4.2}$$

Define the convex annular sector

$$\Theta_j = \{\omega \in \mathbb{R}^2 : 0 < \omega_1 < 2^{-10}\omega_2,\ 2^{j-1} \le |\omega| \le 2^{j+1}\}.$$

If $b_j(t,\xi,\eta) \neq 0$, then both η and $\xi-\eta$ lie in Θ_j (see (3.2)). By convexity, we also have $\frac12\xi = \frac12\eta + \frac12(\xi-\eta) \in \Theta_j$. Observe that the cutoff in (4.2) effects an angular localization, since

$$|\det(\xi,\eta)| = |\xi|\cdot|\eta|\cdot \sin(\measuredangle(\xi,\eta)). \tag{4.3}$$

Note that

$$\mathcal{T}_j F(x,t) = \int_{\mathbb{R}^2} S[F,b_j](\xi,t) e^{i\langle \xi,x\rangle} d\xi.$$

Hence by Plancherel's theorem, Proposition 3.1 follows if we can show

$$\|S[F,b_j]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le c \min\big(\tfrac{j^{1/2}}{2\gamma-1}, j\big)\big[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})\big]^{\frac12} 2^j \|F\|_{L^2(w_\gamma)}. \tag{4.4}$$

We will require three different estimates for the objects $S[F,b]$ to prove this estimate. To clarify various dependencies we introduce the following terminology.

Definition. Let $j,m \ge 1$. We say that the symbol b is (j,m)-*adapted* if b is smooth and $b(t,\xi,\eta)=0$ unless

$$\eta, \xi-\eta \in \Theta_j \quad \text{and} \quad \measuredangle(\xi,\eta) \le 2^{-m+5}. \tag{4.5}$$

We call b *strictly* (j,m)-*adapted* if in addition $b(t,\xi,\eta)=0$ unless

$$\measuredangle(\xi,\eta) \ge 2^{-m-5}.$$

Observe that $b_{j,m}$ is strictly (j,m)-adapted.

Proposition 4.1 (Trivial estimate). *If b is (j,m)-adapted, then for every finite $\mathcal{E} \subset [1,2]$,*

$$\|S[F,b]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le c\,\|b\|_\infty 2^{j-\frac{m}{2}} (\#\mathcal{E})^{\frac12} \|F\|_{L^2(\mathbb{R}^4)}. \tag{4.6}$$

Proof. This follows by an application of the Cauchy–Schwarz inequality to the integration over η in (4.1). □

The next ingredient is the following crucial improvement of (4.6).

Proposition 4.2 (Almost orthogonality). *Suppose that b is strictly (j,m)-adapted and satisfies the differential inequality*

$$|\langle \tfrac{\eta}{|\eta|}, \nabla\rangle^N b(t,\xi,\eta)| \le B\, 2^{-jN} \text{ for } N=0,1,2. \tag{4.7}$$

Suppse $\mathcal{E}$ is uniformly 2^{-j}–separated. Then

$$\|S[F,b]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le c\,B\,2^{j-\frac{m}{2}} \Big(\sup_{|I|=2^{-j+2m}} \#(\mathcal{E}\cap I)\Big)^{\frac12} \|F\|_{L^2(\mathbb{R}^4)}. \tag{4.8}$$

The proof of this proposition is contained in §4.2. It relies on an observation of almost orthogonality of $S[F,b](\cdot,t)$ and $S[F,b](\cdot,t')$ for sufficiently separated t,t'. Finally we will need a certain tail estimate.

Proposition 4.3 (Off–diagonal decay)**.** *Let $0 \le m \le j/2$. Suppose that b is (j,m)-adapted and satisfies the differential inequality*

$$|\partial_\eta^\alpha b(t,\xi,\eta)| \le B\, 2^{-(j-m)|\alpha|} \text{ for } |\alpha| \le 9. \tag{4.9}$$

Assume that F is supported on the set

$$\{(y,z) \,:\, |y-z| \ge 2^{-m+\ell+20}\}$$

for some $\ell \ge 0$. Then for every finite $\mathcal{E} \subset [1,2]$,

$$\|S[F,b]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le c\, B\, 2^{\frac{3}{2}m-\ell} (\#\mathcal{E})^{\frac{1}{2}} \|F\|_{L^2(\mathbb{R}^4)}. \tag{4.10}$$

The proof of this proposition is contained in §4.4. It may be helpful to recognize (4.10) as the special case $N = 1$ of the stronger estimate

$$\|S[F,b]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le c_N\, B\, 2^{j-\frac{m}{2}} 2^{-(j-2m+\ell)N} (\#\mathcal{E})^{\frac{1}{2}} \|F\|_{L^2(\mathbb{R}^4)},$$

which also holds (as long as (4.9) holds for large enough $|\alpha|$), but will not be needed for our purpose.

Note that (4.9) features derivatives taken in arbitrary directions, whereas the derivatives in (4.7) are taken in the radial direction only. The difference between these two estimates reflects the fact that for fixed ξ, the η–support of b is contained in a rectangle of dimensions, say $2^{j+10} \times 2^{j-m+10}$, with its long side aligned radially.

4.1. *Proof of Proposition 3.1 given Propositions 4.2 and 4.3.* It suffices to show (4.4). We first show the uniform estimate

$$\|S[F,b_j]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le c\, j 2^j [\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac{1}{2}} \|F\|_{L^2(w_\gamma)}. \tag{4.11}$$

Observe that since $\mathcal{E}$ is uniformly 2^{-j}–separated,

$$\#\mathcal{E} \lesssim 2^{j\gamma} \chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j}) \quad \text{and} \quad \sup_{|I|=2^{-j+2m}} \#(\mathcal{E}\cap I) \lesssim 2^{2m\gamma} \chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j}). \tag{4.12}$$

Thus, the almost orthogonality estimate (4.8) only beats the trivial estimate (4.6) if $m < j/2$. This motivates the definition of the remainder term

$$R_j F = S\Big[F, b_j - \sum_{0<m<j/2} b_{j,m}\Big]$$

so that $S[F,b_j] = \sum_{0<m<j/2} S[F,b_{j,m}] + R_j F$. Observe that the symbol of R_j is $(j,j/2)$–adapted. Estimate

$$\|S[F,b_j]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le \sum_{0<m<j/2} \|S[F,b_{j,m}]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} + \|R_j F\|_{L^2(\mathbb{R}^2\times\mathcal{E})}. \tag{4.13}$$

We estimate both terms separately. To do this we write F as

$$F = \sum_{k\in\mathbb{Z}} F_k = \sum_{k\le -m} F_k + \sum_{\ell=1}^{\infty} F_{-m+\ell},$$

with each F_k supported on the set

$$\{(y,z) \in \mathbb{R}^2\times\mathbb{R}^2 \,:\, 2^k \le 2^{-20}|y-z| < 2^{k+1}\}.$$

Note that $b_{j,m}$ is strictly (j,m)-adapted and satisfies both, (4.7) and (4.9). Proposition 4.2 and (4.12) imply

$$\|S\big[\textstyle\sum_{k\le -m} F_k, b_{j,m}\big]\|_{L^2(\mathbb{R}^2\times\mathcal{E}_j)} \lesssim 2^j[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F\|_{L^2(w_\gamma)},$$

where we have used that $|y-z|^{-(1-2\gamma)/2} \lesssim 2^{-m(\gamma-\frac12)}$ on the support of $\sum_{k\le -m} F_k$. On the other hand, for $\ell\ge 1$, Proposition 4.3 and (4.12) yield

$$\|S[F_{-m+\ell}, b_{j,m}]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \lesssim 2^{-\ell+\frac32 m+\frac{\gamma}{2}j}[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F_{-m+\ell}\|_{L^2(\mathbb{R}^4)}.$$

Since $|y-z|^{-(1-2\gamma)/2}\approx 2^{(\ell-m)(\gamma-\frac12)}$ on the support of $F_{-m+\ell}$ the quantity on the right hand side is comparable to

$$\begin{aligned}2^{-\ell(\frac32-\gamma)+m(2-\gamma)+j\frac{\gamma}{2}}&[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F_{-m+\ell}\|_{L^2(w_\gamma)}\\ &\lesssim 2^{-\frac{\ell}{2}+j}[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F\|_{L^2(w_\gamma)},\end{aligned}$$

where we used that $m\le j/2$ and $\gamma\le 1$ in the last step. Together we obtain

$$\sum_{0<m<j/2}\|S[F,b_{j,m}]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \lesssim j2^j[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F\|_{L^2(w_\gamma)}.$$

The estimate for the second term in (4.13) is similar: Proposition 4.1 implies

$$\|R_j\big[\sum_{k\le -m} F_k\big]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \lesssim 2^j[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F\|_{L^2(w_\gamma)}.$$

On the other hand, applying Proposition 4.3 with $m=j/2$ gives

$$\|R_j[F_{\lceil -j/2\rceil+\ell}]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \lesssim 2^{-\frac{\ell}{2}+j}[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F\|_{L^2(w_\gamma)}$$

for all $\ell\ge 1$. The previous two displays combined give

$$\|R_jF\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \lesssim 2^j[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F\|_{L^2(w_\gamma)}, \tag{4.14}$$

as required.

In order to finish the proof we need to establish an improvement over (4.11) in the range $j>(2\gamma-1)^{-2}$, namely

$$\|S[F,b_j]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \le c\,(2\gamma-1)^{-1}j^{1/2}2^j[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F\|_{L^2(w_\gamma)}. \tag{4.15}$$

Estimate

$$\begin{aligned}&\|S[F,b_j]\|_{L^2(\mathbb{R}^2\times\mathcal{E})}\\ &\qquad\le j^{\frac12}\Big(\sum_{0<m<j/2}\|S[F,b_{j,m}]\|^2_{L^2(\mathbb{R}^2\times\mathcal{E})}\Big)^{1/2}+\|R_jF\|_{L^2(\mathbb{R}^2\times\mathcal{E})}.\end{aligned} \tag{4.16}$$

The second term has already been estimated in (4.14). To treat the first term we first observe that by Proposition 4.3 and (4.12)

$$\|S[F_{-m+\ell}, b_{j,m}]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \lesssim 2^{-\frac{\ell}{2}+j}[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F_{-m+\ell}\|_{L^2(w_\gamma)},\quad \ell\ge 1.$$

On the other hand, by Proposition 4.2 and (4.12) we have

$$\|S[F_{-m+\ell}, b_{j,m}]\|_{L^2(\mathbb{R}^2\times\mathcal{E})} \lesssim 2^{-(\gamma-\frac12)(-\ell)+j}[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\|F_{-m+\ell}\|_{L^2(w_\gamma)}, \ell\le 0.$$

From these estimates we get

$$\Big(\sum_{0<m<j/2}\|S[F,b_{j,m}]\|^2_{L^2(\mathbb{R}^2\times\mathcal{E})}\Big)^{1/2}$$
$$\le\sum_{\ell=-\infty}^{\infty}\Big(\sum_{0<m<j/2}\|S[F_{-m+\ell},b_{j,m}]\|^2_{L^2(\mathbb{R}^2\times\mathcal{E})}\Big)^{1/2}$$
$$\lesssim 2^j[\chi^{\mathcal{E}}_{\mathrm{A},\gamma}(2^{-j})]^{\frac12}\sum_{\ell=-\infty}^{\infty}\min(2^{-\ell/2},2^{\ell(\gamma-\frac12)})\Big(\sum_{0<m<j/2}\|F_{-m+\ell}\|^2_{L^2(w_\gamma)}\Big)^{1/2}.$$

By the disjointness of supports of the F_k, we have for each $\ell\in\mathbb{Z}$

$$\Big(\sum_{0<m<j/2}\|F_{-m+\ell}\|^2_{L^2(w_\gamma)}\Big)^{1/2}\le\|F\|_{L^2(w_\gamma)}$$

and since $\sum_{\ell=-\infty}^{\infty}\min(2^{-\ell/2},2^{\ell(\gamma-\frac12)})\lesssim(2\gamma-1)^{-1}$ for $1/2<\gamma\le1$ we obtain (4.15). □

4.2. *Proof of Proposition 4.2.* We begin by observing that we may assume without loss of generality that $\mathcal{E}$ is uniformly 2^{-j+2m}–separated (in the sense of (3.6)). This is because every uniformly 2^{-j}–separated set $\mathcal{E}\subset(0,\infty)$ can be written as a disjoint union of at most

$$2\sup_{|I|=2^{-j+2m}}\#(\mathcal{E}\cap I)$$

sets each of which is uniformly 2^{-j+2m}–separated.

By duality, $\|S[F,b]\|_{L^2(\mathbb{R}^2\times\mathcal{E})}$ is equal to the supremum over all G with $\|G\|_{L^2(\mathbb{R}^2\times\mathcal{E})}=1$ of

$$\Big|\int_{\mathbb{R}^4}\widehat{F}(\xi-\eta,\eta)\Big[\sum_{t\in\mathcal{E}}G(\xi,t)e^{\pm it(|\xi-\eta|+|\eta|)}b(t,\xi,\eta)\Big]\,d(\xi,\eta)\Big|.$$

By the Cauchy-Schwarz inequality applied in (ξ,η) we estimate the previous by

$$\|F\|_{L^2(\mathbb{R}^4)}\Big(\int_{\mathbb{R}^4}\Big|\sum_{t\in\mathcal{E}}G(\xi,t)e^{\pm it(|\xi-\eta|+|\eta|)}b(t,\xi,\eta)\Big|^2d(\eta,\xi)\Big)^{1/2}\tag{4.17}$$

For each fixed ξ consider

$$\int_{\mathbb{R}^2}\Big|\sum_{t\in\mathcal{E}}G(\xi,t)e^{it(|\xi-\eta|+|\eta|)}b(t,\xi,\eta)\Big|^2d\eta.$$

Passing to polar coordinates $\eta=\rho\theta$ with $\rho\ge0$ and $\theta\in S^1\subset\mathbb{R}^2$ and fixing the angular variable θ we are left with the one-dimensional integral

$$\int_0^\infty\Big|\sum_{t\in\mathcal{E}}G(\xi,t)e^{\pm it(|\xi-\rho\theta|+\rho)}b(t,\xi,\rho\theta)\Big|^2\rho d\rho.$$

Expanding the square we rewrite this integral as

$$\sum_{t,t'\in\mathcal{E}} G(\xi,t)\overline{G(\xi,t')}\Big[\int_0^\infty e^{\pm i(t-t')\tau(\rho)}\nu_{\xi,\theta}(t,t',\rho)d\rho\Big], \tag{4.18}$$

where we have set

$$\tau(\rho)=\tau_{\xi,\theta}(\rho)=|\xi-\rho\theta|+\rho,$$
$$\nu_{\xi,\theta}(t,t',\rho)=b(t,\xi,\rho\theta)\overline{b(t',\xi,\rho\theta)}\rho.$$

Keep in mind that by the assumptions on b, on the support of $\nu_{\xi,\theta}$ we have $\rho\approx 2^j$, $|\xi-\rho\theta|\approx 2^j$ and the angle of θ with $\xi-\rho\theta$ is $\approx 2^{-m}$. Observe that τ is strictly monotone increasing with

$$\tau'(\rho)=1-\big\langle\theta,\frac{\xi-\rho\theta}{|\xi-\rho\theta|}\big\rangle=1-\cos(\measuredangle(\theta,\xi-\rho\theta))\approx 2^{-2m}. \tag{4.19}$$

Similarly, we will show that

$$|\tau^{(N)}(\rho)|\lesssim_N 2^{-2m-(N-1)j} \tag{4.20}$$

for every integer $N\ge 1$. In order to establish (4.20) we verify that for each $N\ge 1$ there are coefficients $(a_{k,N})_{k=0,\dots,N}$ with $\sum_{k=0}^N a_{k,N}=0$ such that

$$\tau^{(N)}(\rho)=|v|^{-(N-1)}\sum_{k=0}^N a_{k,N}w^k, \tag{4.21}$$

where $v=\xi-\rho\theta$ and $w=\frac{\langle v,\theta\rangle}{|v|}=\cos(\measuredangle(\theta,\xi-\rho\theta))$. This claim implies (4.20) (note that the polynomial $\sum_{k=0}^N a_{k,N}w^k$ is divisible by $1-w$). To prove the claim we use induction on N, with $a_{0,1}=1$, $a_{1,1}=-1$ by (4.19).

Calculate $\frac{d}{d\rho}|v|^{1-N}=(N-1)w|v|^{-N}$ and $\frac{d}{d\rho}w^k=|v|^{-1}kw^{k-1}(w^2-1)$. Hence

$$|v|^N\frac{d}{d\rho}\Big(|v|^{1-N}\sum_{k=0}^N a_{k,N}w^k\Big)=\sum_{k=0}^N(N-1+k)a_{k,N}w^{k+1}-\sum_{k=1}^N ka_{k,N}w^{k-1}$$

which is written as $\sum_{k=0}^{N+1}a_{k,N+1}w^k$, which is a polynomial of degree $N+1$, and one checks using the induction hypothesis that the sum of its coefficients are zero. Hence (4.20) is verified.

As a consequence of (4.20),

$$|\partial_\rho^N[\tfrac{1}{\tau'}](\rho)|\lesssim_N 2^{2m-Nj}.$$

Moreover, since b is strictly (j,m)-adapted and satisfies (4.7),

$$|\partial_\rho^N\nu_{\xi,\theta}(t,t',\rho)|\lesssim_N B^2\,2^{j-Nj}\quad\text{for } N\le 2.$$

Integrating by parts twice then shows

$$\Big|\int_0^\infty e^{\pm i(t-t')\tau(\rho)}\nu_{\xi,\theta}(t,t',\rho)d\rho\Big|\lesssim B^2\,2^{2j}(1+2^{-2m+j}|t-t'|)^{-2}.$$

From this we may estimate (4.18) by

$$B^2\, 2^{2j} \sum_{t,t'\in\mathcal{E}} (1+2^{-2m+j}|t-t'|)^{-2}|G(\xi,t)G(\xi,t')|.$$

An application of the Cauchy–Schwarz inequality shows that the previous is dominated by

$$B^2 2^{2j}\Big(\sum_{s\in\mathcal{E}-\mathcal{E}}(1+2^{-2m+j}|s|)^{-2}\Big)\Big(\sum_{t\in\mathcal{E}}|G(\xi,t)|^2\Big).$$

Since $\mathcal{E}$ is uniformly 2^{-j+2m}–separated (see (3.6)), the previous display is

$$\lesssim B^2\, 2^{2j}\sum_{t\in\mathcal{E}}|G(\xi,t)|^2.$$

Hence we have proved that

$$\Big(\int_{\mathbb{R}^4}\Big|\sum_{t\in\mathcal{E}} G(\xi,t)e^{\pm it(|\xi-\eta|+|\eta|)}b(t,\xi,\eta)\Big|^2 d(\eta,\xi)\Big)^{1/2}\lesssim B\, 2^{j-\frac{m}{2}},$$

recalling that $\|G\|_{L^2(\mathbb{R}^2\times\mathcal{E})}=1$. Note that the factor $2^{-\frac{m}{2}}$ stems from integration over the angular variable θ. In view of (4.17) this concludes the proof of Proposition 4.2. □

4.3. *Interlude: Exotic symbols.* In the proof of Proposition 4.3 we use estimates for oscillatory integrals which are equivalent to the L^2 results for pseudo-differential operators of symbol class $S^0_{\rho,\rho}$ considered by Calderón and Vaillancourt [6]; here we take $\rho=1/2$. Consider symbols $a:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{C}$ such that there exists a constant $A>0$ with

$$|\partial_x^\beta\partial_\xi^\alpha a(x,\xi)|\le A(1+|\xi|)^{-(|\alpha|-|\beta|)/2}$$

for multiindices α,β with, say, $|\alpha|\le 4d+1, |\beta|\le 4d+1$. We define the associated operator

$$P_af(x)=\int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle}a(x,\xi)f(\xi)d\xi, \tag{4.22}$$

which is *a priori* defined at least on integrable functions. Then one has the estimate

$$\|P_af\|_{L^2(\mathbb{R}^d)}\le cA\|f\|_{L^2(\mathbb{R}^d)},$$

where c is a constant only depending on the dimension d. The proof (an application of the Cotlar–Stein almost orthogonality lemma) is due to [6], for an exposition see also Stein [26, Ch. VII, §2.5].

4.4. *Proof of Proposition 4.3.* Fix $t\in\mathcal{E}$. From the definition (4.1),

$$S[F,b](\xi,t)=\int_{\mathbb{R}^4} e^{-i\langle\xi,y\rangle}F(y,z)\Big[\int_{\mathbb{R}^2} e^{i\Phi_{t,\xi,y-z}(\eta)}b(t,\xi,\eta)(\eta)d\eta\Big]\,d(y,z), \tag{4.23}$$

where we have set

$$\Phi_{t,\xi,w}(\eta)=\langle\eta,w\rangle\pm t(|\xi-\eta|+|\eta|).$$

Changing variables $z \mapsto y-w$ in (4.23) shows that $\|S[F,b](\cdot,t)\|^2_{L^2(\mathbb{R}^2)}$ equals

$$\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{-i\langle \xi, y\rangle} F(y, y-w) \Big[\int_{\mathbb{R}^2} e^{i\Phi_{t,\xi,w}(\eta)} b(t,\xi,\eta) d\eta \Big] dy\, dw \Big|^2 d\xi. \tag{4.24}$$

Computing the gradient of the phase function with respect to η,

$$\nabla \Phi_{t,\xi,w}(\eta) = w \pm t \Big(\frac{\eta}{|\eta|} - \frac{\xi-\eta}{|\xi-\eta|} \Big). \tag{4.25}$$

The key observation is now that if ξ and η satisfy (4.5), then

$$t \Big| \frac{\eta}{|\eta|} - \frac{\xi-\eta}{|\xi-\eta|} \Big| \le 2\measuredangle(\eta, \xi-\eta) \le 2^{-m+6}. \tag{4.26}$$

Therefore, if $|w| \ge 2^{-m+\ell+20}$ and $\ell \ge 0$, then

$$|\nabla \Phi_{t,\xi,w}(\eta)| \ge 2^{-m+\ell+19}.$$

This tells us that we should integrate by parts in the η–integral. Define a first order differential operator acting on functions $a : \mathbb{R}^2 \to \mathbb{C}$ by

$$\mathcal{L}_{t,\xi,w}[a] = i \operatorname{div}\Big(a \frac{\nabla \Phi_{t,\xi,w}}{|\nabla \Phi_{t,\xi,w}|^2} \Big).$$

Integrating by parts we obtain

$$\int_{\mathbb{R}^2} e^{i\Phi_{t,\xi,w}(\eta)} b(t,\xi,\eta) d\eta = \int_{\mathbb{R}^2} e^{i\Phi_{t,\xi,w}(\eta)} \mathcal{L}_{t,\xi,w}[b(t,\xi,\cdot)](\eta) d\eta.$$

Plugging this back into (4.24), changing the order of integration and applying the Cauchy–Schwarz inequality in η shows that (4.24) is

$$\lesssim 2^{2j-m} \int_{\mathbb{R}^4} \Big| \int_{\mathbb{R}^4} e^{-i\langle \xi, y\rangle + i\langle \eta, w\rangle} F(y, y-w) \mathcal{L}_{t,\xi,w}[b(t,\xi,\cdot)](\eta)\, d(y,w) \Big|^2 d(\eta,\xi).$$

Setting $G_\xi(w) = \int_{\mathbb{R}^2} e^{-i\langle \xi, y\rangle} F(y, y-w) dy$ we consider

$$\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^2} e^{i\langle \eta, w\rangle} G_\xi(w) \mathcal{L}_{t,\xi,w}[b(t,\xi,\cdot)](\eta) dw \Big|^2 d\eta \tag{4.27}$$

for each fixed ξ. Recalling the support assumption on F, let ψ be a smooth function on $\mathbb{R}^2$ that is equal to 1 on $\{|w| \ge 2^{20}\}$ and supported in $\{|w| \ge 2^{19}\}$. Setting

$$a_{t,\xi}(w,\eta) = \mathcal{L}_{t,\xi,w}[b(t,\xi,\cdot)](\eta) \psi(2^{m-\ell} w),$$

we recognize (4.27) as equal to $\|P^*_{a_{t,\xi}} G_\xi\|^2_{L^2(\mathbb{R}^2)}$ (see (4.22)). The product and chain rules show that $\mathcal{L}_{t,\xi,w}[a]$ equals

$$\frac{i\langle \nabla a, \nabla \Phi_{t,\xi,w}\rangle}{|\nabla \Phi_{t,\xi,w}|^2} - \frac{2ia\langle D^2\Phi_{t,\xi,w} \nabla \Phi_{t,\xi,w}, \nabla \Phi_{t,\xi,w}\rangle}{|\nabla \Phi_{t,\xi,w}|^4},$$

where $D^2\Phi_{t,\xi,w}$ denotes the Hessian matrix of $\Phi_{t,\xi,w}$. From this one can deduce the symbol estimate

$$\big|\partial_w^\beta \partial_\eta^\alpha a_{t,\xi}(w,\eta)\big| \lesssim B\, 2^{-(j-2m+\ell)} 2^{m|\beta|} 2^{-(j-m)|\alpha|}. \tag{4.28}$$

Since $0 \le m \le j/2$ and $a_{t,\xi}$ is supported in $\{(w,\eta) : |\eta| \approx 2^j\}$, an application of the Calderón–Vaillancourt result in §4.3 and L^2 duality yield that (4.27) is

$$\|P^*_{a_{t,\xi}} G_\xi\|^2_{L^2(\mathbb{R}^2)} \lesssim B\, 2^{-2(j-2m+\ell)} \|G_\xi\|^2_{L^2(\mathbb{R}^2)},$$

uniformly in ξ. Integrating over ξ and making use of Plancherel's theorem we obtain the bound

$$\|S[F,b](\cdot,t)\|_{L^2(\mathbb{R}^2)} \lesssim B\, 2^{j-\frac{m}{2}} 2^{-(j-2m+\ell)} \|F\|_{L^2(\mathbb{R}^2)},$$

valid for each fixed t. This implies (4.10). □

5. A necessary condition

We shall need a strengthening of a lower bound from [1] which had been stated there for Assouad regular sets.

Lemma 5.1. *Let $E \subset [1,2]$ and $M_E : L^p \to L^q$. Then*

(i) for all $\delta > 0$, for all intervals I with $\delta \le |I| \le 1$,

$$N(E \cap I, \delta)^{1/q} \lesssim \|M_E\|_{L^p \to L^q} \delta^{\frac{1}{p} - \frac{d}{q}} \big(\delta/|I|\big)^{\frac{d-1}{2}(\frac{1}{p}+\frac{1}{q}-1)}. \tag{5.1}$$

(ii) Let Q_1, $Q_{4,\gamma}$ be as in (1.1). The point $(\frac{1}{p}, \frac{1}{pd})$ belongs to $\overline{\mathcal{T}_E}$ if and only if it belongs to the line segment $\overline{Q_1 Q_{4,\gamma}}$ with $\gamma = \dim_{\mathrm{qA}} E$.

Proof. Fix an interval $I = [a,b]$ with $\delta < b - a < 1$ and let $\rho = \delta/|I|$.

Let $f_{\delta,\rho}$ be the characteristic function of a δ-neighborhood of the spherical cap of diameter $\sqrt{\rho}$, specifically

$$\{(y', y_d) \,:\, ||y| - a| \le \delta,\ |y'| \le \sqrt{\rho},\ y_d > 0\}.$$

Then

$$\|f_{\delta,I}\|_p \approx (\delta \rho^{\frac{d-1}{2}})^{1/p}.$$

Choose a covering of $E \cap I$ by a collection $\mathcal{J}_I$ of pairwise disjoint half open intervals of length δ intersecting $E \cap I$. Then $\#\mathcal{J_I} \ge N(E \cap I, \delta)$.

We now argue as in [1] and let $c \in (0,1)$ be a small constant, say $c < 10^{-2}$. We shall verify that for all $t \in \cup_{J \in \mathcal{J}_I} J$ and $x = (x', x_d)$ such that $|x'| \le c\delta\rho^{-1/2}$ and $|x_d + t - a| \le c\delta$,

$$M_E f_{\delta,\rho}(x) \ge A_t f_{\delta,\rho}(x', x_d) \gtrsim \rho^{\frac{d-1}{2}}. \tag{5.2}$$

Fix $y = (y', y_d) \in S^{d-1}$ with $|y'| \le c\sqrt{\rho}$. Then

$$\begin{aligned} |x + ty|^2 &= |x'|^2 + x_d^2 + 2t\langle x', y'\rangle + 2t x_d y_d + t^2 \\ &= |x'|^2 + (x_d + t)^2 + 2t x_d(\sqrt{1 - |y'|^2} - 1) + 2t\langle x', y'\rangle. \end{aligned}$$

Since $|x_d + t - a| \le c\delta$ and $|x'|^2 \le c^2\delta^2\rho^{-1} \le c^2\delta$, we get

$$||x'|^2 + (x_d + t)^2 - a^2| \le 6c\delta,$$

$$|2t\langle x', y'\rangle| \le 4|x'||y'| \le 4c^2\delta,$$

and

$$|2tx_d(\sqrt{1-|y'|^2}-1)| \le 2(|t-a|+c\delta)|y'|^2 \le 2c(|I|+c\delta)\rho \le 4c\delta.$$

Here we used that $|I| = \delta\rho^{-1}$. This implies

$$||x+ty|^2 - a^2| \le 14c\delta,$$

and therefore $||x+ty|-a| \le \delta$ if c is chosen small enough ($c = 10^{-2}$ works). Also, $|x'+ty'| \le |x'|+2|y'| \le \sqrt{\rho}$ so that $f_{\delta,\rho}(x+ty) = 1$. This proves (5.2).

Since the intervals $J \in \mathcal{J}_I$ are disjoint, the corresponding regions of x where (5.2) holds can be chosen disjoint, and therefore

$$\|M_E f_{\delta,\rho}\|_q \gtrsim \rho^{\frac{d-1}{2}} \big[\delta N(E\cap I,\delta)(\delta\rho^{-1/2})^{d-1}\big]^{1/q}.$$

Thus we must have

$$\rho^{\frac{d-1}{2}} \big(N(E\cap I,\delta)\delta\cdot(\delta\rho^{-1/2})^{d-1}\big)^{1/q} \lesssim \|M_E\|_{L^p\to L^q}(\delta\rho^{\frac{d-1}{2}})^{1/p}.$$

which yields (5.1).

Regarding part (ii), by Theorem 1.1 the points $(\frac{1}{p},\frac{1}{pd})$ belong to $\overline{\mathcal{T}_E}$ if they are on the line segment $\overline{Q_1Q_{4,\gamma}}$ with $\gamma = \dim_{\mathrm{qA}} E$. It follows from part (i) that this condition is also necessary. □

Proof of Theorem 1.4. Observe that

$$\sup_{j=1,\dots,m} \|M_{E_j}\|_{p\to q} \le \|M_E\|_{p\to q} \le \sum_{j=1}^m \|M_{E_j}\|_{p\to q}.$$

By Theorem 1.1 we have $\cap_{j=1}^m \mathcal{Q}(\beta_j,\gamma_j) \subset \overline{\mathcal{T}_E}$. Also, since E_j is (β_j,γ_j)-regular, the known necessary conditions (*cf.* (1.6) and Lemma 5.1 above) yield

$$\mathcal{Q}(\beta_j,\gamma_j)^{\complement} \subset \overline{\mathcal{T}_{E_j}}^{\complement} \subset \overline{\mathcal{T}_E}^{\complement}$$

for each $j = 1,\dots,m$. Hence, $\overline{\mathcal{T}_E} \subset \cap_{j=1}^m \mathcal{Q}(\beta_j,\gamma_j)$. □

6. Constructions of Assouad regular sets

In the proof of Theorem 1.2 we rely on a family of (quasi-)Assouad regular sets that is *uniform* in the following sense.

Lemma 6.1. *There exist sets $\{E_{\beta,\gamma} : 0<\beta<\gamma\le 1\}$ such that each $E_{\beta,\gamma}$ is (β,γ)-Assouad regular and there exists $c \ge 1$ such that for all $0<\beta<\gamma\le 1$, $\delta\in(0,1)$ and intervals $I \subset [1,2]$ with $|I|>\delta$,*

$$N(E_{\beta,\gamma},\delta) \le c\,\delta^{-\beta}, \quad N(E_{\beta,\gamma}\cap I,\delta) \le c\,\big(\tfrac{\delta}{|I|}\big)^{-\gamma}. \tag{6.1}$$

To prove the lemma we shall modify a construction in [1, §5]. Additional care is needed because of our requirement of uniformity. In what follows we fix $0<\beta<\gamma\le 1$.

6.1. *Preliminary: Cantor set construction.* Let J be a compact interval. For $0<\mu\le 1/2$ and an integer $m\ge 0$ we let $\mathfrak{C}_{\mu,m}(J)$ denote the union of the mth generation $\{I_{m,\nu} : \nu=1,\dots,2^m\}$ of intervals of length $\mu^m|J|$ that arise from starting with J and repeatedly removing the open middle piece of length $(1-2\mu)|J|$. Then the set $\mathfrak{C}_\mu(J)=\cap_{m\ge 0}\mathfrak{C}_{\mu,m}(J)$ has Hausdorff, Minkowski and Assouad dimensions equal to $\gamma=-\log(2)/\log(\mu)$. Note $\mathfrak{C}_{1/2,m}(J)=J$ for all $m\ge 0$. For every $\delta\in(0,|J|)$ it can be seen that

$$\tfrac{1}{2}|J|^\gamma\delta^{-\gamma}\le N(\mathfrak{C}_\mu(J),\delta)\le 2|J|^\gamma\delta^{-\gamma}$$

Similarly, for intervals $I\subset J$ and $\delta\in(0,|I|)$ one can verify

$$\tfrac{1}{4}|I|^\gamma\delta^{-\gamma}\le N(\mathfrak{C}_\mu(J)\cap I,\delta)\le 4|I|^\gamma\delta^{-\gamma}.$$

However, to construct the Assouad regular sets we will not use the Cantor sets $\mathfrak{C}_\mu(J)$ directly. Instead we will work with

$$\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J)=\{\text{midpoint of}\, I_{m,\nu} : \nu=1,\dots,2^m\}.$$

Write $\delta_m=\mu^m|J|$. Observe that for every $m\ge 0$, $\delta\in(0,|J|)$,

$$N(\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J),\delta)\le \min(\delta_m^{-\gamma},2\delta^{-\gamma})|J|^\gamma. \tag{6.2}$$

If $\delta<\delta_m$ this holds with equality since $\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J)$ consists of $2^m=\delta_m^{-\gamma}|J|^\gamma$ points that are pairwise separated by $\ge\delta_m$. If $\delta\ge\delta_m$ then there exists $j\le m$ such that $\delta\in[\mu^j|J|,\mu^{j-1}|J|)$, so

$$N(\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J),\delta)\le 2^j\le 2\delta^{-\gamma}|J|^\gamma$$

and (6.2) is proved.

We claim that (6.2) implies

$$N(\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J)\cap I,\delta)\le\begin{cases}8\delta^{-\gamma}|I|^\gamma & \text{if } \delta_m\le\delta\le|I|,\\ 4\delta_m^{-\gamma}|I|^\gamma & \text{if } \delta\le\delta_m\le|I|,\\ 1 & \text{if } |I|<\delta_m,\end{cases} \tag{6.3}$$

i.e. $N(\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J)\cap I,\delta)\le\max(1,\min(8\delta^{-\gamma},4\delta_m^{-\gamma})|I|^\gamma)$, valid for all $m\ge 0$, open subintervals $I\subset J$ with $|I|>\delta$. To see this, first note that the inequality holds if $|I|<\delta_m$ (as the points in $\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J)$ have mutual distance $\ge\delta_m$). Let $|I|\ge\delta_m$. Let ℓ be the integer so that $|I|\in[\mu^\ell|J|,\mu^{\ell-1}|J|)$. Then $\ell\le m$. Define

$$\mathcal{V}=\{\nu=1,\dots,2^{\ell-1} : I_{\ell-1,\nu}\cap I\ne\emptyset\}.$$

Observe that $\#\mathcal{V}\le 2$ (if $\mu<1/3$, then even $\#\mathcal{V}\le 1$). Next,

$$\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J)\cap I\subset\bigcup_{\nu\in\mathcal{V}}\mathfrak{C}^{\mathrm{mid}}_{\mu,m}(J)\cap I_{\ell-1,\nu}=\bigcup_{\nu\in\mathcal{V}}\mathfrak{C}^{\mathrm{mid}}_{m-\ell+1}(I_{\ell-1,\nu}).$$

Using (6.2), $\mu^{m-\ell+1}|I_{\ell-1,\nu}|=\mu^m|J|=\delta_m$ and $|I_{\ell-1,\nu}|^\gamma\le 2|I|^\gamma$,

$$N(\mathfrak{C}^{\mathrm{mid}}_{m-\ell+1}(I_{\ell-1,\nu}),\delta)\le\min(2\delta_m^{-\gamma},4\delta^{-\gamma})|I|^\gamma.$$

Since $\#\mathcal{V}\le 2$, this implies the claim.

6.2. *Assouad regular sets.* Let $\lambda = 2^{-1/\beta}$ and $\mu = 2^{-1/\gamma}$, so that $\lambda < \mu \leq 1/2$. Define

$$J_k = [1+\lambda^{k+1}, 1+\lambda^k], \quad \theta = 1-\beta/\gamma, \quad m(k) = \lceil \tfrac{k}{\theta} \rceil.$$

We then set

$$F = \bigcup_{k=1}^{\infty} F_k, \text{ where } F_k = \mathfrak{C}^{\mathrm{mid}}_{\mu, m(k)}(J_k).$$

The length of each the constituent intervals $I_{m(k),\nu}$, $\nu = 1, \dots, 2^{m(k)}$ of $\mathfrak{C}^{\mu}_{m(k)}(J_k)$ is

$$\sigma_k = |J_k| \mu^{m(k)}.$$

Since $\lambda < 1/2$, $2^{-k/\beta - 1} \leq |J_k| = 2^{-k/\beta}(1-\lambda) \leq 2^{-k/\beta}$. The choice of $m(k)$ is made so that $\sigma_k^{\theta} \approx 2^{-k/\beta} \approx |J_k|$. More precisely,

$$(2^{-1}\mu)^{\theta} 2^{-k/\beta} \leq \sigma_k^{\theta} \leq 2^{-k/\beta}. \tag{6.4}$$

For open intervals $I \subset [1,2]$, $\delta \in (0,1)$, $|I| > \delta$ we claim that

$$N(F \cap I, \delta) \leq 48\, (\delta/|I|)^{-\gamma}. \tag{6.5}$$

This estimate immediately yields $\dim_{\mathrm{A}} F \leq \gamma$.

To prove (6.5) first take $I \subset J_k$ open with $|I| > \delta$. Then by (6.3),

$$N(F_k \cap I, \delta) \leq \max(1, 8 \min(\sigma_k^{-\gamma}, \delta^{-\gamma}) |I|^{\gamma}). \tag{6.6}$$

Then, for an arbitrary open interval $I \subset [1,2]$ and $\delta < |I|$,

$$\begin{aligned} N(F \cap I, \delta) &\leq \sum_{k \geq 0} N(F_k \cap (J_k \cap I), \delta) \\ &\leq \sum_{\substack{k: |J_k| \geq |I|, \\ J_k \cap I \neq \emptyset}} N(F_k \cap (J_k \cap I), \delta) + \sum_{k: |J_k| \leq |I|} N(F_k \cap J_k, \delta). \end{aligned}$$

By (6.6) this is bounded by

$$8 \sum_{\substack{k: |J_k| \geq |I|, \\ J_k \cap I \neq \emptyset}} \delta^{-\gamma} |I|^{\gamma} + 8 \sum_{k: |J_k| \leq |I|} \delta^{-\gamma} |J_k|^{\gamma} \leq 48\, \delta^{-\gamma} |I|^{\gamma}$$

which finishes the proof of (6.5). Next we turn to $\overline{\dim}_{\mathrm{A},\theta} F$. We have

$$N(F \cap J_k, \sigma_k) = N(F_k, \sigma_k) = 2^{m(k)} = \mu^{-\gamma m(k)} = \sigma_k^{-\gamma} |J_k|^{\gamma}, \tag{6.7}$$

which implies $\overline{\dim}_{\mathrm{A},\theta} F \geq \gamma$, because $|J_k| \approx \sigma_k^{\theta}$. Since $\overline{\dim}_{\mathrm{A},\theta} F \leq \dim_{\mathrm{A}} F$ this proves

$$\dim_{\mathrm{A}} F = \overline{\dim}_{\mathrm{A},\theta} F = \gamma.$$

Regarding $\dim_{\mathrm{M}} F$ we see that because of $\sigma_k^{-\gamma} |J_k|^{\gamma} \approx \sigma_k^{-\beta}$ one gets $\dim_{\mathrm{M}} F \geq \beta$. Finally, for every $\delta \in (0,1)$,

$$N(F, \delta) \leq c_F \delta^{-\beta}, \tag{6.8}$$

where the constant $c_F \geq 1$ depends only on β, γ (and may blow up as β, γ tend to 0). This follows since

$$N(F,\delta) \leq N(\cup_{k:\delta\geq |J_k|} F_k, \delta) + \sum_{k:\sigma_k<\delta<|J_k|} N(F_k,\delta) + \sum_{k:\sigma_k\geq\delta} N(F_k,\delta),$$

which by choice of $m(k)$ and (6.6) is

$$\lesssim_{\beta,\gamma} 1 + \sum_{k\,:\,\sigma_k<\delta<\sigma_k^\theta} \delta^{-\gamma}\sigma_k^{\gamma-\beta} + \sum_{k\geq 0:\sigma_k\geq\delta} \sigma_k^{-\gamma}\sigma_k^{\gamma-\beta} \lesssim_{\beta,\gamma} \delta^{-\beta}.$$

To resolve the blowup in (6.8) we use an affine transformation. Define

$$E_{\beta,\gamma} = 1 + c_F^{-1/\beta} F \subset [1,2].$$

Then by (6.8), $N(E_{\beta,\gamma},\delta) = N(F, c_F^{1/\beta}\delta) \leq c_F (c_F^{1/\beta}\delta)^{-\beta} = \delta^{-\beta}$ for all $\delta < c_F^{-1/\beta}$ and $N(E_{\beta,\gamma},\delta) \leq 1 \leq \delta^{-\beta}$ for all $\delta \in [c_F^{-1/\beta}, 1)$. Similarly, by (6.5), $N(E_{\beta,\gamma}\cap I,\delta) \leq c\,(\delta/|I|)^{-\gamma}$ for all $\delta\in(0,1)$ and intervals $I\subset[1,2]$ with $|I|>\delta$. This concludes the proof of Lemma 6.1. □

6.3. *Zero Minkowski and positive Assouad dimension.* We consider $E \subset [1,2]$ with $\dim_{\mathrm{M}} E = 0$. Note that $\overline{\dim}_{\mathrm{A},\theta} E = 0$ for all $\theta < 1$ ([10, Cor. 3.3]), hence also $\dim_{\mathrm{qA}} E = 0$. However (as already remarked in [12]) the Assouad dimension can be any given $\gamma \in [0,1]$. We prove this for the sake of completeness.

Lemma 6.2. *For every $\alpha \in [0,1]$ there exists a set $G_\alpha \subset [1,2]$ such that $\dim_{\mathrm{M}} G_\alpha = 0$ and $\dim_{\mathrm{A}} G_\alpha = \alpha$.*

Proof. If $\alpha = 0$ let $G_0 = \{\text{point}\}$. For $\alpha = 1$ let $G_1 = \{1 + 2^{-\sqrt{\ell}} : \ell \geq 1\}$.

It remains to consider the case $0 < \alpha < 1$. Let

$$J_n = [1 + 2^{-2^n-1}, 1 + 2^{-2^n}], \quad F_n = \mathfrak{C}^{\mathrm{mid}}_{\mu,n}(J_n), \quad \mu = 2^{1/\alpha}.$$

Next, let $n_0 = n_0(\alpha)$ be the smallest non-negative integer so that $\alpha 2^{n_0} \geq 1$ and define

$$G_\alpha = \bigcup_{n\geq n_0} F_n \subset [1,2].$$

We first consider the Minkowski dimension. Pick $\delta \in (0,1)$ and observe that $\cup_{n:2^{-2^n}\leq\delta} F_n$ is covered by $[0,\delta]$ and that the cardinality of $\cup_{n:2^{-2^n}>\delta} F_n$ is bounded by $\sum_{n:2^{-2^n}>\delta} 2^n \leq 2\log_2(\delta^{-1})$. Hence $\dim_{\mathrm{M}} G_\alpha = 0$.

It remains to show that $\dim_{\mathrm{A}} G_\alpha = \alpha$. Each of the intervals that make up $\mathfrak{C}_{\mu,n}(J_n)$ has length

$$\sigma_n = \mu^n |J_n| = \mu^n 2^{-2^n-1}.$$

Note by definition that

$$N(F_n \cap J_n, \sigma_n) = N(F_n,\sigma_n) = 2^n = \sigma_n^{-\alpha}|J_n|^\alpha.$$

Thus we get $\dim_{\mathrm{A}} G_\alpha \geq \alpha$. It remains to show $\dim_{\mathrm{A}} G_\alpha \leq \alpha$. Here we proceed as in §6.2. If $I \subset J_n$ with $|I| > \delta$ then by (6.3),

$$N(F_n \cap I, \delta) \leq 8\delta^{-\alpha}|I|^\alpha.$$

This implies for an arbitrary open interval $I \subset [1,2]$ with $|I| > \delta$ that

$$N(G_\alpha \cap I, \delta) \leq 8 \sum_{\substack{n:|J_n|\geq|I| \\ J_n\cap I\neq\emptyset}} \delta^{-\alpha}|I|^\alpha + 8\delta^{-\alpha} \sum_{n\geq n_0:|J_n|\leq|I|} |J_n|^\alpha \leq 48\,\delta^{-\alpha}|I|^\alpha.$$

This is because the first sum in this display has at most two terms while

$$\begin{aligned}\sum_{\substack{n\geq n_0: \\ |J_n|\leq|I|}} |J_n|^\alpha &\leq \sum_{\substack{n\geq n_0: \\ 2^{-2^n-1}\leq|I|}} (2^{-2^n-1})^\alpha \leq 2^{-\alpha} \sum_{n\geq 0: 2^{-2^n}\leq(2|I|)^{2^{-n_0}}} 2^{-2^n} \\ &\leq 2^{-\alpha} \sum_{\substack{n\geq 0: \\ 2^{-n}\leq(2|I|)^{2^{-n_0}}}} 2^{-n} \leq 2^{-\alpha+1}(2|I|)^{2^{-n_0}} \leq 4|I|^\alpha. \qquad \square\end{aligned}$$

7. Convex sets occurring as closures of type sets

In this section we prove Theorem 1.2. Let $\mathcal{W} \subset [0,1]^2$ be a closed convex set and let $0 \leq \beta \leq \gamma \leq 1$ be such that

$$\mathcal{Q}(\beta,\gamma) \subset \mathcal{W} \subset \mathcal{Q}(\beta,\beta) \tag{7.1}$$

and suppose that γ is minimal with this property. If $\overline{\mathcal{T}_E} = \mathcal{W}$ for some $E \subset [1,2]$ then we have $\beta = \dim_{\mathrm{M}} E$, moreover it follows from part (ii) of Lemma 5.1 that $\gamma = \dim_{\mathrm{qA}} E$. In what follows it thus suffices to prove the existence of E satisfying $\overline{\mathcal{T}_E} = \mathcal{W}$.

If $\beta = \gamma$, then we may take E to be a self similar Cantor set of Minkowski dimension β. If $\beta = 0$, then $\mathcal{W} = \mathcal{Q}(0,0)$, so the single average example $E = \{\text{point}\}$ works. It remains to consider the case $0 < \beta < \gamma \leq 1$.

We may also assume that $\mathcal{Q}(\beta,\gamma) \subsetneq \mathcal{W} \subsetneq \mathcal{Q}(\beta,\beta)$ (if $\mathcal{W} = \mathcal{Q}(\beta,\gamma)$ then we choose any (β,γ)-regular set for E, such as the one from Lemma 6.1, and if $\mathcal{W} = \mathcal{Q}(\beta,\beta)$ we again choose a Cantor set).

Let $\mathfrak{L}$ denote the set of lines that pass through at least one point in $\partial\mathcal{W}$ but are disjoint from the interior of $\mathcal{W}$. Each line $\ell \in \mathfrak{L}$ divides the plane into two half-spaces. We denote by $\mathfrak{H}(\ell)$ the closed half-space that contains $\mathcal{W}$. Then

$$\mathcal{W} = \bigcap_{\ell\in\mathfrak{L}} \mathfrak{H}(\ell). \tag{7.2}$$

There exists a countable subset $\mathfrak{L}' \subset \mathfrak{L}$ such that $\mathcal{W} = \cap_{\ell\in\mathfrak{L}'}\mathfrak{H}(\ell)$. We further select a subset $\mathfrak{L}^\flat \subset \mathfrak{L}'$ consisting only of those lines that do not contain any of the edges of $\mathcal{Q}(\beta,\gamma)$. $\mathfrak{L}^\flat$ must be non-empty because $\mathcal{W} \supsetneq \mathcal{Q}(\beta,\gamma)$.

Since $\mathfrak{L}^\flat$ is countable we may write $\mathfrak{L}^\flat = \{\ell_1, \ell_2, \dots\}$. The line ℓ_n intersects the line segment connecting $Q_{3,\beta}$ and $Q_{3,0}$ in a point Q_{3,β_n} for some

$\beta_n \in [0, \beta]$. The line ℓ_n also intersects the line segment connecting $Q_{4,\gamma}$ and $Q_{4,\beta}$ in a point Q_{4,γ_n} for some $\gamma_n \in [\beta, \gamma]$. This is illustrated in Figure 3.

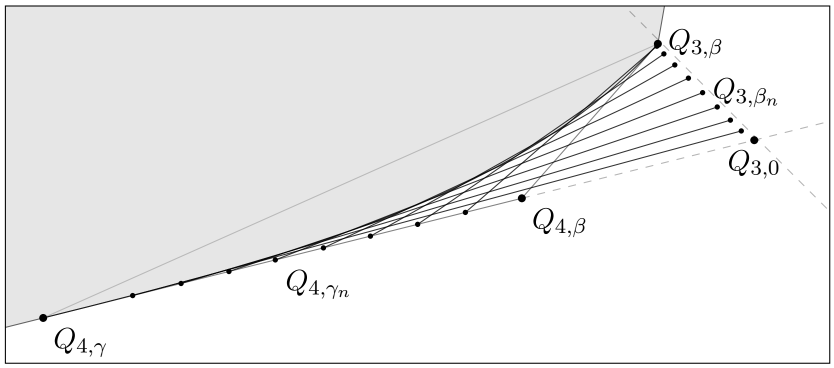

FIGURE 3. $d = 4$, $\beta = 0.3$, $\gamma = 1$

Then from (7.2),

$$\mathcal{W} = \bigcap_{n \geq 1} \mathcal{Q}(\beta_n, \gamma_n). \tag{7.3}$$

Let $E_{\beta,\gamma} \subset [1, 2]$ be as in Lemma 6.1. Let $L : \mathbb{N} \to \mathbb{N}$ be strictly increasing (we may just choose $L(n) = n$ in this proof but we will need to make a different choice to prove Remark 1.3). Define

$$\begin{aligned} E_n &= 1 + 2^{-L(n)-1} E_{\beta_n,\gamma_n} \subset [1 + 2^{-L(n)-1}, 1 + 2^{-L(n)}], \\ E &= \bigcup_{n=1}^{\infty} E_n \subset [1, 2]. \end{aligned} \tag{7.4}$$

It remains to show that $\overline{\mathcal{T}_E} = \mathcal{W}$. First we have $\|M_E\|_{p\to q} \geq \|M_{E_n}\|_{p\to q}$ for every $n \geq 1$. Since E_n is (β_n, γ_n)-regular, we have $\overline{\mathcal{T}_E} \subset \overline{\mathcal{T}_{E_n}} = \mathcal{Q}(\beta_n, \gamma_n)$, *cf.* (1.6). Hence $\overline{\mathcal{T}_E} \subset \mathcal{W}$ by (7.3).

To prove $\mathcal{W} \subset \overline{\mathcal{T}_E}$ let us take a point (p_*^{-1}, q_*^{-1}) in the interior of $\mathcal{W}$. It suffices to show that M_E is bounded $L^{p_*} \to L^{q_*}$. By the construction of E_n and (6.1),

$$N(E_n, \delta) \leq c\delta^{-\beta_n}, \quad N(E_n \cap I, \delta) \leq c\big(\tfrac{\delta}{|I|}\big)^{-\gamma_n} \tag{7.5}$$

for every $n \geq 1, \delta \in (0, 2^{-n})$ and every interval $I \subset [1, 2]$ with $|I| > \delta$. Here it is important that c does not depend on n.

Crucially, for each j all E_n with $n > j$ are contained in $[1, 1 + 2^{-L(j)}]$ and thus in $[1, 1 + 2^{-j}]$. Therefore we can estimate

$$\big\| \sup_{t\in E} |\mathcal{A}_t^j f| \big\|_{q_*} \leq \sum_{n\leq j} \big\| \sup_{t\in E_n} |\mathcal{A}_t^j f| \big\|_{q_*} + \big\| \sup_{1\leq t\leq 1+2^{-j}} |\mathcal{A}_t^j f| \big\|_{q_*}. \tag{7.6}$$

The second term on the right hand side is dominated by

$$\|\mathcal{A}_1 f\|_{q_*} + \int_0^{2^{-j}} \big\| \tfrac{d}{dt}\mathcal{A}_t^j f\big\|_{q_*} dt \lesssim 2^{-aj}\|f\|_{p_*}$$

for some $a > 0$ because (p_*^{-1}, q_*^{-1}) is in the interior of $\mathcal{Q}(0,0) \supset \mathcal{W}$.

It remains to estimate the first term in (7.6). Since (p_*^{-1}, q_*^{-1}) is in the interior of $\mathcal{W}$, it is away from the boundary of each $\mathcal{Q}(\beta_n, \gamma_n)$ by a positive distance independent of n. By Corollary 2.2 we now obtain $\varepsilon = \varepsilon(p_*, q_*) > 0$ not depending on n such that

$$\big\| \sup_{t\in E_n} |\mathcal{A}_t^j f|\big\|_{q_*} \lesssim_{d,p_*,q_*} 2^{-j\varepsilon(p_*,q_*)}\|f\|_{p_*} \tag{7.7}$$

for all $j \geq 0$ and $n \geq 1$. To see that the implicit constant does not depend on n one uses (2.16) and that (7.5) implies

$$[\chi^{E_n}_{\mathrm{A},\gamma_n}(2^{-j})]^{b_1}[\chi^{E_n}_{\mathrm{M},\beta_n}(2^{-j})]^{b_2} \leq c$$

with c depending only on the dimension d. Hence, the first term on the right hand side of (7.6) is $\lesssim j2^{-j\varepsilon}\|f\|_{p_*}$. This concludes the proof that $(p_*^{-1}, q_*^{-1}) \in \mathcal{T}_E$.

Proof of Remark 1.3. Without loss of generality let $\gamma_* = \gamma$.

We need to make a judicious choice of the sequence $L(n)$ to achieve $\dim_{\mathrm{A}} E = \gamma$. Define $L(n)$ iteratively such that for $n = 2, 3, \dots$

$$L(n) \geq \max\{L(k) + \gamma_n^{-1}(n-k) :\ 1 \leq k \leq n-1\}. \tag{7.8}$$

We then claim that the construction actually yields

$$\dim_{\mathrm{A}} E \leq \gamma. \tag{7.9}$$

and since $\gamma = \dim_{\mathrm{qA}} E \leq \dim_{\mathrm{A}} E$ we actually get equality in (7.9).

We now show (7.9). Let $J_n = [1 + 2^{-L(n)-1}, 1 + 2^{-L(n)}]$. Let $\delta < |I| \leq 1$ and observe that

$$\sum_{\substack{n:|J_n|\geq|I|,\\ J_n\cap I\neq\emptyset}} N(E_n \cap I, \delta) \leq c \sum_{\substack{n:|J_n|\geq|I|,\\ J_n\cap I\neq\emptyset}} (\delta/|I|)^{-\gamma_n} \leq 2c(\delta/|I|)^{-\gamma}$$

using that $\gamma_n \leq \gamma$ and there are at most two n such that J_n intersects I and $|J_n| \geq |I|$. Next let $n_\circ$ be the smallest n for which $|J_n| < |I|$. By (7.8), $L(n)\gamma_n - L(n_\circ)\gamma_n \geq n - n_\circ$. We obtain

$$\begin{aligned}\sum_{n:|J_n|<|I|} N(E_n \cap (J_n \cap I), \delta) &\leq c \sum_{n:|J_n|<|I|} (\delta/|J_n|)^{-\gamma_n}\\ &= c \sum_{n:|J_n|<|I|} (\delta/|I|)^{-\gamma_n}(|I|/|J_n|)^{-\gamma_n} \leq c(\delta/|I|)^{-\gamma} \sum_{n\geq n_\circ} (|I|/|J_n|)^{-\gamma_n},\end{aligned}$$

and

$$\sum_{n\ge n_\circ}(|I|/|J_n|)^{-\gamma_n}=\sum_{n\ge n_\circ}2^{-(L(n)+1)\gamma_n}|I|^{-\gamma_n}$$
$$\le\sum_{n\ge n_\circ}2^{-L(n)\gamma_n}2^{L(n_\circ)\gamma_n}\le\sum_{n\ge n_\circ}2^{-(n-n_\circ)}\le 2.$$

Hence $\sum_{n:|J_n|<|I|}N(E_n\cap(J_n\cap I),\delta)\lesssim 2c(\delta/|I|)^{-\gamma}$. The two cases imply (7.9), and this settles the case $\gamma=\gamma_*$ in Remark 1.3.

Finally consider the case $\gamma_*\in(\gamma,1]$. Take a set E' as constructed above with $\dim_{\mathrm{M}}E'=\beta$, $\dim_{\mathrm{qA}}E'=\dim_{\mathrm{A}}E'=\gamma$ and $\overline{\mathcal{T}_{E'}}=\mathcal{W}$ and a set G_{γ_*} as in Lemma 6.2 with $\dim_{\mathrm{M}}G_{\gamma_*}=0$ and $\dim_{\mathrm{A}}G_{\gamma_*}=\gamma_*$. Define $E=E'\cup G_{\gamma_*}$. Then $\dim_{\mathrm{A}}E=\max(\dim_{\mathrm{A}}E',\dim_{\mathrm{A}}G_{\gamma_*})=\gamma_*$ and since $\overline{\mathcal{T}_{G_{\gamma_*}}}=\mathcal{Q}(0,0)\supset\mathcal{W}$ we see that $\overline{\mathcal{T}_E}=\mathcal{W}$. □

References

[1] T.C. Anderson, Kevin Hughes, Joris Roos, Andreas Seeger. *$L^p\to L^q$-estimates for spherical maximal operators.* arXiv:1909.05389. Math. Z. 297 (2021), no. 3-4, 1057–1074.

[2] P. Assouad. *Étude d'une dimension métrique liée à la possibilité de plongements dans $\mathbb{R}^n$.* C. R. Acad. Sci., Sér. A-B. (1979) 288 (15): A731–A734.

[3] F. Bernicot, D. Frey, S. Petermichl. *Sharp weighted norm estimates beyond Calderón-Zygmund theory.* arXiv:1510.00973. Anal. PDE 9 (2016), no. 5, 1079–1113.

[4] J. Bourgain. *Estimations de certaines fonctions maximales.* C. R. Acad. Sci. Paris Sér. I 301 (1985), 499–502.

[5] J. Bourgain. *Averages in the plane over convex curves and maximal operators.* Journal d'Analyse 47 (1986), 69–85.

[6] A.P. Calderón, Rémi Vaillancourt. *A class of bounded pseudo-differential operators.* Proc. Nat. Acad. Sci. U.S.A. 69 (1972), 1185–1187.

[7] C.P. Calderón. *Lacunary spherical means.* Ilinois J. Math. 23 (1979), 476?484.

[8] R. R. Coifman, G. Weiss. Review: R. E. Edwards and G. I. Gaudry, *Littlewood-Paley and multiplier theory.* Bull. Amer. Math. Soc. 84 (1978), 242?250.

[9] J.M. Fraser. *Assouad type dimensions and homogeneity of fractals.* arXiv:1301.2934. Trans. Amer. Math. Soc. 366 (2014), no. 12, 6687–6733.

[10] J.M. Fraser, H. Yu. *New dimension spectra: finer information on scaling and homogeneity.* arXiv:1610.02334. Adv. Math. 329 (2018), 273–328.

[11] J.M. Fraser, Han Yu. *Assouad-type spectra for some fractal families.* arXiv:1611.08857. Indiana Univ. Math. J. 67 (2018), no. 5, 2005–2043.

[12] J.M. Fraser, K.E. Hare, K.G. Hare, S. Troscheit, H. Yu. *The Assouad spectrum and the quasi-Assouad dimension: a tale of two spectra.* arXiv:1804.09607. Ann. Acad. Sci. Fenn. Math. 44, (2019), 379–387.

[13] S. Klainerman, M. Machedon. *Space-time estimates for null forms and the local existence theorem.* Comm. Pure Appl. Math. 46 (1993), no. 9, 1221–1268.

[14] M.T. Lacey. *Sparse bounds for spherical maximal functions.* arXiv:1702.08594v6. J. Anal. Math. 139 (2019), no. 2, 613–635.

[15] S. Lee. *Endpoint estimates for the circular maximal function.* Proc. Amer. Math. Soc. 131 (2003), no. 5, 1433–1442.

[16] W. Littman. *L^p-L^q-estimates for singular integral operators arising from hyperbolic equations.* Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ.

California, Berkeley, Calif., 1971), pp. 479–481. Amer. Math. Soc., Providence, R.I., 1973.

[17] F. Lü and L.-F. Xi. *Quasi-Assouad dimension of fractals.* J. Fractal Geom., 3, (2016), 187–215.

[18] G. Mockenhaupt, A. Seeger, C.D. Sogge. *Wave front sets and Bourgain's circular maximal theorem.* Ann. Math. 134 (1992), 207–218.

[19] J. Roos, A. Seeger. *L^p improving for spherical maximal operators on radial functions.* In preparation.

[20] W. Schlag. *A generalization of Bourgain's circular maximal theorem.* J. Amer. Math. Soc. 10 (1997), no. 1, 103–122.

[21] W. Schlag, C.D. Sogge. *Local smoothing estimates related to the circular maximal theorem.* Math. Res. Lett. 4 (1997), no. 1, 1–15.

[22] A. Seeger, T. Tao, J. Wright. *Endpoint mapping properties of spherical maximal operators.* arXiv:math/0205153. J. Inst. Math. Jussieu 2 (2003), no. 1, 109–144.

[23] A. Seeger, S. Wainger, J. Wright. *Pointwise convergence of spherical means.* Math. Proc. Camb. Phil. Soc. 118 (1995), 115–124.

[24] A. Seeger, S. Wainger, J. Wright. *Spherical maximal operators on radial functions.* arXiv:math/9601220. Math. Nachr. 187 (1997), 95–105.

[25] E.M. Stein. *Maximal functions: spherical means.* Proc. Natl. Acad. Sci. U.S.A. 73 (1976), 2174–2175.

[26] E.M. Stein. Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton University Press, Princeton, NJ, 1993.

[27] T. Tao, *Endpoint bilinear restriction theorems for the cone, and some sharp null form estimates.* arXiv:math/9909066. Math. Z. 238 (2001), no. 2, 215–268.

[28] S. Wainger. *Applications of Fourier transforms to averages over lower dimensional sets.* Proc. Symp. Pure Math. 35 I (1979), American Mathematical Society, Providence, R.I., 85–94.

[29] T. Wolff. *A sharp bilinear cone restriction estimate.* arXiv:math/0105266. Ann. of Math. (2) 153 (2001), no. 3, 661–698.

Joris Roos, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA

Current address: Department of Mathematical Sciences, University of Massachusetts Lowell, USA & School of Mathematics, The University of Edinburgh, Scotland, UK

Email address: `jroos.math@gmail.com`

Andreas Seeger, Department of Mathematics, University of Wisconsin, 480 Lincoln Drive, Madison, WI 53706, USA

Email address: `seeger@math.wisc.edu`