

THE THIRD HOMOLOGY OF SYMPLECTIC GROUPS AND ALGEBRAIC K -THEORY

HUSNEY PARVEZ SARWAR AND MARCO SCHLICHTING

ABSTRACT. We improve the homology stability range for the 3rd integral homology of symplectic groups over commutative local rings with infinite residue field. As an application, we show that for local commutative rings containing an infinite field of characteristic not 2 the symbol map from Milnor-Witt K -theory to higher Grothendieck-Witt groups is an isomorphism in degrees 2 and 3.

CONTENTS

1. Introduction	2
2. The complex of non-degenerate unimodular sequences	4
3. The Homology spectral sequence and its E^1 -page	10
4. Triviality of $d_{p,q}^r$ for q even	14
5. A formula for $d_{0,2n+1}^2$	16
6. Surjectivity of $d_{0,5}^2$	22
7. Localising homology groups	29
8. Homology stability	31
9. The Hurewicz map	35
10. The KO -degree map	36
Appendix A. The Pfaffian of some matrices	43
A.1. The skew-symmetric matrix of Lemma 6.2	44
A.2. The skew-symmetric matrix of Lemma 6.19	45
A.3. The skew-symmetric matrix of Lemma 6.20	46
References	47

1991 *Mathematics Subject Classification.* 20J05 (19B14, 19D45, 19G38).

Key words and phrases. Homology of symplectic groups, Homology stability, symplectic K -theory, Milnor-Witt K -theory.

Sarwar acknowledges support from the Science and Engineering Research Board (S.E.R.B), India.

Schlichting acknowledges support from EPSRC grant EP/M001113/1 and the Leverhulme Trust.

1. INTRODUCTION

In this paper we improve homology stability ranges for the symplectic groups over commutative local rings with infinite residue field and use this to give explicit presentations of some algebraic K -groups.

Recall that for an integer $n \geq 0$ and a commutative ring R , the symplectic group $\mathrm{Sp}_{2n}(R)$ is the group of R -linear automorphisms of R^{2n} that preserve the standard symplectic form $\langle x, y \rangle = \sum_{i=0}^{n-1} (x_{2i+1}y_{2i+2} - x_{2i+2}y_{2i+1})$, $x, y \in R^{2n}$. We consider $\mathrm{Sp}_{2n}(R)$ as a subgroup of $\mathrm{Sp}_{2n+2}(R)$ via the embedding $A \mapsto \begin{pmatrix} 1_{R^2} & 0 \\ 0 & A \end{pmatrix}$. The following is part of Theorem 8.3 in the text. Unless otherwise stated, all homology groups $H_n(G)$ of a group G are taken with integer coefficients.

Theorem 1.1. *Let R be a local ring with infinite residue field. Then inclusion of groups induces a surjection $H_3 \mathrm{Sp}_2(R) \twoheadrightarrow H_3 \mathrm{Sp}_4(R)$ and for $k \geq 2$ isomorphisms $H_3 \mathrm{Sp}_{2k}(R) \cong H_3 \mathrm{Sp}_{2k+2}(R)$:*

$$H_3(\mathrm{Sp}_2(R)) \twoheadrightarrow H_3(\mathrm{Sp}_4(R)) \xrightarrow{\cong} H_3(\mathrm{Sp}_6(R)) \xrightarrow{\cong} H_3(\mathrm{Sp}_8(R)) \xrightarrow{\cong} \dots$$

Theorem 1.1 is optimal in the sense that the first map $H_3 \mathrm{Sp}_2(R) \twoheadrightarrow H_3 \mathrm{Sp}_4(R)$ is not injective, in general; see Remark 8.4. Theorem 1.1 answers a question raised by Hutchinson and Wendt in [HW15, Remark 9.6]. An important consequence of Theorem 1.1 is the following relative homology stability result proved in Theorem 8.5.

Theorem 1.2. *Let R be a local ring with infinite residue field. Then inclusion of groups induces isomorphisms of relative integral homology groups for $i \leq 3$*

$$H_i(\mathrm{SL}_3(R), \mathrm{Sp}_2(R)) \xrightarrow{\cong} H_i(\mathrm{SL}_4, \mathrm{Sp}_4) \xrightarrow{\cong} H_i(\mathrm{SL}_6, \mathrm{Sp}_6) \xrightarrow{\cong} \dots \xrightarrow{\cong} H_i(\mathrm{SL}(R), \mathrm{Sp}(R)).$$

The importance of Theorem 1.2 lies in the fact that in degree 3, the left group $H_3(\mathrm{SL}_3(R), \mathrm{Sp}_2(R))$ was identified in [Sch17a] with the third Milnor-Witt K -group $K_3^{MW}(R)$ of R and the right group $H_3(\mathrm{SL}(R), \mathrm{Sp}(R))$ is the first non-vanishing homotopy group of the fibre from symplectic K -theory to algebraic K -theory.

We generalise Theorem 1.1 to homology degree $n \geq 3$ by showing the following in Theorem 8.6 though the stability range here is probably not optimal when $n \geq 4$.

Theorem 1.3. *Let R be a local ring with infinite residue field, and let $n \geq 3$ be an integer. Then inclusion of groups induces a surjection $H_n \mathrm{Sp}_{2n-4}(R) \twoheadrightarrow H_n \mathrm{Sp}_{2n-2}(R)$ and for $k \geq 0$ isomorphisms $H_n \mathrm{Sp}_{2n+2k-2}(R) \cong H_n \mathrm{Sp}_{2n+2k}(R)$:*

$$H_n(\mathrm{Sp}_{2n-4}(R)) \twoheadrightarrow H_n(\mathrm{Sp}_{2n-2}(R)) \xrightarrow{\cong} H_n(\mathrm{Sp}_{2n}(R)) \xrightarrow{\cong} H_n(\mathrm{Sp}_{2n+2}(R)) \xrightarrow{\cong} \dots$$

This improves on the homology stability ranges for infinite fields F due to Essert [Ess13] who proves for $n \geq 0$ the following surjection and isomorphisms

$$H_n(\mathrm{Sp}_{2n}(F)) \twoheadrightarrow H_n(\mathrm{Sp}_{2n+2}(F)) \xrightarrow{\cong} H_n(\mathrm{Sp}_{2n+4}(F)) \xrightarrow{\cong} H_n(\mathrm{Sp}_{2n+6}(F)) \xrightarrow{\cong} \dots$$

See [Ess13, Theorem 3.9] and [SW20, Theorem A] which improve on [Mir05].

As application of Theorems 1.1 and 1.2 we generalise some results of Asok-Fasel [AF17] from fields to local rings replacing the use of \mathbb{A}^1 -homotopy theory with group homology computations.

We consider the KO -degree map [AF17]

$$(1.1) \quad K_n^{MW}(R) \rightarrow GW_n^{[n]}(R)$$

from the Milnor-Witt K -groups of Hopkins-Morel [Mor12], [Sch17a], [GSZ16] to the higher Grothendieck-Witt-groups of [Kar73] in the form of [Sch17b]. The following is a combination of Theorems 10.5 and 10.8 in the text.

Theorem 1.4. *Let R be a local ring containing an infinite field of characteristic not 2. Then the KO -degree map (1.1) is an isomorphism for $n = 2, 3$.*

The restriction to rings that contain an infinite field of characteristic not 2 in Theorem 1.4 comes from our use of [GSZ16]. This is probably unnecessary in light of Theorems 9.1 and 9.2.

Finally, we obtain an interpretation of the indecomposable part of K_3 in terms of orthogonal K -theory. Recall that when $\frac{1}{2} \in R$, the orthogonal K -group $KO_3(R)$ is the higher Grothendieck-Witt group $GW_3^{[4]}(R) = \pi_3 BO(R)^+$ where $O(R) = \bigcup_{n \geq 0} O_{2n}(R)$ is the infinite orthogonal group of R , that is, the union of the groups of R -linear automorphisms preserving the standard hyperbolic quadratic form of rank $2n$. The following is Corollary 10.10 in the text.

Theorem 1.5. *Let R be a commutative local ring containing an infinite field of characteristic not 2. Then*

$$K_3^{ind}(R) \cong KO_3(R).$$

Using \mathbb{A}^1 -homotopy theory, Theorems 1.4 and 1.5 were proven for infinite fields of characteristic not 2 by Asok–Fasel in [AF17].

As is the case for most papers on homology stability of groups, we construct a highly connected complex on which our groups act, and we analyse the resulting spectral sequence. Our innovation is the use of the complex of non-degenerate unimodular sequences, a subcomplex of the complex of unimodular sequences used in [NS89] and [HT10], [Sch17a] to prove optimal homology stability for general linear and special linear groups. This leads to the introduction of odd rank symplectic groups $Sp_{2n-1}(R)$, and we prove a homology stability result in Theorem 8.3 for the extended sequence of groups

$$Sp_{-1}(R) = Sp_0(R) \subset Sp_1(R) \subset Sp_2(R) \subset \cdots \subset Sp_n(R) \subset Sp_{n+1}(R) \subset \cdots$$

Finally, we use the localisation techniques introduced by the second author in [Sch17a, §2] and [Sch21, Appendix D] to relate the homology of $\mathrm{Sp}_{2n+1}(R)$ to that of $\mathrm{Sp}_{2n}(R)$ in Proposition 7.1.

For most of the paper, we fix a commutative local ring R with infinite residue field and often suppress R from the notation. For instance, Sp_n , U_q , Skew_q^+ will mean $\mathrm{Sp}_n(R)$, $U_q(R)$, $\mathrm{Skew}_q^+(R)$.

We would like to thank the referee for his/her careful reading of the manuscript and for his/her useful suggestions.

2. THE COMPLEX OF NON-DEGENERATE UNIMODULAR SEQUENCES

Throughout this paper, ring means commutative ring. Let R be a ring. Let R^* denote the group of units of R under multiplication and $\mathrm{GL}_n(R)$ the group of all invertible $n \times n$ matrices with entries in R .

Let $\psi_{2n} = \psi_2 \oplus \cdots \oplus \psi_2$ be the standard hyperbolic symplectic form of rank $2n$

$$\psi_{2n} = \begin{pmatrix} \psi_2 & & & \\ & \psi_2 & & \\ & & \ddots & \\ & & & \psi_2 \end{pmatrix} = \bigoplus_1^n \psi_2, \quad \psi_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a ring R , the *symplectic group* $\mathrm{Sp}_{2n}(R) \subset \mathrm{GL}_{2n}(R)$, is the subgroup

$$\mathrm{Sp}_{2n}(R) = \{A \in \mathrm{GL}_{2n}(R) \mid {}^tA \psi_{2n} A = \psi_{2n}\}$$

of R -linear automorphisms preserving the form ψ_{2n} where tA denotes the transpose matrix of A . We will always consider $\mathrm{Sp}_{2n}(R)$ as a subgroup of $\mathrm{Sp}_{2n+2}(R)$ via the embedding

$$(2.1) \quad \mathrm{Sp}_{2n}(R) \subset \mathrm{Sp}_{2n+2}(R) : A \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A \end{pmatrix}.$$

For $n \geq 0$, the *symplectic group of rank $2n + 1$* is the subgroup

$$\mathrm{Sp}_{2n+1}(R) = \{A \in \mathrm{Sp}_{2n+2}(R) \mid Ae_1 = e_1\}$$

of $\mathrm{Sp}_{2n+2}(R)$ fixing the first standard basis vector e_1 . This is the group of matrices under multiplication

$$(2.2) \quad \begin{pmatrix} 1 & c & {}^t u \psi M \\ 0 & 1 & 0 \\ 0 & u & M \end{pmatrix}$$

where $\psi = \psi_{2n}$, $M \in \mathrm{Sp}_{2n}(R)$, $u \in R^{2n}$, $c \in R$. We let $\mathrm{Sp}_{-1}(R) = \{1\}$ be the trivial group. The inclusions (2.1) refine to the sequence of inclusions of groups

$$(2.3) \quad \mathrm{Sp}_{-1}(R) = \mathrm{Sp}_0(R) \subset \mathrm{Sp}_1(R) \subset \mathrm{Sp}_2(R) \subset \cdots \subset \mathrm{Sp}_n(R) \subset \mathrm{Sp}_{n+1}(R) \subset \cdots$$

where

$$(2.4) \quad \mathrm{Sp}_{2n}(R) \subset \mathrm{Sp}_{2n+1}(R) : M \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & M \end{pmatrix}, \quad \mathrm{Sp}_{2n-1}(R) \subset \mathrm{Sp}_{2n}(R) : M \mapsto M.$$

We shall denote the inclusions $\mathrm{Sp}_r(R) \subset \mathrm{Sp}_s(R)$ by ε_r^s , or simply by ε if source and target group are understood, $r \leq s$. Small rank symplectic groups are as follows

$$\mathrm{Sp}_{-1}(R) = \mathrm{Sp}_0(R) = \{1\}, \quad \mathrm{Sp}_1(R) = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \mid c \in R \right\}, \quad \mathrm{Sp}_2(R) = \mathrm{SL}_2(R).$$

We will study homology stability for the sequence of groups (2.3).

A *space* over a ring R is a projective R -module of finite rank. A map of spaces is a map of R -modules between spaces. A submodule $M \subset V$ of a space V is called *subspace* if it is a direct factor. A map of spaces $V \rightarrow W$ is called *split* if its image is a subspace. An element $x \in V$ is called *unimodular* if it generates a subspace $Rx \subset V$.

A *bilinear space* V over a commutative ring R is a space over R equipped with an R -bilinear form $V \times V \rightarrow R : (x, y) \mapsto \langle x, y \rangle$. It is called *symplectic* if $\langle x, x \rangle = 0$ for all $x \in V$. Note that symplectic forms satisfy $\langle x, y \rangle = -\langle y, x \rangle$. A symplectic space V is called *non-degenerate* or *regular* if the adjoint map $V \rightarrow V^* = \mathrm{Hom}_R(V, R) : x \mapsto \langle x, \cdot \rangle$ is split with kernel of minimal possible rank, i.e., if V has even rank then the kernel is required to be 0 (in which case the adjoint map is an isomorphism), and if V has odd rank, then the kernel is required to have rank 1.

From now on, let R be a commutative local ring. A basis $v = (v_1, \dots, v_q)$ of a space V over R of rank q determines the dual basis $v^\vee = (v_1^\vee, \dots, v_q^\vee)$ of V^* by the property $v_i^\vee(v_j) = \delta_{i,j}$. If V is equipped with a symplectic form then the adjoint map $V \rightarrow V^*$ in the basis v and v^\vee is the *Gram matrix*

$$\Gamma(v) = (\langle v_i, v_j \rangle)_{i,j=1}^q$$

of v . In particular, the form is non-degenerate if and only if $\Gamma(v)$ is split with kernel of rank 0 for q even and of rank 1 for q odd. A *symplectic basis* of a non-degenerate symplectic space V of rank $2n$ is an ordered basis $v_1, v_2, \dots, v_{2n-1}, v_{2n}$ of V such that $\langle v_{2r-1}, v_{2r} \rangle = 1$ and $\langle v_i, v_j \rangle = 0$ for $r = 1, \dots, n$, $1 \leq i < j \leq 2n$, $(i, j) \neq (2r-1, 2r)$. That is, a symplectic basis is an ordered basis whose Gram matrix is ψ_{2n} . Recall [MH73, Corollary 3.5] that every non-degenerate symplectic space of even rank over a commutative local ring has a symplectic basis.

Remark 2.1. Let $v = (v_1, \dots, v_q)$ be a basis of a symplectic space $(V, \langle \cdot, \cdot \rangle)$ of odd rank q . If (v_1, \dots, v_{q-1}) generates a non-degenerate subspace $W \subset V$ then V is non-degenerate. This is because $V = W \perp W^\perp$ as W is non-degenerate, and the orthogonal complement $W^\perp = \{x \in V \mid \langle x, y \rangle = 0 \ \forall y \in W\}$ of W in V is $W^\perp = Rx$ for some unimodular $x \in V$. Now, the Gram matrix in the basis (v_1, \dots, v_{q-1}, x) is $\begin{pmatrix} \Gamma(v_1, \dots, v_{q-1}) & 0 \\ 0 & 0 \end{pmatrix}$ which is split of rank $q-1$.

Lemma 2.2. *Let R be a local ring and $(R^{2n}, \langle \cdot, \cdot \rangle)$ a non-degenerate symplectic space. Let $V \subset R^{2n}$ be a non-degenerate subspace of rank $2r+1$. Then*

- (1) $V \cap V^\perp = Rx$ for some unimodular $x \in R^{2n}$.
- (2) V contains a non-degenerate subspace of rank $2r$.
- (3) If $V_0 \subset V$ is a non-degenerate subspace of rank $2r$, then $V = V_0 \perp Rx$ with $Rx = V \cap V^\perp$, x unimodular, and there is $y \in R^{2n}$ such that $y \in V_0^\perp$ and $\langle x, y \rangle = 1$.

Proof. (1) The R -linear map $\rho : V \rightarrow V^* : v \mapsto \langle v, \cdot \rangle$ has kernel $\ker(\rho) = V \cap V^\perp$ which is a rank 1 subspace as V is non-degenerate of odd rank.

(2) The form on V induces a unique non-degenerate form on $V/(V \cap V^\perp)$ such that the quotient map $V \rightarrow V/(V \cap V^\perp)$ preserves forms. Any $V_0 \subset V$ mapping isomorphically onto $V/(V \cap V^\perp)$ is a non-degenerate subspace of rank $2r$.

(3) Any non-degenerate $V_0 \subset V$ of rank $2r$ induces a map $V_0 \rightarrow V/(V \cap V^\perp)$ preserving forms. Since V_0 is non-degenerate and both have the same rank, this map is an isometry, and $V = V_0 \perp (V \cap V^\perp)$. We have $V \cap V^\perp = Rx$ for some unimodular x . Note that $x \in V_0^\perp$. Since the symplectic form on R^{2n} is non-degenerate its restriction to V_0^\perp is non-degenerate, too. In particular, there is $y \in V_0^\perp$ such that $\langle x, y \rangle = 1$. □

The following is well-known when the subspaces have even rank.

Corollary 2.3 (Witt's Theorem). *Let R be a local ring and $(R^{2n}, \langle \cdot, \cdot \rangle)$ a non-degenerate symplectic space. Let $V, W \subset R^{2n}$ be non-degenerate subspaces. If V and W have the same rank then there is an isometry $V \cong W$. Moreover, any isometry $g_0 : V \cong W$ extends to an isometry $g : R^{2n} \rightarrow R^{2n}$ such that $g|_V = g_0$.*

Proof. If V and W have even rank, they and their orthogonals V^\perp and W^\perp all have symplectic basis. Hence there are isometries $V \cong W$ and $V^\perp \cong W^\perp$. Any isometry $g_0 : V \cong W$ extends to an isometry $g = g_0 \perp g_1$ of $R^{2n} = V \perp V^\perp = W \perp W^\perp$ by choosing an isometry $g_1 : V^\perp \cong W^\perp$.

Now assume that V and W have odd rank $2r + 1$. By Lemma 2.2(1), we have $V \cap V^\perp = Rx$, $W \cap W^\perp = Ry$ for some unimodular $x, y \in R^{2n}$. By Lemma 2.2(2), we can choose non-degenerate subspaces $V_0 \subset V$ and $W_0 \subset W$ of rank $2r$. Note that the linear map sending a symplectic basis of V_0 to a symplectic basis of W_0 and x to y is an isometry $V \cong W$. Now, let $g_0 : V \cong W$ be any isometry. By Lemma 2.2(3), we can choose $x', y' \in R^{2n}$ such that $x' \in V_0^\perp$ and $y' \in W_0^\perp$ and such that $\langle x, x' \rangle = \langle y, y' \rangle = 1$. Then the isometry $g_0 : V \cong W$ extends to an isometry $V \perp Rx' \cong W \perp Ry'$ by sending x' to y' . Since the latter two spaces are non-degenerate of even rank, the last isometry extends to an isometry g of R^{2n} . □

Let $q \geq 0$ be an integer. A *unimodular sequence* of length q in R^{2n} is a sequence (v_1, \dots, v_q) of q vectors v_1, \dots, v_q in R^{2n} such that each subsequence

of length $r \leq \min(q, 2n)$ generates a subspace of rank r . Unimodular sequences were used in [NS89] and [HT10], [Sch17a] to prove optimal homology stability for general linear and special linear groups.

From now on consider R^{2n} equipped with the standard symplectic form where the standard basis e_1, \dots, e_{2n} has Gram matrix $\Gamma(e_1, \dots, e_{2n}) = \psi_{2n}$. A unimodular sequence (v_1, \dots, v_q) of length q in R^{2n} is called *non-degenerate* if any subsequence of length $r \leq \min(q, 2n)$ is a basis of a non-degenerate subspace of the symplectic space R^{2n} . For an integer $q \geq 0$, let

$$U_q(R^{2n}) = \{v = (v_1, \dots, v_q) \mid v \text{ non-degenerate unimodular in } R^{2n}\}$$

be the set of non-degenerate unimodular sequences of length q in R^{2n} . The set $U_0(R^{2n})$ is the singleton set consisting of the empty sequence, and the set $U_q(R^0)$ is the singleton set with unique element the sequence $(0, 0, \dots, 0)$ of length q . For a q -tuple $v = (v_1, \dots, v_q)$ of vectors $v_i \in R^{2n}$ and an ordered subset $I = \{i_1 < \dots < i_r\} \subset \{1, \dots, q\}$, we write $v_I = (v_{i_1}, \dots, v_{i_r})$. In view of Remark 2.1, a sequence $v = (v_1, \dots, v_q)$ of vectors in R^{2n} is in $U_q(R^{2n})$ if and only if it is unimodular and for any subset $I \subset \{1, \dots, q\}$ of even cardinality $\leq \min(q, 2n)$, the Gram matrix $\Gamma(v_I)$ is invertible.

We define a chain complex $(C_*(R^{2n}), d)$ where $C_q(R^{2n}) = \mathbb{Z}[U_q(R^{2n})]$ is the free abelian group generated by the set $U_q(R^{2n})$ and the \mathbb{Z} -linear differential is defined on the basis by

$$(2.5) \quad d_q(v_1, \dots, v_q) = \sum_{i=1}^q (-1)^{i+1} (v_1, \dots, \hat{v}_i, \dots, v_q).$$

Let $v = (v_1, \dots, v_q) \in U_q(R^{2n})$. A vector $x \in R^{2n}$ is said to be *in good position with respect to v* if $(v, x) \in U_{q+1}(R^{2n})$.

Remark 2.4. Let (R, m) be a local ring. If $k = R/m$ denotes the residue field of R , then x is in good position with respect to v if and only if its class $\bar{x} \in k^{2n}$ modulo m is in good position with respect to $\bar{v} = (\bar{v}_1, \dots, \bar{v}_q) \in U_q(k^{2n})$ over k .

In this paper, a matrix $A = (a_{ij}) \in M_q(R)$ is called *skew symmetric* if $a_{ij} = -a_{ji}$ and $a_{ii} = 0$ for all $1 \leq i, j \leq q$. For $q \geq 0$, let $\text{Skew}_q(R) \subset M_q(R)$ be the set of skew symmetric $q \times q$ -matrices with entries in R . For a skew symmetric matrix $A = (a_{ij}) \in M_{2n}(R)$, we denote by $\text{Pf}(A)$ its *Pfaffian*. It can be recursively defined by the formula

$$(2.6) \quad \text{Pf}(A) = \sum_{i=1}^{2n-1} (-1)^{i+1} a_{i,2n} \text{Pf}(A_{\widehat{i,2n}})$$

where $A_{\widehat{i,j}}$ denotes the matrix A with both the i -th and j -th rows and columns removed, and the Pfaffian of the unique 0×0 matrix is 1. For instance,

$$\text{Pf} \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a, \quad \text{Pf} \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

The Pfaffian satisfies $\text{Pf}(A)^2 = \det(A)$, $\text{Pf}({}^tUAU) = \det(U)\text{Pf}(A)$, $\text{Pf}(cA) = c^n \text{Pf}(A)$ and $\text{Pf}(\psi_{2n}) = 1$ for all $A \in \text{Skew}_{2n}(R)$ and $c \in R$.

Lemma 2.5. *Let k be a field, $q \geq 0$, $n \geq 1$ integers, and let $v = (v_1, \dots, v_q) \in U_q(k^{2n})$ be a non-degenerate unimodular sequence of length q in k^{2n} equipped with the standard symplectic form $\langle \cdot, \cdot \rangle$. Then there is a finite set of subspaces $V_1, \dots, V_s \subset k^{2n}$ of rank $< 2n$ such that every $x \in k^{2n} - \bigcup_{j=1}^s V_j$ is in good position with respect to v .*

Proof. Let $I \subset \{1, \dots, q\}$ be a subset of cardinality $< 2n$. If I has even cardinality, let $V_I \subset k^{2n}$ be the subspace generated by v_I . Then for any $x \in k^{2n} \setminus V_I$, the vectors v_i , $i \in I$ and x generate a non-degenerate subspace of k^{2n} , and $\dim V_I < 2n$. If I has odd cardinality, consider the k -linear map $k^{2n} \rightarrow k$:

$$x \mapsto \text{Pf}(\Gamma(v_I, x)) = \langle u, x \rangle$$

where $u = \sum_{i \in I} \varepsilon_i \text{Pf}(\Gamma(v_{I-\{i\}}))v_i$ and ε_i is 1 or -1 ; see (2.6). Let $V_I = u^\perp$ be the kernel of that map. Since each $\text{Pf}(\Gamma(v_{I-\{i\}}))$ is a unit ($|I|$ being odd), the vector u is unimodular and $\dim(u^\perp) < 2n$. Then for any $x \in k^{2n} \setminus V_I$, the vectors v_i , $i \in I$ and x generate a non-degenerate subspace of k^{2n} , and $\dim V_I < 2n$. Now any $x \in k^{2n} \setminus \bigcup_{I \subset \{1, \dots, q\}, |I| < 2n} V_I$ is in good position with respect to v . \square

Corollary 2.6. *Let (R, m) be a local ring with infinite residue field. Then the chain complex $C_*(R^{2n})$ is acyclic, that is, for all $q \geq 0$, we have*

$$H_q(C_*(R^{2n})) = 0.$$

Proof. Let $\xi = \sum_{i=1}^r n_i v_i \in C_q(R^{2n})$ be a cycle where $n_i \in \mathbb{Z}$ and $v_i \in U_q(R^{2n})$. Since R/m is infinite, there is $x \in R^{2n}$ which is in good position with respect to all v_i , $i = 1, \dots, r$, by Lemma 2.5 and Remark 2.4. Then $(x, \xi) \in C_{q+1}$ and $d(x, \xi) = \xi - (x, d\xi) = \xi$, that is, ξ is a boundary. \square

Let $\text{Skew}_q^+(R) \subset \text{Skew}_q(R)$ be the subset of those skew symmetric matrices A such that for all $\emptyset \neq I \subset \{1, \dots, q\}$ of even cardinality, the submatrix A_I with columns and rows in I is invertible. For $q = 0, \dots, 2n + 1$, the map

$$\Gamma : U_q(R^{2n}) / \text{Sp}_{2n}(R) \rightarrow \text{Skew}_q(R) : v = (v_1, \dots, v_q) \mapsto \Gamma(v) = (\langle v_i, v_j \rangle)_{i,j}$$

sending a non-degenerate unimodular sequence to its Gram matrix has image in $\text{Skew}_q^+(R)$.

For integers $q \geq j \geq 1$, a sequence v of length q and $A \in \text{Skew}_q$ a skew-symmetric matrix, denote by $v_{\hat{j}}$ the sequence of length $q - 1$ obtained from v by removing the j -th entry, and denote by $A_{\hat{j}} \in \text{Skew}_{q-1}$ the skew-symmetric matrix obtained from A by removing the j -th row and column.

Construction 2.7. For integers $0 \leq i \leq 2n + 1$ there are unique sections

$$V_{i, 2n} : \text{Skew}_i^+(R) \rightarrow U_i(R^{2n})$$

of Γ such that

- (1) $V_{0,2n}$ is the empty sequence, and $V_{1,0} = (0)$.
- (2) $V_{i,2n}(A) = V_{i,2m}(A)$ for $i \leq 2n \leq 2m$ under the standard embedding $R^{2n} \subset R^{2m} : e_i \mapsto e_i$.
- (3) $V_{i+1,2n}(A)_{\widehat{i+1}} = V_{i,2n}(A_{\widehat{i+1}})$, $A \in \text{Skew}_{i+1}^+(R)$, $0 \leq i \leq 2n$.
- (4) For $A \in \text{Skew}_{2n+1}^+(R)$ we have

$$V_{2n+1,2n}(A) = (v_1, \dots, v_{2n}, w_{2n+1}), \quad V_{2n+1,2n+2}(A) = (v_1, \dots, v_{2n}, v_{2n+1})$$

where $(v_1, \dots, v_{2n}) = V_{2n,2n}(A_{\widehat{2n+1}})$, $w_{2n+1} \in R^{2n}$ is the unique element such that $\langle v_i, w_{2n+1} \rangle = A_{i,2n+1}$ for all $0 \leq i \leq 2n$, and

$$v_{2n+1} = w_{2n+1} + e_{2n+1}.$$

- (5) For $A \in \text{Skew}_{2n+2}^+(R)$, write $V_{2n+1,2n+2}(A_{\widehat{2n+2}}) = (v_1, \dots, v_{2n}, v_{2n+1})$ and $V_{2n+1,2n}(A_{\widehat{2n+1}}) = (v_1, \dots, v_{2n}, w_{2n+2})$. Then $V_{2n+2,2n+2}(A) = (v_1, \dots, v_{2n+2})$ with

$$v_{2n+2} = w_{2n+2} + (A_{2n+1,2n+2} - \langle v_{2n+1}, w_{2n+2} \rangle) e_{2n+2}.$$

From the recursive nature of the construction satisfying (1) - (5) it is clear that the sections exist and are unique.

Lemma 2.8. *For $0 \leq q \leq 2n + 1$ the following map is bijective*

$$\Gamma : U_q(R^{2n}) / \text{Sp}_{2n}(R) \rightarrow \text{Skew}_q^+(R).$$

Proof. Surjectivity for $0 \leq q \leq 2n + 1$ follows from the existence of the sections in Construction 2.7.

For injectivity, let $v = (v_1, \dots, v_q)$ and $w = (w_1, \dots, w_q)$ be in $U_q(R^{2n})$ and assume that $\Gamma(v) = \Gamma(w)$. First assume $q \leq 2n$. Then v and w span non-degenerate subspaces V and W of R^{2n} . Since $\Gamma(v) = \Gamma(w)$, the linear map sending v_i to w_i is an isometry $V \cong W$. By Witt's theorem (Corollary 2.3), this isometry extends to an isometry of R^{2n} . In particular, $[v] = [w] \in U_q(R^{2n}) / \text{Sp}_{2n}(R)$.

Now assume $q = 2n + 1$. There is a unique $g \in \text{Sp}_{2n}(R)$ such that $(v_1, \dots, v_{2n}) = g(w_1, \dots, w_{2n})$. So, we can assume $(v_1, \dots, v_{2n}) = (w_1, \dots, w_{2n})$. Now the bijectivity of the map

$$(2.7) \quad R^{2n} \xrightarrow{\cong} R^{2n} : v \mapsto (\langle v_i, v \rangle)_{i=1}^{2n}.$$

shows that we also have $v_{2n+1} = w_{2n+1}$. □

For $i = 1, \dots, q$, we define the maps

$$\text{Skew}_q^+(R) \rightarrow \text{Skew}_{q-1}^+(R) : A \mapsto A_i^\wedge$$

omitting the i -th row and column. We make the graded abelian group $\mathbb{Z}[\text{Skew}_*^+(R)]$ into a chain complex with the differentials

$$(2.8) \quad d_q : \mathbb{Z}[\text{Skew}_q^+(R)] \rightarrow \mathbb{Z}[\text{Skew}_{q-1}^+(R)] : [A] \mapsto \sum_{i=1}^q (-1)^{i+1} [A_i^\wedge].$$

It is easy to check that $d_q d_{q+1} = 0$.

Lemma 2.9. *Let R be a local ring with infinite residue field. Then the chain complex $(\mathbb{Z}[\text{Skew}_*^+(R)], d_*)$ is acyclic. That is, for all $p \geq 0$ we have*

$$H_p(\mathbb{Z}[\text{Skew}_*^+(R)]) = 0.$$

For $A \in \text{Skew}_q^+(R)$ and $v \in R^q$, denote by $A * v$ the matrix

$$A * v = \begin{pmatrix} A & v \\ -v & 0 \end{pmatrix} \in \text{Skew}_{q+1}(R).$$

Note that $A * v \in \text{Skew}_{q+1}^+(R)$ if and only if for all $I = \{i_1 < \dots < i_r\} \subset \{1, \dots, q\}$ of odd cardinality r , the Pfaffian $\text{Pf}(A_I * v_I)$ of $A_I * v_I$ is a unit in R , or equivalently non zero in the residue field k of R .

Proof of Lemma 2.9. Let $\xi = \sum_{j=1}^m n_j [A_j] \in \mathbb{Z}[\text{Skew}_q^+(R)]$. We need to show that if ξ is a cycle then it is a boundary. Since $\text{Skew}_1^+(R) = \text{Skew}_0^+(R)$ is the singleton set, this is clear for $q = 0$, and we can assume $q \geq 1$.

Let k be the residue field of R and let $I = \{i_1 < \dots < i_r\} \subset \{1, \dots, q\}$ have odd cardinality. The recursive formula for the Pfaffian (2.6) shows that the map $k^q \rightarrow k$:

$$\bar{v} \mapsto \text{Pf}(A_I * \bar{v}_I) = (\pm \text{Pf}(A_{I-i_1}), \dots, \pm \text{Pf}(A_{I-i_r})) \cdot \bar{v}_I$$

is k -linear. Since $A \in \text{Skew}_q^+(R)$, all entries $\pm \text{Pf}(A_{I-i_s})$ are units and the k -linear map is non-zero. Thus, its kernel $V_{A,I}$ has codimension 1 in k^q .

Since the residue field k of R is infinite, the set

$$k^q \setminus \bigcup_{j=1, \dots, m, I \subset \{1, \dots, q\}, |I| \text{ odd}} V_{A_j, I}$$

is non-empty. In particular, there is $v \in R^q$ which is mapped into that set under the map $R \rightarrow k$. By the discussion above, $A_j * v \in \text{Skew}_{q+1}^+(R)$ for all $j = 1, \dots, m$. Then $\xi * v = \sum_{j=1}^m n_j [A_j * v] \in \mathbb{Z}[\text{Skew}_{q+1}^+(R)]$ and $d(\xi * v) = (-1)^q \xi + (d\xi) * v$. In particular, if ξ is a cycle then it is a boundary. \square

Corollary 2.10. *Let R be a local ring with infinite residue field. Then the complex $\mathbb{Z}[U_*(R^{2n})/\text{Sp}_{2n}(R)]$ is acyclic in degrees $\leq 2n$, that is,*

$$H_p(\mathbb{Z}[U_*(R^{2n})/\text{Sp}_{2n}(R)]) = 0, \quad 0 \leq p \leq 2n.$$

Proof. Combine Lemmas 2.8 and 2.9. \square

3. THE HOMOLOGY SPECTRAL SEQUENCE AND ITS E^1 -PAGE

For a group G , let $\mathbb{Z}[G]$ denote its integral group ring. Most of our computations are concerned with the homology [Bro94]

$$H_*(G; M) = \text{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M) = H_*(\mathbb{Z}[EG] \otimes_G M)$$

of G with coefficient in a (bounded below) complex of left G -modules M . Here $\mathbb{Z}[EG]$ is the complex of right G -modules which in degree n is the free \mathbb{Z} -module on the right G -set $E_n G = G^{n+1}$ and differential d_n defined on basis elements by $d_n(g_0, \dots, g_n) = \sum_{i=0}^n (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_n)$. The complex $\mathbb{Z}[EG]$

is a resolution of the trivial G -module \mathbb{Z} by the free right $\mathbb{Z}[G]$ -modules $\mathbb{Z}[E_n G]$ with basis $B_n G = G^n$.

Example 3.1. If $M = \mathbb{Z}[U]$ is the free group on a left G -set U , the low degree homology groups $H_i(G, \mathbb{Z}[U])$, $i = 0, 1$, are the homology groups of the complex

$$\mathbb{Z}[G^2 \times U] \xrightarrow{d_2} \mathbb{Z}[G \times U] \xrightarrow{d_1} \mathbb{Z}[U] \xrightarrow{d_0} 0$$

where $d_2[g, h, u] = [h, u] - [gh, u] + [g, hu]$ and $d_1[g, u] = [u] - [gu]$, $g, h \in G$, $u \in U$. In particular, $H_0(G, \mathbb{Z}[U]) = \mathbb{Z}[G \setminus U]$ and $H_1(G, \mathbb{Z}[U])$ is the group of elements $\sum_{i=1}^n m_i [g_i, u_i] \in \mathbb{Z}[G \times U]$ satisfying $\sum_{i=1}^n m_i [g_i u_i] = \sum_{i=1}^n m_i [u_i] \in \mathbb{Z}[U]$ modulo the relation $[gh, u] = [h, u] + [g, hu]$.

Example 3.2. Continuing the previous example, if $U = *$ is the one-element set, then we have the isomorphism

$$H_1(G, \mathbb{Z}) \xrightarrow{\cong} G^{ab} : \sum_{i=1}^n m_i [g_i] \mapsto \prod_{i=1}^n g_i^{m_i}.$$

Consider the category \mathcal{C} whose objects are the pairs (G, M) , where G is a group and M is a bounded below complex of left G -modules. The arrows $(G, M) \rightarrow (G', M')$ in \mathcal{C} are pairs (α, f) where $\alpha : G \rightarrow G'$ is a group homomorphism and $f : M \rightarrow M'$ is a chain map such that $f(a \cdot x) = \alpha(a) \cdot f(x)$ for $a \in G$ and $x \in M$. Composition in \mathcal{C} is composition of the α 's and f 's. An arrow $(\alpha, f) : (G, M) \rightarrow (G', M')$ defines a map on homology

$$(\alpha, f)_* : H_*(G; M) \rightarrow H_*(G'; M')$$

induced by the chain map

$$\mathbb{Z}[EG] \otimes_G M \rightarrow \mathbb{Z}[EG'] \otimes_{G'} M' : (a_0, \dots, a_n) \otimes x \mapsto (\alpha(a_0), \dots, \alpha(a_n)) \otimes f(x).$$

Let $(\alpha_0, f_0), (\alpha_1, f_1) : (G, M) \rightarrow (G', M')$ be two arrows in \mathcal{C} . Assume for simplicity that M and M' are G and G' -modules, respectively. If there is an element $h \in G'$ such that $\alpha_1(a) = h\alpha_0(a)h^{-1}$ and $f_1(x) = hf_0(x)$ for all $a \in G$ and $x \in M$, then the induced maps on homology agree:

$$(3.1) \quad (\alpha_0, f_0)_* = (\alpha_1, f_1)_* : H_*(G; M) \rightarrow H_*(G'; M').$$

For a complex M of left G -modules with $M_i = 0$ for $i < 0$, the stupid filtration $M_{\leq 0} \subset M_{\leq 1} \subset M_{\leq 2} \subset \dots \subset M$ of M defined by

$$(3.2) \quad (M_{\leq r})_i = \begin{cases} M_i & i \leq r \\ 0 & i > r \end{cases}$$

has quotients $M_{\leq q}/M_{\leq q-1}$ the G -module M_q placed in homological degree q . This defines the spectral sequence

$$(3.3) \quad E_{p,q}^1 = H_p(G; M_q) \Rightarrow H_{p+q}(G; M)$$

with differential d^r of bidegree $(r-1, -r)$. The spectral sequence is functorial for maps in \mathcal{C} .

Let us recall the chain complex $(C_*(R^{2n}), d)$ of left $\mathrm{Sp}_{2n}(R)$ -modules from Section 2 and its truncation (3.2), the subcomplex $C_{\leq 2n+1}(R^{2n})$, which is the free abelian group $C_q(R^{2n}) = \mathbb{Z}[U_q(R^{2n})]$ on the set of non-degenerate unimodular sequences of length q in R^{2n} for $0 \leq q \leq 2n+1$ and which is 0 otherwise. The differential was defined in (2.5).

Lemma 3.3. *Let R be a local ring with infinite residue field. Then for all integers n, r with $0 \leq r \leq 2n$ we have*

$$H_r(\mathrm{Sp}_{2n}(R); C_{\leq 2n+1}(R^{2n})) = 0.$$

Proof. For a group G and a bounded below complex of left G -modules M , the stupid filtration $\mathbb{Z}[EG]_{\leq 0} \subset \mathbb{Z}[EG]_{\leq 1} \subset \mathbb{Z}[EG]_{\leq 2} \subset \dots$ of $\mathbb{Z}[EG]$ defined by

$$(\mathbb{Z}[EG]_{\leq q})_r = \begin{cases} \mathbb{Z}[E_r G], & r \leq q \\ 0, & r > q \end{cases}$$

induces a spectral sequence

$$E_{p,q}^1 = H_p(\mathbb{Z}[E_q G] \otimes_G M) = \mathbb{Z}[B_q G] \otimes H_p(M) \Rightarrow H_{p+q}(G; M)$$

where $B_q G = G^q$. For $G = \mathrm{Sp}_{2n}(R)$ and $M = C_{\leq 2n+1}(R^{2n})$, we have $E_{p,q}^1 = 0$ for $p \leq 2n$ in view of Corollary 2.6. The spectral sequence then implies that $H_r(\mathrm{Sp}_{2n}(R); C_{\leq 2n+1}(R^{2n})) = 0$ for all $r \in \mathbb{Z}$ with $r \leq 2n$. \square

For $G = \mathrm{Sp}_{2n}(R)$ and $M = C_{\leq 2n+1}(R^{2n})$, the spectral sequence (3.3) has the form

$$(3.4) \quad E_{p,q}^1(2n) \Rightarrow H_{p+q}(\mathrm{Sp}_{2n}(R), C_{\leq 2n+1}(R^{2n}))$$

where the differential d^r is of bidegree $(r-1, -r)$ and

$$E_{p,q}^1(2n) = \begin{cases} H_p(\mathrm{Sp}_{2n}(R), \mathbb{Z}[U_q(R^{2n})]) & q \leq 2n+1 \\ 0 & q > 2n+1. \end{cases}$$

The spectral sequence converges to 0 for $p+q \leq 2n$ in view of Lemma 3.3. We need to determine explicitly the d^1 -differentials $d_{p,q}^1 : E_{p,q}^1(2n) \rightarrow E_{p,q-1}^1(2n)$ which, for $q \leq 2n+1$, are the maps

$$d_{p,q}^1 = (1, d)_* : H_p(\mathrm{Sp}_{2n}(R), \mathbb{Z}[U_q(R^{2n})]) \rightarrow H_p(\mathrm{Sp}_{2n}(R), \mathbb{Z}[U_{q-1}(R^{2n})])$$

where $d : \mathbb{Z}[U_q(R^{2n})] \rightarrow \mathbb{Z}[U_{q-1}(R^{2n})]$ is the differential of the complex $C_*(R^{2n})$. In order to do so, we recall Shapiro's Lemma. Let G be a group acting on a set S from the left. Shapiro's Lemma gives an isomorphism

$$\bigoplus (i_x, x)_* : \bigoplus_{[x] \in G \backslash S} H_*(G_x; \mathbb{Z}) \xrightarrow{\cong} H_*(G; \mathbb{Z}[S])$$

of homology groups where the direct sum is over a set of representatives $x \in S$ of equivalence classes $[x] \in G \backslash S$, the group $G_x = \{a \in G \mid ax = x\}$ is the stabiliser of G at $x \in S$, the homomorphism $i_x : G_x \subset G$ is the inclusion, and x also denotes the homomorphism of abelian groups $\mathbb{Z} \rightarrow \mathbb{Z}[S] : 1 \mapsto x$.

We will apply Shapiro's Lemma to $G = \mathrm{Sp}_{2n}(R)$ and $S = U_q(R^{2n})$. To ease notation we will write $\mathrm{St}_{2n}(v)$ for the stabiliser of $\mathrm{Sp}_{2n}(R)$ at a sequence $v = (v_1, \dots, v_q)$ of vectors v_i in R^{2n} . For the sections of Γ defined in Construction 2.7 we have $\mathrm{St}_{2n}(V_{q,2n}(A)) = \mathrm{St}_{2n}(e_1, \dots, e_q) = \mathrm{Sp}_{2n-q}(R)$ for $q = 0, \dots, 2n$ since $V_{q,2n}(A) = (v_1, \dots, v_q)$ and (e_1, \dots, e_q) span the same subspace of R^{2n} . Moreover, $\mathrm{St}_{2n}(V_{2n+1,2n}(A)) = \mathrm{St}_{2n}(e_1, \dots, e_{2n}) = \{1\} = \mathrm{Sp}_{-1}(R)$ is the trivial group.

Recall from Lemma 2.8 that the Gram matrix defines a bijection of sets

$$\Gamma : \mathrm{Sp}_{2n}(R) \backslash U_q(R^{2n}) \xrightarrow{\cong} \mathrm{Skew}_q^+(R) : [v] \mapsto \Gamma(v), \quad 0 \leq q \leq 2n+1.$$

In the following we shall denote the inclusions $\mathrm{Sp}_r(R) \subset \mathrm{Sp}_s(R)$ generically by ε for $r \leq s$. By Shapiro's lemma, the following map is an isomorphism

(3.5)

$$H_*(\mathrm{Sp}_{2n-q}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Skew}_q^+] \xrightarrow{\cong} H_*(\mathrm{Sp}_{2n}; \mathbb{Z}[U_q(R^{2n})]), \quad 0 \leq q \leq 2n+1$$

where $\alpha \otimes A$ is sent to $(\varepsilon, V_{q,2n}(A))_*(\alpha)$.

Lemma 3.4. *For $0 \leq q \leq 2n$ the following diagram is commutative*

(3.6)

$$\begin{array}{ccc} H_*(\mathrm{Sp}_{2n-q-1}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Skew}_{q+1}^+] & \xrightarrow[\cong]{(3.5)} & H_*(\mathrm{Sp}_{2n}; \mathbb{Z}[U_{q+1}(R^{2n})]) \\ \varepsilon_* \otimes d \downarrow & & \downarrow (1,d)_* \\ H_*(\mathrm{Sp}_{2n-q}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Skew}_q^+] & \xrightarrow[\cong]{(3.5)} & H_*(\mathrm{Sp}_{2n}; \mathbb{Z}[U_q(R^{2n})]). \end{array}$$

Proof. Recall that $d = \sum_{i=1}^{q+1} (-1)^{i+1} d_i$ where d_i omits the i -th entry. Write V_q for the section $V_{q,2n}$ defined in Construction 2.7. We will show that for all $A \in \mathrm{Skew}_{q+1}^+(R)$ and all $1 \leq i \leq q+1$, the two maps

$$(\varepsilon, d_i V_{q+1}(A))_*, (\varepsilon, V_q(d_i A))_* : H_*(\mathrm{Sp}_{2n-q-1}; \mathbb{Z}) \longrightarrow H_*(\mathrm{Sp}_{2n}; \mathbb{Z}[U_q(R^{2n})])$$

are equal. For $A \in \mathrm{Skew}_{q+1}^+(R)$ we will construct $B \in \mathrm{Sp}_{2n}(R)$ such that

$$B(d_i V_{q+1}(A)) = V_q(d_i A) \text{ and } BC = CB \text{ for all } C \in \mathrm{Sp}_{2n-q-1}(R) \subset \mathrm{Sp}_{2n}(R).$$

The existence of such B implies $(\varepsilon, d_i V_{q+1}(A))_* = (\varepsilon, V_q(d_i A))_*$; see (3.1).

Assume q is odd. All entries of $V_q(d_i A)$ and $V_{q+1}(A)$, and hence of $d_i V_{q+1}(A)$, are vectors in the non-degenerate subspace $R^{q+1} \subset R^{2n}$ of even rank $q+1$. The unimodular sequences $d_i V_{q+1}(A)$ and $V_q(d_i A)$ are basis of non-degenerate subspaces W_1 and W_2 of R^{q+1} of rank q . Since both sequences have the same Gram matrix, the linear isomorphism $W_1 \cong W_2$ sending $d_i V_{q+1}(A)$ to $V_q(d_i A)$ is an isometry. By Corollary 2.3 this isometry extends to an isometry $B : R^{q+1} \rightarrow R^{q+1}$ which we extend to all of R^{2n} by the requirement $Be_i = e_i$ for $i = q+2, \dots, 2n$. Then $B \in \mathrm{Sp}_{2n}(R)$ satisfies $B(d_i V_{q+1}(A)) = V_q(d_i A)$ and commutes with any $C \in \mathrm{Sp}_{2n-q-1}(R) \subset \mathrm{Sp}_{2n}(R)$.

Assume q even. If $q = 0$ then $dV_{q+1}(A) = V_q(dA)$ is the empty sequence and we are done. If $q = 2n$, then $d_i V_{q+1}(A)$ and $V_q(d_i A)$ are both basis of R^{2n} . Since $\Gamma(dV_{q+1}(A)) = \Gamma(V_q(dA))$, the linear isomorphism $B : R^{2n} \rightarrow R^{2n}$ sending $d_i V_{q+1}(A)$ to $V_q(d_i A)$ is an isometry which commutes with every element of $\mathrm{Sp}_{-1}(R) = \{1\}$.

Assume q is even, $2 \leq q \leq 2n-2$ and $i = q+1$. Then $d_i V_{q+1}(A) = V_q(d_i A)$ and we can choose B to be the identity matrix.

Finally, assume $q = 2r$ is even, $2 \leq q \leq 2n-2$ and $1 \leq i \leq q$. Recall that $A \in \mathrm{Skew}_{2r+1}^+$. Writing $V_{q+1}(A) = (v_1, v_2, \dots, v_{2r+1})$, we have $d_i V_{q+1}(A) = (v_1, \dots, \hat{v}_i, \dots, v_{2r+1})$ where $v_{2r+1} = w_{2r+1} + e_{2r+1}$ as in Construction 2.7 (4) with $v_1, \dots, v_{2r}, w_{2r+1} \in R^{2r}$. Since e_{2r+1} is perpendicular to R^{2r} , the two sequences $(v_1, \dots, \hat{v}_i, \dots, v_{2r+1})$ and $(v_1, \dots, \hat{v}_i, \dots, w_{2r+1})$ have the same (invertible) Gram matrix. In particular, the latter sequence defines a basis of R^{2r} . It follows that $v_+ = (v_1, \dots, \hat{v}_i, \dots, v_{2r+1}, e_{2r+1})$ is a basis of R^{2r+1} . Write $V_q(d_i A) = (u_1, \dots, u_{2r})$. By Construction 2.7, this is a basis of R^{2r} . It follows that v_+ and $u_+ = (u_1, \dots, u_{2r}, e_{2r+1})$ both define a basis of R^{2r+1} . Since $\Gamma(v_+) = (d_i A) * 0 = \Gamma(u_+)$, the linear endomorphism of R^{2r+1} that sends v_+ to u_+ is an isometry. By Corollary 2.3, this isometry extends to an isometry of R^{2r+2} which we can extend to an isometry $B : R^{2n} \rightarrow R^{2n}$ such that $Be_i = e_i$, $i = 2r+3, \dots, 2n$. Note that $B(d_i V_{q+1}(A)) = V_q(d_i A)$ and $Be_{2r+1} = e_{2r+1}$. Now, this $B \in \mathrm{Sp}_{2n}(R)$ and any $C \in \mathrm{Sp}_{2n-2r-1}(R) \subset \mathrm{Sp}_{2n}(R)$ have matrix representations

$$B = \begin{pmatrix} M & 0 & u & 0 \\ {}^t v & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{2n-2r-2} \end{pmatrix}, \quad C = \begin{pmatrix} 1_{2r} & 0 & 0 & 0 \\ 0 & 1 & c & {}^t y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x & N \end{pmatrix} \in \mathrm{Sp}_{2n}(R)$$

where $M \in \mathrm{Sp}_{2r}(R)$, $N \in \mathrm{Sp}_{2n-2r-2}(R)$, $x, y \in R^{2n-2r-2}$, $u, v \in R^{2r}$ and $b, c \in R$. Any two such matrices commute because

$$\begin{pmatrix} M & 0 & u & 0 \\ {}^t v & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & {}^t y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x & N \end{pmatrix} = \begin{pmatrix} M & 0 & u & 0 \\ {}^t v & 1 & b+c & {}^t y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x & N \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & {}^t y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & x & N \end{pmatrix} \begin{pmatrix} M & 0 & u & 0 \\ {}^t v & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This finishes the proof. \square

Corollary 3.5. *For $n \geq 1$ and $p \geq 0$, the following differential is trivial:*

$$0 = d_{p,2}^1 : E_{p,2}^1(R^{2n}) \rightarrow E_{p,1}^1(R^{2n}).$$

Proof. By Lemma 3.4, this differential is the map

$$\varepsilon_* \otimes d : H_*(\mathrm{Sp}_{2n-2}(R)) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Skew}_2^+(R)] \longrightarrow H_*(\mathrm{Sp}_{2n-1}(R)) \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Skew}_1^+(R)]$$

But $d : \mathbb{Z}[\mathrm{Skew}_2^+(R)] \rightarrow \mathbb{Z}[\mathrm{Skew}_1^+(R)]$ is the zero map. \square

4. TRIVIALITY OF $d_{p,q}^r$ FOR q EVEN

The goal in this section is to show that the differentials $d_{p,q}^r$ of the spectral sequence (3.4) vanish for $r \geq 2$ and $q < 2n$ even. Since $d \circ d = 0$ in the

complex $C_*(R^{2n})$, the diagram

$$\begin{array}{ccccc}
 0 & \longrightarrow & \mathbb{Z}[U_{2r+1}(R^{2r})] & \longrightarrow & 0 \\
 \downarrow & & \downarrow d & & \downarrow \\
 \mathbb{Z}[U_{2r+1}(R^{2n})] & \xrightarrow{d} & \mathbb{Z}[U_{2r}(R^{2n})] & \xrightarrow{d} & \mathbb{Z}[U_{2r-1}(R^{2n})]
 \end{array}$$

commutes and defines a map of complexes

$$\varphi : \mathbb{Z}[U_{2r+1}(R^{2r})][-2r] \rightarrow C_{\leq 2n+1}(R^{2n})$$

of $\mathrm{Sp}_{2n-2r}(R)$ -modules where $\mathrm{Sp}_{2n-2r}(R)$ acts trivially on the source complex $\mathbb{Z}[U_{2r+1}(R^{2r})][-2r]$ and via its inclusion $\varepsilon : \mathrm{Sp}_{2n-2r}(R) \subset \mathrm{Sp}_{2n}(R)$ on the target complex. The pair

$$(\varepsilon, \varphi) : (\mathrm{Sp}_{2n-2r}(R), \mathbb{Z}[U_{2r+1}(R^{2r})][-2r]) \longrightarrow (\mathrm{Sp}_{2n}(R), C_{\leq 2n+1}(R^{2n}))$$

defines a map of associated group homology spectral sequences

$$(4.1) \quad E_{p,q}^s(2n; r) \longrightarrow E_{p,q}^s(2n)$$

resulting from the stupid filtrations of the coefficient complexes; see (3.3). By definition, we have

$$E_{p,q}^1(2n; r) = \begin{cases} 0, & q \neq 2r \\ H_p(\mathrm{Sp}_{2n-2r}, \mathbb{Z}[U_{2r+1}(R^{2r})]), & q = 2r. \end{cases}$$

Proposition 4.1. *For integers $0 \leq r < n$, $s = 2$, and all $0 \leq p$, the map (4.1) is surjective in bidegree $(p, 2r)$:*

$$E_{p,2r}^2(2n; r) \rightarrow E_{p,2r}^2(2n).$$

Proof. The map $E_{p,2r}^1(2n; r) \rightarrow E_{p,2r}^1(2n)$ is the diagonal arrow in the diagram

$$\begin{array}{ccc}
 H_p(\mathrm{Sp}_{2n-2r}) \otimes \mathbb{Z}[U_{2r+1}(R^{2r})] & \xrightarrow{1 \otimes d \circ \Gamma} & H_p(\mathrm{Sp}_{2n-2r}) \otimes \mathbb{Z}[\mathrm{Skew}_{2r}^+] \\
 & \searrow & \downarrow (3.5) \\
 & & H_p(\mathrm{Sp}_{2n}, \mathbb{Z}[U_{2r}(R^{2n})])
 \end{array}$$

sending $\alpha \otimes v$ to $(\varepsilon, dv)_*(\alpha)$. This diagram commutes because $d_i v$ and $V_{2r,2n}(\Gamma d_i v)$ are two basis of the subspace $R^{2r} \subset R^{2n}$ that have the same Gram matrix. In particular, the automorphism $B : R^{2r} \rightarrow R^{2r}$ sending $d_i v$ to $V_{2r,2n}(\Gamma d_i v)$ is an isometry of R^{2r} which we extend to $B \in \mathrm{Sp}_{2n}(R)$ by requiring $B e_i = e_i$ for $i = 2r + 1, \dots, 2n$. Since B commutes with every element in $\mathrm{Sp}_{2n-2r}(R) \subset \mathrm{Sp}_{2n}(R)$ the diagram commutes. It follows that under the isomorphism (3.5), the map $E_{p,2r}^1(2n; r) \rightarrow E_{p,2r}^1(2n)$ is the first

map in the complex

$$\begin{array}{ccc} H_p(\mathrm{Sp}_{2n-2r}) \otimes \mathbb{Z}[U_{2r+1}] & \xrightarrow{1 \otimes d \circ \Gamma} & H_p(\mathrm{Sp}_{2n-2r}) \otimes \mathbb{Z}[\mathrm{Skew}_{2r}^+] \\ & & \downarrow \varepsilon_* \otimes d \\ & & H_p(\mathrm{Sp}_{2n-2r+1}) \otimes \mathbb{Z}[\mathrm{Skew}_{2r-1}^+]. \end{array}$$

In view of Lemma 3.4, the second map in that complex is $d_{p,2r}^1 : E_{p,2r}^1(2n) \rightarrow E_{p,2r-1}^1(2n)$. Since $\varepsilon_* : H_p(\mathrm{Sp}_{2n-2r}) \rightarrow H_p(\mathrm{Sp}_{2n-2r+1})$ is (split) injective, Lemmas 2.8 and 2.9 imply that this complex is exact. It follows that $E_{p,2r}^1(2n; r)$ surjects onto the kernel of the right vertical map which surjects onto $E_{p,2r}^2(2n)$. In particular, $E_{p,2r}^2(2n; r)$ surjects onto $E_{p,2r}^2(2n)$. \square

Corollary 4.2. *Let R be a local ring with infinite residue field. Then for $q < 2n$ even and $s \geq 2$, the spectral sequence (3.4) satisfies $d_{p,q}^s = 0$.*

Proof. The spectral sequence $E(2n; r)$ has all differentials $d_{p,q}^s = 0$, by definition. Thus, if $E_{p,q}^s(2n; r) \rightarrow E_{p,q}^s(2n)$ is surjective, then the differential $d_{p,q}^s$ of $E(2n)$ vanishes and the map $E_{p,q}^{s+1}(2n; r) \rightarrow E_{p,q+2r}^{s+1}(2n)$ is surjective. Therefore, Proposition 4.1 implies that all differentials leaving $E_{p,q}^s(2n)$ vanish for $s \geq 2$ and $q < 2n$ even. \square

5. A FORMULA FOR $d_{0,2n+1}^2$

Our aim is to show that the differential

$$d_{0,2n+1}^2 : E_{0,2n+1}^2(2n) \rightarrow E_{1,2n-1}^2(2n) = H_1(\mathrm{Sp}_{2n}, \mathbb{Z}[U_{2n-1}])$$

in the spectral sequence (3.4) is surjective at least when $2n = 4$. This will be achieved in Proposition 6.21. In this section we will give an explicit formula for this differential in Proposition 5.1. Note that $E_{1,2n-1}^2(2n) \subset E_{1,2n-1}^1(2n)$ since $E_{1,2n}^1(2n) = 0$.

Recall from Lemma 2.9 the surjection

$$d : \mathbb{Z}[\mathrm{Skew}_{2n+2}^+] \twoheadrightarrow E_{0,2n+1}^2(2n) = \ker \left(\mathbb{Z}[\mathrm{Skew}_{2n+1}^+] \xrightarrow{d} \mathbb{Z}[\mathrm{Skew}_{2n}^+] \right).$$

For every matrix $A \in \mathrm{Skew}_{2n-1}^+(R)$ we chose

$$v(A) = (v_1, \dots, v_{2n-1}) \in U_{2n-1}(R^{2n-1}) \subset GL_{2n-1}(R)$$

with $\Gamma(v(A)) = A$, $\det v(A) = 1$ and $v_i \in R^i$ for $i = 1, \dots, 2n-1$ where R^i is considered a subspace of R^{2n-1} via $e_j \mapsto e_j$, $j = 1, \dots, i$. This is possible, for if we write the section $V_{2n-1,2n-2}(A)$ of Construction 2.7 as $V_{2n-1,2n-2}(A) = v'(A) = (v_1, \dots, v_{2n-2}, v'_{2n-1})$ then $\Gamma(v'(A)) = A$ and $v(A) = (v_1, \dots, v_{2n-2}, v_{2n-1})$ has the required properties where $v_{2n-1} = v'_{2n-1} + \det_{R^{2n-2}}^{-1}(v_1, \dots, v_{2n-2}) \cdot e_{2n-1}$.

Using the identification $H_1 \mathrm{Sp}_1(R) \cong \mathrm{Sp}_1(R) \cong R : \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mapsto x$ of Example 3.2, Shapiro's Lemma yields the isomorphism

$$(5.1) \quad \begin{aligned} R [\mathrm{Skew}_{2n-1}^+(R)] &\xrightarrow{\cong} H_1(\mathrm{Sp}_{2n}(R), \mathbb{Z}[U_{2n-1}(R^{2n})]) \\ a \cdot [A] &\mapsto [e_{2n-1,2n}(a), v(A)] \end{aligned}$$

where $e_{ij}(a)$ denotes the standard elementary matrix with 1's on the diagonal, a at the (i, j) -spot and zero elsewhere, $i \neq j$.

For $A \in \mathrm{Skew}_{2n+2}^+(A)$ and subset $I \subset \{1, 2, \dots, 2n+2\}$, we denote by A_I^\wedge the skew-symmetric matrix obtained from A by removing all rows and columns in I . Now we can state the explicit formula for the differential under consideration.

Proposition 5.1. *Under the isomorphism (5.1), the composition*

$$\gamma : \mathbb{Z}[\mathrm{Skew}_{2n+2}^+(R)] \xrightarrow{d} E_{0,2n+1}^2(2n) \xrightarrow{d_{0,2n+1}^2} E_{1,2n-1}^2(2n) \subset E_{1,2n-1}^1(2n) = R[\mathrm{Skew}_{2n-1}^+(R)]$$

sends a generator $A \in \mathrm{Skew}_{2n+2}^+(R)$ to

$$\gamma(A) = \sum_{1 \leq i < j < k \leq 2n+2} (-1)^{i+j+k} \frac{\mathrm{Pf}(A)}{\mathrm{Pf}(A_{ij}^\wedge) \mathrm{Pf}(A_{ik}^\wedge) \mathrm{Pf}(A_{jk}^\wedge)} \cdot [A_{ijk}^\wedge].$$

Proof. The spectral sequence $E_{p,q}(2n)$ of (3.4) is the spectral sequence

$$E_{p,q}^2 = H_q(H_p(C, d^h), d^v) \Rightarrow H_{p+q}(\mathrm{Tot} C)$$

associated with the double complex $C_{p,q} = \mathbb{Z}[G^p \times U_q]$ where $G = \mathrm{Sp}_{2n}(R)$ and $U_q = U_q(R^{2n})$, $q \leq 2n+1$. Horizontal and vertical differentials d^h and d^v are induced by the differential in the Bar complex $\mathbb{Z}[G^* \times U_q]$ and the complex $\mathbb{Z}[U_*]$, respectively. For a spectral sequence arising from a double complex as above, the differential $d_{p,q}^2 : E_{p,q}^2 \rightarrow E_{p+1,q-2}^2$ is defined as follows. An element $[x] \in E_{p,q}^2$ is represented by an element $x \in C_{p,q}$ such that $d^h x = 0 \in C_{p-1,q}$ and there is $y \in C_{p+1,q-1}$ such that $d^h y = d^v x \in C_{p,q-1}$. Then $d_{p,q}^2[x] = [d^v y] \in E_{p+1,q-2}^2$.

In our case, let $A \in \mathrm{Skew}_{2n+2}^+(R)$. Then $d(A) = \sum_{i=1}^{2n+2} (-1)^{i+1} [A_i^\wedge] \in E_{0,2n+1}^2$ is represented by

$$\alpha = \sum_{i=1}^{2n+2} (-1)^{i+1} [(u_i)_i^\wedge] \in \mathbb{Z}[U_{2n+1}(R^{2n})] \in C_{0,2n+1}$$

where $u_i \in U_{2n+2}(R^{2n})$ is such that $\Gamma(u_i)_i^\wedge = A_i^\wedge$. To find such u_i , use Lemma 2.8 to find $(u_i)_i^\wedge \in U_{2n+1}(R^{2n})$ and then Lemma 2.5 to extend it to $u_i \in U_{2n+2}(R^{2n})$. For $1 \leq j \leq 2n+2$, $j \neq i$, the unimodular sequences $(u_i)_ij^\wedge \in U_{2n}(R^{2n}) \subset \mathrm{GL}_{2n}(R)$ are invertible matrices and we set

$$g_{ji} = (u_j)_ij^\wedge \circ ((u_i)_ij^\wedge)^{-1}.$$

Since $\Gamma\left((u_i)_{ij}^\wedge\right) = A_{i,j}^\wedge = \Gamma\left((u_j)_{ij}^\wedge\right)$, the change of basis matrix g_{ji} is an isometry. In particular, $g_{ji} \in \mathrm{Sp}_{2n}(R)$, and we set

$$\beta = \sum_{1 \leq i < j \leq 2n+2} (-1)^{i+j+1} [g_{ji}, (u_i)_{ij}^\wedge] \in \mathbb{Z}[\mathrm{Sp}_{2n} \times U_{2n}] = C_{1,2n}.$$

Using the equality

$$d^h [g_{ji}, (u_i)_{ij}^\wedge] = (u_i)_{ij}^\wedge - g_{ji} \cdot (u_i)_{ij}^\wedge = (u_i)_{ij}^\wedge - (u_j)_{ij}^\wedge,$$

we check that $d^h(\beta) = d^v(\alpha) \in C_{0,2n}$:

$$\begin{aligned} d^h(\beta) &= \sum_{1 \leq i < j \leq 2n+2} (-1)^{i+j+1} (u_i)_{ij}^\wedge + \sum_{1 \leq i < j \leq 2n+2} (-1)^{i+j} (u_j)_{ij}^\wedge \\ &= \sum_{i=1}^{2n+2} \sum_{j=i+1}^{2n+2} (-1)^{i+j+1} d_{j-1}((u_i)_i^\wedge) + \sum_{1 \leq j < i \leq 2n+2} (-1)^{i+j} (u_i)_{ij}^\wedge \\ &= \sum_{i=1}^{2n+2} \left(\sum_{j=i}^{2n+1} (-1)^{i+j} d_j((u_i)_i^\wedge) + \sum_{j=1}^{i-1} (-1)^{i+j} d_j((u_i)_i^\wedge) \right) \\ &= \sum_{j=1}^{2n+1} (-1)^{j+1} \sum_{i=1}^{2n+2} (-1)^{i+1} d_j((u_i)_i^\wedge) \\ &= d^v(\alpha). \end{aligned}$$

Thus, we have

$$\begin{aligned}
\gamma(A) &= d^v(\beta) \\
&= \sum_{k=1}^{2n} \sum_{1 \leq i < j \leq 2n+2} (-1)^{i+j+k} [g_{ji}, d_k(u_i)_{\widehat{ij}}] \\
&= \sum_{1 \leq k < i < j \leq 2n+2} (-1)^{i+j+k} [g_{ji}, (u_i)_{\widehat{ijk}}] \\
&\quad - \sum_{1 \leq i < k < j \leq 2n+2} (-1)^{i+j+k} [g_{ji}, (u_i)_{\widehat{ijk}}] \\
&\quad + \sum_{1 \leq i < j < k \leq 2n+2} (-1)^{i+j+k} [g_{ji}, (u_i)_{\widehat{ijk}}] \\
&= \sum_{1 \leq i < j < k \leq 2n+2} (-1)^{i+j+k} \left([g_{kj}, (u_j)_{\widehat{ijk}}] - [g_{ki}, (u_i)_{\widehat{ijk}}] + [g_{ji}, (u_i)_{\widehat{ijk}}] \right).
\end{aligned}$$

Note that $[g_{kj}, (u_j)_{\widehat{ijk}}] - [g_{ki}, (u_i)_{\widehat{ijk}}] + [g_{ji}, (u_i)_{\widehat{ijk}}]$ is a cycle for the horizontal differential and thus represents an element of $H_1(\mathrm{Sp}_{2n}, U_{2n-1})$. Using the identity $[gh, u] = [h, u] + [g, hu]$ in $H_1(G, U)$ together with $(u_j)_{\widehat{ijk}} = g_{ji} \cdot (u_i)_{\widehat{ijk}}$, this cycle is

$$\begin{aligned}
& [g_{kj}, g_{ji} \cdot (u_i)_{\widehat{ijk}}] - [g_{ki}, (u_i)_{\widehat{ijk}}] + [g_{ji}, (u_i)_{\widehat{ijk}}] \\
&= [g_{kj}g_{ji}, (u_i)_{\widehat{ijk}}] - [g_{ji}, (u_i)_{\widehat{ijk}}] - [g_{ki}, (u_i)_{\widehat{ijk}}] + [g_{ji}, (u_i)_{\widehat{ijk}}] \\
&= [g_{kj}g_{ji}, (u_i)_{\widehat{ijk}}] - [g_{ki}, (u_i)_{\widehat{ijk}}] \\
&= [g_{ik}g_{kj}g_{ji}, (u_i)_{\widehat{ijk}}]
\end{aligned}$$

where we also used $[g, u] - [h, u] = [h^{-1}g, u]$ if $gu = hu$, and $g_{ik} = g_{ki}^{-1}$. Hence,

$$\gamma(A) = \sum_{1 \leq i < j < k \leq 2n+2} (-1)^{i+j+k} [g_{ik}g_{kj}g_{ji}, (u_i)_{\widehat{ijk}}].$$

The proposition now follows from Lemma 5.2 below. \square

Lemma 5.2. *Let $A \in \mathrm{Skew}_{2n+2}^+(R)$ and $1 \leq i < j < k \leq 2n+2$. Then the element*

$$[g_{ik}g_{kj}g_{ji}, (u_i)_{\widehat{ijk}}] \in H_1(\mathrm{Sp}_{2n}(R), U_{2n-1}(R^{2n}))$$

is independent of the choice of $u_i, u_j, u_k \in U_{2n+2}(R^{2n})$ as long as $\Gamma(u_r)_r^\wedge = A_r^\wedge$ and $g_{rs} = (u_r)_r^\wedge \circ ((u_s)_r^\wedge)^{-1}$ for $r, s \in \{i, j, k\}$. Under the isomorphism (5.1), we have

$$(5.2) \quad [g_{ik}g_{kj}g_{ji}, (u_i)_{ijk}^\wedge] = \frac{\text{Pf}(A)}{\text{Pf}(A_{ij}^\wedge)\text{Pf}(A_{ik}^\wedge)\text{Pf}(A_{jk}^\wedge)} \cdot [A_{ijk}^\wedge].$$

Proof. Let $\tilde{u}_i, \tilde{u}_j, \tilde{u}_k \in U_{2n+2}(R^{2n})$ be another set of unimodular sequences with $\Gamma(\tilde{u}_r)_r^\wedge = A_r^\wedge$ and set $\tilde{g}_{rs} = (\tilde{u}_r)_{rs}^\wedge \circ ((\tilde{u}_s)_{rs}^\wedge)^{-1}$ for $r, s \in \{i, j, k\}$. Since $\Gamma : \text{Sp}_{2n} \setminus U_{2n+1}(R^{2n}) \cong \text{Skew}_{2n+1}^+(R)$ is a bijection, there is $h_r \in \text{Sp}_{2n}(R)$ such that $(\tilde{u}_r)_{rs}^\wedge = h_r(u_r)_{rs}^\wedge$ for $r \in \{i, j, k\}$. Then $\tilde{g}_{rs} = h_r \cdot g_{rs} \cdot h_s^{-1}$ and

$$[\tilde{g}_{ik}\tilde{g}_{kj}\tilde{g}_{ji}, (\tilde{u}_i)_{ijk}^\wedge] = [h_i(g_{ik}g_{kj}g_{ji})h_i^{-1}, h_i(u_i)_{ijk}^\wedge] = [g_{ik}g_{kj}g_{ji}, (u_i)_{ijk}^\wedge]$$

using $[hgh^{-1}, hu] = [g, u] \in H_1(G, U)$ for all $h, g \in G$ and $u \in U$ with $gu = u$. This proves independence of choices.

Next we want to reduce checking equation (5.2) for all indices $i < j < k$ to checking it just for $(i, j, k) = (2n, 2n+1, 2n+2)$. To that end, if $k < 2n+2$, let $B \in \text{Skew}_{2n+2}^+(R)$ be obtained from A by exchanging k -th row and column with the $k+1$ -st row and column, that is, let σ be the permutation of $\{1, \dots, 2n+2\}$ that exchanges k and $k+1$ and fixes everything else, then the entries of B satisfy $B_{r,s} = A_{\sigma(r), \sigma(s)}$. Note that $A_{i,j,k}^\wedge = B_{i,j,k+1}^\wedge$. Similarly, let v_i, v_j, v_{k+1} be obtained from u_i, u_j, u_k by exchanging the k -th and $k+1$ -st columns. That is, $v_r = (u_{\sigma(r)}) \circ \sigma$ where σ also denotes the $2n+2 \times 2n+2$ permutation matrix corresponding to the permutation σ above, $r \in \{i, j, k+1\}$. Then $\Gamma(v_r)_r^\wedge = B_r^\wedge$ for $r \in \{i, j, k+1\}$. The right hand side of (5.2) doesn't change if we replace A with B and i, j, k with $i, j, k+1$ since $\text{Pf}(A) = -\text{Pf}(B)$, $\text{Pf}(A_{i,j}^\wedge) = -\text{Pf}(B_{i,j})$, $\text{Pf}(A_{i,k}^\wedge) = \text{Pf}(B_{i,k+1})$ and $\text{Pf}(A_{j,k}^\wedge) = \text{Pf}(B_{j,k+1})$. For the left hand side, write $f_{r,s} = (v_r)_{r,s}^\wedge \circ ((v_s)_{r,s}^\wedge)^{-1}$ where $r, s \in \{i, j, k+1\}$. Then $f_{r,s} = g_{\sigma(r), \sigma(s)}$ for $r \neq s \in \{i, j, k+1\}$. In particular,

$$[f_{i,k+1}f_{k+1,j}f_{ji}, (v_i)_{ijk+1}^\wedge] = [g_{ik}g_{kj}g_{ji}, (u_i)_{ijk}^\wedge],$$

and the left hand side has not changed. Similarly, if $k = 2n+2$ and $j < 2n+1$ one shows that both sides of (5.2) remain unchanged if we replace A with the matrix B obtained from A by exchanging the j -th and $j+1$ st row and column. Finally, if $(j, k) = (2n+1, 2n+2)$ and $i < 2n$ we can exchange the i -th and $i+1$ st row and column in A without changing the sides of the equation (5.2).

Now we are reduced to checking equation (5.2) for $(i, j, k) = (2n, 2n+1, 2n+2)$. So, let $(i, j, k) = (2n, 2n+1, 2n+2)$ and set $I = \{i, j, k\}$. Recall that $v(A_I^\wedge) \in GL_{2n-1}(R)$ is an upper triangular matrix of determinant 1 and thus of the form

$$v(A_I^\wedge) = \begin{pmatrix} w & w_{2n-1} \\ 0 & \det^{-1} w \\ 0 & 0 \end{pmatrix} \in M_{2n, 2n-1}(R)$$

when the columns are considered as lying in R^{2n} . Note that $w \in GL_{2n-2}(R)$.

For $r \in I$, construct $u_r \in U_{2n+2}(R^{2n})$ as follows. Set $(u_r)_I^\wedge = v(A_I^\wedge)$, extend it first to $(u_r)_r^\wedge \in U_{2n+1}(R^{2n})$ with $\Gamma((u_r)_r^\wedge) = A_r^\wedge$ as in Construction 2.7 and then extend it to $u_r \in U_{2n+2}(R^{2n})$ using Lemma 2.5 and Remark 2.4. Then g_{rs} fixes $(u_r)_I^\wedge = v(A_I^\wedge) = (u_s)_I^\wedge$. Since the columns of $(u_r)_I^\wedge = v(A_I^\wedge)$ generate R^{2n-1} , the map g_{rs} fixes every vector in R^{2n-1} . Therefore, $g_{rs} = e_{2n-1, 2n}(c_{rs}) \in \mathrm{Sp}_1(R) \subset \mathrm{Sp}_{2n}(R)$ for unique $c_{rs} \in R$ and all $r, s \in I$. Note that $c_{rs} = -c_{sr}$ since $g_{rs} = g_{sr}^{-1}$. Under the identification (5.1) we then have

$$[g_{ik}g_{kj}g_{ji}, (u_i)_I^\wedge] = (c_{ik} + c_{kj} + c_{ji}) \cdot [A_I^\wedge].$$

Let $r, s, t \in I$ be such that $\{r, s, t\} = I$, and let u_s^r denote the s -th vector in the unimodular sequence $u^r := u_r$ and $u_{q,s}^r$ its q -th row entry. Then

$$u_s^r = \begin{pmatrix} w_s \\ d_s^r \\ \mathrm{Pf}(A_{rt}^\wedge) \end{pmatrix}$$

where $d_s^r \in R$, and $w_s \in R^{2n-2}$ is the unique solution to

$${}^t w \circ \psi \circ w_s = (A_{q,s})_{1 \leq q \leq 2n-2}$$

which expresses part of the equality $\Gamma(u^r)_r^\wedge = A_r^\wedge$. The last entry of u_s^r follows from the identity

$$\mathrm{Pf}(A_{rt}^\wedge) = \det(u^r)_{rt}^\wedge = u_{2n,s}^r \cdot \det v(A_I^\wedge) = u_{2n,s}^r$$

since the matrix $(u^r)_{rt}^\wedge = (v(A_I^\wedge), u_s^r)$ is upper triangular and $\det v(A_I^\wedge) = 1$.

For $s < t$ we have

$$(u^r)_{\widehat{2n-1}, r} = \begin{pmatrix} w & w_s & w_t \\ 0 & d_s^r & d_t^r \\ 0 & \mathrm{Pf}(A_{rt}^\wedge) & \mathrm{Pf}(A_{rs}^\wedge) \end{pmatrix},$$

and therefore,

(5.3)

$$\begin{aligned} \mathrm{Pf}(A_{\widehat{2n-1}, r}) &= \det(u^r)_{\widehat{2n-1}, r} \\ &= \det w \cdot (d_s^r \mathrm{Pf}(A_{rs}^\wedge) - d_t^r \mathrm{Pf}(A_{rt}^\wedge)) \\ &= \mathrm{Pf}(A_{\widehat{2n-1}, I}) \cdot (d_s^r \mathrm{Pf}(A_{rs}^\wedge) - d_t^r \mathrm{Pf}(A_{rt}^\wedge)), \quad s < t. \end{aligned}$$

For all r, s, t with $\{r, s, t\} = I$, the equation $u_t^r = g_{rs}(u_t^s)$ yields

$$d_t^r = d_t^s + c_{rs} \mathrm{Pf}(A_{rs}^\wedge).$$

Therefore,

$$\begin{aligned}
& c_{ik} + c_{kj} + c_{ji} \\
&= \frac{d_j^i - d_j^k}{\text{Pf}(A_{ik}^\wedge)} + \frac{d_k^j - d_k^i}{\text{Pf}(A_{jk}^\wedge)} + \frac{d_k^j - d_k^i}{\text{Pf}(A_{ij}^\wedge)} \\
&= \frac{d_j^i \text{Pf}(A_{ij}^\wedge) - d_k^i \text{Pf}(A_{ik}^\wedge)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge)} + \frac{d_k^j \text{Pf}(A_{ik}^\wedge) - d_j^k \text{Pf}(A_{jk}^\wedge)}{\text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge)} + \frac{d_k^j \text{Pf}(A_{jk}^\wedge) - d_i^j \text{Pf}(A_{ij}^\wedge)}{\text{Pf}(A_{jk}^\wedge) \text{Pf}(A_{ij}^\wedge)} \\
(5.3) \quad &= \frac{\text{Pf}(A_{\widehat{2n-1,i}})}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{\widehat{2n-1,I}})} + \frac{\text{Pf}(A_{\widehat{2n-1,k}})}{\text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge) \text{Pf}(A_{\widehat{2n-1,I}})} - \frac{\text{Pf}(A_{\widehat{2n-1,j}})}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{jk}^\wedge) \text{Pf}(A_{\widehat{2n-1,I}})} \\
&= \frac{\text{Pf}(A_{\widehat{2n-1,i}}) \text{Pf}(A_{j,k}^\wedge) - \text{Pf}(A_{\widehat{2n-1,j}}) \text{Pf}(A_{i,k}^\wedge) + \text{Pf}(A_{\widehat{2n-1,k}}) \text{Pf}(A_{i,j}^\wedge)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge) \text{Pf}(A_{\widehat{2n-1,I}})} \\
&= \frac{\text{Pf}(A) \text{Pf}(A_{\widehat{2n-1,I}})}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge) \text{Pf}(A_{\widehat{2n-1,I}})}
\end{aligned}$$

where the last equation follows from [DW95, Theorem 1] (in the notation of that paper, choose $I_1 = \{1, 2, \dots, 2n-2, 2n, 2n+1, 2n+2\}$ and $I_2 = \{1, 2, \dots, 2n-1\}$). \square

6. SURJECTIVITY OF $d_{0,5}^2$

The goal of this section is to show in Proposition 6.21 below that the map $\gamma : \mathbb{Z}[\text{Skew}_6^+(R)] \rightarrow R[\text{Skew}_3^+(R)]$ of Proposition 5.1 is surjective. Recall that, unless stated otherwise, (R, m) is a local ring with infinite residue field R/m .

Notation 6.1. A skew-symmetric matrix $A = (a_{ij})$ with entries in R will be specified by giving its upper triangular part, the lower triangular part being determined by the requirement $a_{ij} = -a_{ji}$. For instance,

$$\begin{pmatrix} 0 & a & b \\ & 0 & c \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

For $a, b, c \in R^*$, we set

$$[a \ b] = \begin{pmatrix} 0 & a & b \\ & 0 & c \\ & & 0 \end{pmatrix} \quad \text{and} \quad [a] = \begin{pmatrix} a & a \\ & a \end{pmatrix}$$

These are elements of $\text{Skew}_3^+(R)$. For units $a, b, c \in R^*$ with $a^{-1} - b^{-1} + c^{-1} \in R^*$ and $x \in R$ we write

$$x \{a \ b\} = \frac{x}{(a^{-1} - b^{-1} + c^{-1})^2} \begin{bmatrix} a^{-1} & b^{-1} \\ & c^{-1} \end{bmatrix}$$

and for $a \in R^*$, $x \in R$ we set

$$x \{a\} = x \{a \ a\} = a^2 x [a^{-1}]$$

considered as elements of $R[\text{Skew}_3^+(R)]$. Note that

$$x \{a \ c\} = xc^2 \begin{bmatrix} a^{-1} & a^{-1} \\ & c^{-1} \end{bmatrix}$$

for all $a, c \in R^*$. Finally, for $\xi, \zeta \in R [\text{Skew}_3^+(R)]$, we write

$$\xi \equiv \zeta$$

to mean $\xi = \zeta$ in $\text{coker}(\gamma)$ where $\gamma : \mathbb{Z}[\text{Skew}_6^+(R)] \rightarrow R [\text{Skew}_3^+(R)]$ is the map in Proposition 5.1 for $n = 2$. Our goal is to show $\xi \equiv 0$ for all $\xi \in R [\text{Skew}_3^+(R)]$.

Lemma 6.2. *For all $a, b, c, d, e \in R^*$ we have in $R [\text{Skew}_3^+(R)]$ the relation*

$$(6.1) \quad \begin{aligned} 0 \equiv & \frac{a^2}{b^2} \frac{(d^2 - c^2)}{e} \{ \begin{smallmatrix} a & a \\ & b \end{smallmatrix} \} - \frac{a^2 d^2}{c^2 e} \{ \begin{smallmatrix} a & a \\ & c \end{smallmatrix} \} - \frac{a^2}{c} \{ \begin{smallmatrix} a & a \\ & e \end{smallmatrix} \} + \frac{b^2 d^2}{c^2 e} \{ \begin{smallmatrix} b & b \\ & c \end{smallmatrix} \} \\ & + \frac{b^2}{c} \{ \begin{smallmatrix} b & b \\ & e \end{smallmatrix} \} + b \{ \begin{smallmatrix} d & d \\ & e \end{smallmatrix} \} - b \{ \begin{smallmatrix} c & c \\ & e \end{smallmatrix} \}. \end{aligned}$$

Proof. The 6×6 skew-symmetric matrix

$$A = \left(\begin{array}{ccc|ccc} 0 & a & a & a & a & a \\ & 0 & b & b & b & b \\ & & 0 & c & c & c \\ \hline & & & 0 & d & d \\ & & & & 0 & e \\ & & & & & 0 \end{array} \right)$$

has Pfaffian $\text{Pf}(A) = ace$ and is in $\text{Skew}_6^+(R)$ if and only if all its entries a, \dots, e are units. We compute

$$(6.2) \quad \begin{aligned} \gamma(A) = & \left(\frac{e}{a^2 d^2} - \frac{e}{a^2 c^2} \right) \left[\begin{smallmatrix} a & a \\ & b \end{smallmatrix} \right] \\ & - \frac{e}{a^2 d^2} \left[\begin{smallmatrix} a & a \\ & c \end{smallmatrix} \right] - \frac{c}{a^2 e^2} \left[\begin{smallmatrix} a & a \\ & e \end{smallmatrix} \right] + \frac{e}{b^2 d^2} \left[\begin{smallmatrix} b & b \\ & c \end{smallmatrix} \right] \\ & + \frac{c}{b^2 e^2} \left[\begin{smallmatrix} b & b \\ & e \end{smallmatrix} \right] + \frac{1}{be^2} \left[\begin{smallmatrix} d & d \\ & e \end{smallmatrix} \right] - \frac{1}{be^2} \left[\begin{smallmatrix} c & c \\ & e \end{smallmatrix} \right]. \end{aligned}$$

See Appendix A.1 for more details. The result follows by replacing a, \dots, e with their inverses a^{-1}, \dots, e^{-1} . \square

Lemma 6.3. *For all $a, c, d, e \in R^*$ we have in $R [\text{Skew}_3^+(R)]$ the relation*

$$(6.3) \quad \frac{d^2 - c^2}{e} \{a\} \equiv a \left(\left\{ \begin{smallmatrix} c & c \\ & e \end{smallmatrix} \right\} - \left\{ \begin{smallmatrix} d & d \\ & e \end{smallmatrix} \right\} \right).$$

Proof. Put $a = b$ in Lemma 6.2. \square

Lemma 6.4. *The function $R \times R^* \rightarrow \text{coker}(\gamma) : (x, a) \mapsto x \{a\}$ is linear in both variables x and a (as long as the second variable only involves units), that is,*

$$(x + y) \{a\} \equiv x \{a\} + y \{a\} \quad \text{and} \quad x \{a + b\} \equiv x \{a\} + x \{b\}$$

for all $a, b, a + b \in R^*$, $x, y \in R$.

Proof. The first relation is actually an equality in $R [\text{Skew}_3^+(R)]$:

$$(x + y) \{a\} = (x + y)a^2[a^{-1}] = xa^2[a^{-1}] + ya^2[a^{-1}] = x \{a\} + y \{a\}.$$

For the second relation, for fixed $c, d, e \in R^*$, the right hand side of (6.3) is linear in a , hence the left hand side is, as long as it is defined. If $x \in R$ is a unit, choose $c, d \in R^*$ such that $d^2 - c^2 \in R^*$. This is possible since for given $c \in R^*$, d only needs to avoid finitely many elements of R/m . Setting $e = (d^2 - c^2)x^{-1}$ shows that the function $R^* \rightarrow \text{coker}(\gamma) : a \mapsto x\{a\}$ is linear in a . It follows that for any two units $x, y \in R$, the expression $(x + y) \{a\} = x \{a\} + y \{a\}$ is linear in a . Since every element of R is a sum of two units, we are done. \square

We will extend the map in Lemma 6.4 to a \mathbb{Z} -bilinear map defined on all of $R \times R$ using the following.

Lemma 6.5. *Let (R, m) be a local ring with infinite residue field, and let A be an abelian group. Then any function $f : R^* \rightarrow A$ satisfying $f(a) + f(b) = f(a + b)$ for all $a, b, a + b \in R^*$ extends to a unique \mathbb{Z} -linear function $R \rightarrow A$.*

Proof. If $x \in m$ and $a, b \in R^*$ then $f(x + a) - f(a) = f(x + b) - f(b)$. For we can choose $c \in R^*$ such that $a + c, b + c, a + b + c \in R^*$ as c only needs to avoid a finite number of elements of R/m . Then $f(x + a) + f(b) + f(c) = f(x + a + c) + f(b) = f(x + a + b + c) = f(x + b) + f(a + c) = f(x + b) + f(a) + f(c)$. Thus, we can set $f(x) = f(a + x) - f(a)$ for $a \in R^*$, and this expression is independent of $a \in R^*$. This defines a function $f : R \rightarrow A$ which we need to check is \mathbb{Z} -linear. So, let $x, y \in R$. If $x, y \in m$ then choose $a, b \in R^*$ such that $a + b \in R^*$. Then $f(x + y) = f(x + y + a + b) - f(a + b) = f(x + a) + f(y + b) - f(a) - f(b) = f(x) + f(y)$. If $x \in m$ and $y \in R^*$ then $f(x + y) = f(x) + f(y)$, by definition of $f(x)$. The rest is clear. \square

Definition 6.6. We define the map $R \times R \rightarrow \text{coker}(\gamma) : (x, y) \mapsto \langle x, y \rangle$ by

$$\langle x, y \rangle = \begin{cases} x\{y\} & y \in R^*, \\ x\{y + a\} - x\{a\} & y \in m, a \in R^*. \end{cases}$$

By Lemmas 6.4 and 6.5, this is well-defined (that is, independent of the choice of $a \in R^*$), and the map is \mathbb{Z} -linear in both variables x and $y \in R$.

Lemma 6.7. *For $c, e \in R^*$, the expression*

$$a^2e^{-1} \left\{ \begin{smallmatrix} a & a \\ & c \end{smallmatrix} \right\} + a^2c^{-1} \left\{ \begin{smallmatrix} a & a \\ & e \end{smallmatrix} \right\}$$

in $\text{coker}(\gamma)$ is independent of $a \in R^$.*

Proof. The claim follows by setting $c = d$ in Lemma 6.2 \square

Lemma 6.8. *For all $c, d, f \in R^*$ we have*

$$f \left\{ \begin{smallmatrix} c & c \\ & d \end{smallmatrix} \right\} \equiv d\{f\} + f\{d\} - c^2d^{-1}\{f\}.$$

Proof. This is Lemma 6.3 with $e = d$ replacing a with f . \square

Lemma 6.9. *For all $a, c, e \in R^*$ we have in $\text{coker}(\gamma)$*

$$0 \equiv \langle a^2c^{-1} - c, a^2e^{-1} - e \rangle + \langle a^2e^{-1} - e, a^2c^{-1} - c \rangle.$$

Proof. We rewrite the expression in Lemma 6.7 using Lemma 6.8:

$$\begin{aligned} & \frac{a^2}{e} \{ a \ c \} + \frac{a^2}{c} \{ a \ e \} \\ (6.4) \quad & \equiv \frac{a^2}{e} \{c\} + c \{ \frac{a^2}{e} \} - \frac{a^2}{c} \{ \frac{a^2}{e} \} + \frac{a^2}{c} \{e\} + e \{ \frac{a^2}{c} \} - \frac{a^2}{e} \{ \frac{a^2}{c} \} \\ & = \frac{a^2}{e} \{c\} + \frac{a^2}{c} \{e\} + \left(e - \frac{a^2}{e} \right) \{ \frac{a^2}{c} \} + \left(c - \frac{a^2}{c} \right) \{ \frac{a^2}{e} \}. \end{aligned}$$

By Lemma 6.7, these expressions are independent of a . Setting $a = e$, the last expression equals

$$e \{c\} + c \{e\} - \frac{e^2}{c} \{e\} + \frac{e^2}{c} \{e\} + e \{ \frac{e^2}{c} \} - e \{ \frac{e^2}{c} \} = e \{c\} + c \{e\}.$$

Taking the difference with (6.4) gives

$$\begin{aligned} (6.5) \quad 0 & \equiv - \left(e - \frac{a^2}{e} \right) \{c\} - \left(c - \frac{a^2}{c} \right) \{e\} + \left(e - \frac{a^2}{e} \right) \{ \frac{a^2}{c} \} + \left(c - \frac{a^2}{c} \right) \{ \frac{a^2}{e} \} \\ & = \left(e - \frac{a^2}{e} \right) \left(\{ \frac{a^2}{c} \} - \{c\} \right) + \left(c - \frac{a^2}{c} \right) \left(\{ \frac{a^2}{e} \} - \{e\} \right). \end{aligned}$$

Rewriting the last expression in terms of $\langle \ , \ \rangle$ using Definition 6.6 and bilinearity yields the lemma. \square

Lemma 6.10. *Let R be a local ring with infinite residue field. Then the set of elements of the form $\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3}$ generates R as an abelian group where $u_i \in R^*$ such that $u_1 + u_2 + u_3 = 0$.*

Proof. Let $A \subset R$ be the abelian subgroup generated by elements of the form $\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3}$ as in the lemma. We need to show $A = R$. We will show that there is a triple (u_1, u_2, u_3) such that $u_1 + u_2 + u_3 = 0$ and $w = \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} \in R^*$ is a unit. Then $tw \in A$ for all $t \in R^*$, in particular, $R^* \subset A$ which implies $R \subset A$. We need to find $u_1, u_2 \in R^*$ such that $u_1 + u_2, \frac{1}{u_1} + \frac{1}{u_2} - \frac{1}{u_1 + u_2} \in R^*$ that is, such that $u_1, u_2, u_1 + u_2, u_1^2 + u_1u_2 + u_2^2 \in R^*$. This only needs to be checked for $F = R/m$. For any given $0 \neq u_1 \in F$ there are only finitely many u_2 which do not satisfy this requirement. Since F is infinite, there is $u_2 \in F$ such that $u_2, u_1 + u_2, u_1^2 + u_1u_2 + u_2^2 \neq 0$. \square

Lemma 6.11. *Let R be a local ring with infinite residue field. Then for $a \in R^*$ the following map of abelian groups (defined on generators) is surjective*

$$\overline{\mathbb{Z}[R^*]} \rightarrow \overline{R} : c \mapsto \frac{a^2}{c} - c.$$

Proof. The image contains the set of elements $a^2(\frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3})$ where $u_i \in R^*$ and $u_1 + u_2 + u_3 = 0$. By lemma 6.10, this set generates \overline{R} as an abelian group. \square

Lemma 6.12. *Let R be a local ring with infinite residue field. Then the \mathbb{Z} -bilinear map $R \times R \rightarrow \text{coker}(\gamma) : (x, y) \mapsto \langle x, y \rangle$ of Definition 6.6 is anti-symmetric, that is, for all $x, y \in R$ we have*

$$\langle x, y \rangle + \langle y, x \rangle \equiv 0.$$

In particular, for all $a, b \in R^$ we have*

$$a \{b\} + b \{a\} \equiv 0.$$

Proof. This follows from Lemmas 6.11 and 6.9, and the bilinearity of $\langle \cdot, \cdot \rangle$. \square

Lemma 6.13. *For all $c, d, f \in R^*$ we have*

$$f \{ \begin{smallmatrix} c & c \\ d & d \end{smallmatrix} \} \equiv -c^2 d^{-1} \{f\} \equiv f \{c^2 d^{-1}\}.$$

Proof. Combine Lemmas 6.8 and 6.12. \square

Lemma 6.14. *For all $a, b, c \in R^*$ we have*

$$(6.6) \quad 0 \equiv a^2 \left(\frac{c^2}{b^3} + \frac{1}{c} \right) \left\{ \frac{a^2}{b} \right\} - b^2 \left(\frac{c^2}{b^3} + \frac{1}{c} \right) \{b\}.$$

Proof. We simplify expression (6.1) using Lemma 6.13, and we obtain

$$(6.7) \quad \begin{aligned} 0 &\equiv \frac{a^2}{b^2} \frac{(d^2 - c^2)}{e} \left\{ \frac{a^2}{b} \right\} - \frac{a^2 d^2}{c^2 e} \left\{ \frac{a^2}{c} \right\} - \frac{a^2}{c} \left\{ \frac{a^2}{e} \right\} + \frac{b^2 d^2}{c^2 e} \left\{ \frac{b^2}{c} \right\} \\ &\quad + \frac{b^2}{c} \left\{ \frac{b^2}{e} \right\} + b \left\{ \frac{d^2}{e} \right\} - b \left\{ \frac{c^2}{e} \right\}. \end{aligned}$$

Using anti-symmetry of the bilinear form $\langle \cdot, \cdot \rangle$ (Lemma 6.12) on the 3rd, 5th, 6th and 7th summand yields

$$(6.8) \quad \begin{aligned} 0 &\equiv \frac{a^2}{b^2} \frac{(d^2 - c^2)}{e} \left\{ \frac{a^2}{b} \right\} - \frac{a^2 d^2}{c^2 e} \left\{ \frac{a^2}{c} \right\} + \frac{a^2}{e} \left\{ \frac{a^2}{c} \right\} + \frac{b^2 d^2}{c^2 e} \left\{ \frac{b^2}{c} \right\} \\ &\quad - \frac{b^2}{e} \left\{ \frac{b^2}{c} \right\} - \frac{d^2}{e} \{b\} + \frac{c^2}{e} \{b\} \\ &= \frac{d^2 - c^2}{e} \left(\frac{a^2}{b^2} \left\{ \frac{a^2}{b} \right\} - \frac{a^2}{c^2} \left\{ \frac{a^2}{c} \right\} + \frac{b^2}{c^2} \left\{ \frac{b^2}{c} \right\} - \{b\} \right). \end{aligned}$$

For any $c \in R^*$ we can find $d \in R^*$ such that $d^2 - c^2 \in R^*$. Since $e \in R^*$ is an arbitrary unit we conclude that

$$(6.9) \quad 0 \equiv x \left(\frac{a^2}{b^2} \left\{ \frac{a^2}{b} \right\} - \frac{a^2}{c^2} \left\{ \frac{a^2}{c} \right\} + \frac{b^2}{c^2} \left\{ \frac{b^2}{c} \right\} - \{b\} \right).$$

for all $a, b, c \in R^*$ and $x \in R$. In particular for $x = c^2/b$ we obtain

$$(6.10) \quad 0 \equiv \frac{a^2 c^2}{b^3} \left\{ \frac{a^2}{b} \right\} - \frac{a^2}{b} \left\{ \frac{a^2}{c} \right\} + b \left\{ \frac{b^2}{c} \right\} - \frac{c^2}{b} \{b\}$$

Using anti-symmetry on the 2nd and 3rd term (Lemma 6.12) we obtain

$$\begin{aligned}
 (6.11) \quad 0 &\equiv \frac{a^2 c^2}{b^3} \left\{ \frac{a^2}{b} \right\} + \frac{a^2}{c} \left\{ \frac{a^2}{b} \right\} - \frac{b^2}{c} \{b\} - \frac{c^2}{b} \{b\} \\
 &= a^2 \left(\frac{c^2}{b^3} + \frac{1}{c} \right) \left\{ \frac{a^2}{b} \right\} - b^2 \left(\frac{c^2}{b^3} + \frac{1}{c} \right) \{b\}.
 \end{aligned}$$

□

Lemma 6.15. *Let R be a local ring with infinite residue field and $b \in R^*$ a unit. Then the following map of abelian groups (defined on generators) is surjective*

$$\mathbb{Z}[R^*] \rightarrow R : c \mapsto \frac{c^2}{b^3} + \frac{1}{c}.$$

Proof. Choose units u_1, u_2, u_3 such that all non-empty partial sums are units as well as

$$w = \frac{1}{u_1 + u_2 + u_3} - \frac{1}{u_1 + u_2} - \frac{1}{u_1 + u_3} - \frac{1}{u_2 + u_3} + \frac{1}{u_1} + \frac{1}{u_2} + \frac{1}{u_3} \in R^*.$$

This is possible since we only need to find such $u_1, u_2, u_3 \in R/m$ and for any $u_1, u_2 \in R/m$ such that $u_1, u_2, u_1 + u_2 \neq 0 \in R/m$ there are only finitely many $u_3 \in R/m$ such that $u_1 + u_3$ or $u_2 + u_3$, or $u_1 + u_2 + u_3$ or w is zero in R/m (for w to be zero in R/m , u_3 has to be a solution of a non-zero polynomial in R/m).

Now, let $u_1, u_2, u_3 \in R$ as above. The element

$$s = (u_1 + u_2 + u_3)^2 - (u_1 + u_2)^2 - (u_1 + u_3)^2 - (u_2 + u_3)^2 + u_1^2 + u_2^2 + u_3^2$$

of R is zero. Hence, $t^2 s b^{-3} + t^{-1} w = t^{-1} w$ is in the image of the map for any $t \in R^*$. Hence R^* is in the image and so is R . □

Lemma 6.16. *For all $a \in R^*$ and $x, y \in R$.*

$$\langle a^2 x, y \rangle \equiv \langle x, a^{-2} y \rangle.$$

Proof. Using Lemma 6.15, Lemma 6.14 implies

$$(6.12) \quad a^2 x \left\{ \frac{a^2}{b} \right\} \equiv b^2 x \{b\}$$

for all $a, b \in R^*$ and $x \in R$ in view of Lemma 6.15. In particular, the left hand side is independent of a (as the right hand side is) and thus equals its value for $a = 1$. In other words (replacing b with b^{-1}), we have

$$(6.13) \quad a^2 x \{a^2 b\} \equiv x \{b\}.$$

Replacing b with $a^{-2} b$ yields $a^2 x \{b\} \equiv x \{a^{-2} b\}$ for all $a, b \in R^*$ and $x \in R$. That is, the Lemma holds for $y = b$ a unit. Since $\langle \cdot, \cdot \rangle$ is linear in the second variable, we are done. □

Lemma 6.17. *For all $x \in R$ and $a \in R^*$, we have $x[a] \equiv 0$.*

Proof. As in the proof of Lemma 6.15, choose units u_1, u_2, u_3 such that all non-empty partial sums are units as well as

$$w = \frac{1}{(u_1 + u_2 + u_3)^2} - \frac{1}{(u_1 + u_2)^2} - \frac{1}{(u_1 + u_3)^2} - \frac{1}{(u_2 + u_3)^2} + \frac{1}{u_1^2} + \frac{1}{u_2^2} + \frac{1}{u_3^2} \in R^*.$$

The element

$$s = (u_1 + u_2 + u_3)^2 - (u_1 + u_2)^2 - (u_1 + u_3)^2 - (u_2 + u_3)^2 + u_1^2 + u_2^2 + u_3^2$$

of R is zero. Therefore,

$$0 \equiv \langle sx, y \rangle \equiv \langle x, wy \rangle$$

by bilinearity of $\langle \cdot, \cdot \rangle$ and Lemma 6.16. Since $y \in R$ is arbitrary and $w \in R^*$, we have $\langle x, y \rangle = 0$ for all $x, y \in R$ which translates into the statement of the Lemma. \square

Lemma 6.18. *For all $x \in R$ and $a, b \in R^*$, we have $x \begin{bmatrix} a & a \\ & b \end{bmatrix} \equiv 0$.*

Proof. This is Lemma 6.17 together with Lemma 6.13. \square

Lemma 6.19. *For all $x \in R$ and $a, b, c \in R^*$ such that $a - b + c \in R^*$, we have $x \begin{bmatrix} a & b \\ & c \end{bmatrix} \equiv 0$.*

Proof. The skew symmetric matrix

$$A = \left(\begin{array}{ccc|ccc} 0 & d & d & d & d & d \\ & 0 & d & d & d & d \\ & & 0 & d & d & d \\ \hline & & & 0 & a & b \\ & & & & 0 & c \\ & & & & & 0 \end{array} \right)$$

has Pfaffian $\text{Pf}(A) = d^2(a - b + c)$ and is in $\text{Skew}_6^+(R)$ if and only if $a, b, c, d, (a - b + c) \in R^*$. Modulo terms that are $\equiv 0$ by Lemmas 6.18 and 6.17, we have

$$\gamma(A) \equiv \frac{1}{(a-b+c)^2 d} \begin{bmatrix} a & b \\ & c \end{bmatrix};$$

see Appendix A.2. Since d is an arbitrary unit in R , we are done. \square

Lemma 6.20. *For all $x \in R$ and $a, b, c \in R^*$, we have $x \begin{bmatrix} a & b \\ & c \end{bmatrix} \equiv 0$.*

Proof. The 6×6 skew symmetric matrix

$$A = \left(\begin{array}{ccc|ccc} 0 & a & b & d & d & d \\ & 0 & b & e & e & e \\ & & 0 & f & f & f \\ \hline & & & 0 & a & b \\ & & & & 0 & b \\ & & & & & 0 \end{array} \right)$$

has Pfaffian $\text{Pf}(A) = (af - be + bd)a$ and is in $\text{Skew}_6^+(R)$ if and only if $\text{Pf}(A)$, a, b, d, e, f are units. Given $a, b, d, f \in R^*$ arbitrary, choose e such

that $a - d + e, b - e + f, af - be + bd \in R^*$. This is possible since R/m is infinite. Modulo terms that are $\equiv 0$ by Lemmas 6.18, 6.17 and 6.19, we have

$$0 \equiv \gamma(A) \equiv -\frac{a}{b^4} \begin{bmatrix} b & d \\ & f \end{bmatrix};$$

see Appendix A.3. Since $a \in R^*$ is an arbitrary unit, we are done. \square

Proposition 6.21. *Let R be a local ring with infinite residue field. Then the map $\gamma : \mathbb{Z}[\text{Skew}_6^+(R)] \rightarrow R[\text{Skew}_3^+(R)]$ of Proposition 5.1 is surjective for $n = 2$.*

Proof. This is a reformulation of Lemma 6.20. \square

7. LOCALISING HOMOLOGY GROUPS

The goal in this section is to show that if $\text{Sp}_{2n+1}(R) \rightarrow \text{Sp}_{2n+2}(R)$ is a surjection (injection, isomorphism) in homology then so is $\text{Sp}_{2n}(R) \rightarrow \text{Sp}_{2n+2}(R)$; see Proposition 7.1.

Recall that $\text{Sp}_{2n+1}(R)$ is the subgroup of $\text{Sp}_{2n+2}(R)$ of matrices

$$(7.1) \quad \begin{pmatrix} 1 & c & {}^t u \psi M \\ 0 & 1 & 0 \\ 0 & u & M \end{pmatrix}$$

where $\psi = \psi_{2n}$, $M \in \text{Sp}_{2n}(R)$, $u \in R^{2n}$, $c \in R$. We let the group R^* of units of R act from the left on $\text{Sp}_{2n+2}(R)$ by conjugation with the matrix $T_b \in \text{Sp}_{2n+2}(R)$ for $b \in R^*$ where

$$T_b = \begin{pmatrix} b & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & 1_n \end{pmatrix}$$

and 1_n denotes the $n \times n$ identity matrix. Note that $T_b \cdot A \cdot T_b^{-1} = A$ for $A \in \text{Sp}_{2n}(R)$ and $T_b \cdot A \cdot T_b^{-1} \in \text{Sp}_{2n+1}(R)$ for $A \in \text{Sp}_{2n+1}(R)$ since

$$T_b \cdot \begin{pmatrix} 1 & c & {}^t u \psi M \\ 0 & 1 & 0 \\ 0 & u & M \end{pmatrix} \cdot T_b^{-1} = \begin{pmatrix} 1 & b^2 c & b {}^t u \psi M \\ 0 & 1 & 0 \\ 0 & bu & M \end{pmatrix}.$$

By functoriality, this defines an R^* -action (hence a left $\mathbb{Z}[R^*]$ -module structure) on the homology groups $H_t(\text{Sp}_q(R))$ for $q = 2n, 2n+1, 2n+2$. The action is trivial for Sp_{2n} and Sp_{2n+2} , the latter because $T_b \in \text{Sp}_{2n+2}(R)$, but that action is non-trivial for Sp_{2n+1} , in general. Let m be an integer such that $m > 2t$. We choose units $a_1, \dots, a_m \in R^*$ such that for every non-empty subset $I \subset \{1, \dots, m\}$ the partial sum $a_I = \sum_{i \in I} a_i$ is a unit in R . This is possible since R has infinite residue field. Let $s_m \in \mathbb{Z}[R^*]$ be the element

$$s_m = - \sum_{\emptyset \neq I \subset \{1, \dots, m\}} (-1)^{|I|} \langle a_I \rangle \in \mathbb{Z}[R^*]$$

first considered in [Sch17a, §2] where $\langle u \rangle \in \mathbb{Z}[R^*]$ denotes the element of the group ring corresponding to $u \in R^*$. Since R^* acts trivially on $H_t(\text{Sp}_{2n}(R))$ and $H_t(\text{Sp}_{2n+2}(R))$, multiplication by s_m on those groups is the identity map [Sch17a, p. 7] in view of the equality

$$1 = - \sum_{\emptyset \neq I \subset \{1, \dots, m\}} (-1)^{|I|}.$$

In particular, we have

$$H_t(\mathrm{Sp}_{2n}(R)) = s_m^{-1}H_t(\mathrm{Sp}_{2n}(R)), \quad H_t(\mathrm{Sp}_{2n+2}(R)) = s_m^{-1}H_t(\mathrm{Sp}_{2n+2}(R)).$$

Proposition 7.1. *Let $m, n, t \geq 0$ be integers with $m > 2t$, and let R be a local ring with infinite residue field. Then localisation of the maps $H_t(\mathrm{Sp}_{2n}(R)) \rightarrow H_t(\mathrm{Sp}_{2n+1}(R)) \rightarrow H_t(\mathrm{Sp}_{2n+2}(R))$ at $s_m \in \mathbb{Z}[R^*]$ induces a commutative diagram of abelian groups*

$$\begin{array}{ccccc} H_t(\mathrm{Sp}_{2n}(R)) & \longrightarrow & H_t(\mathrm{Sp}_{2n+1}(R)) & \longrightarrow & H_t(\mathrm{Sp}_{2n+2}(R)) \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ s_m^{-1}H_t(\mathrm{Sp}_{2n}(R)) & \xrightarrow{\cong} & s_m^{-1}H_t(\mathrm{Sp}_{2n+1}(R)) & \longrightarrow & s_m^{-1}H_t(\mathrm{Sp}_{2n+2}(R)) \end{array}$$

in which the outer two vertical arrows and the lower left horizontal arrow are isomorphisms.

In particular, if the map $H_t(\mathrm{Sp}_{2n+1}(R)) \rightarrow H_t(\mathrm{Sp}_{2n+2}(R))$ is a surjection (injection, isomorphism) then so is $H_t(\mathrm{Sp}_{2n}(R)) \rightarrow H_t(\mathrm{Sp}_{2n+2}(R))$.

Proof. We have already seen that the two outer vertical maps are isomorphisms. Also, a localisation of a surjection (injection, isomorphism) is a surjection (injection, isomorphism). So, all we have to prove is that the lower left horizontal map is an isomorphism. The inclusion of groups $\varepsilon : \mathrm{Sp}_{2n}(R) \rightarrow \mathrm{Sp}_{2n+1}(R)$ has a retraction

$$(7.2) \quad \rho : \mathrm{Sp}_{2n+1}(R) \rightarrow \mathrm{Sp}_{2n}(R) : \begin{pmatrix} 1 & c & {}^t u \psi M \\ 0 & 1 & 0 \\ 0 & u & M \end{pmatrix} \mapsto M.$$

This defines an exact sequence of groups

$$1 \rightarrow G \longrightarrow \mathrm{Sp}_{2n+1}(R) \xrightarrow{\rho} \mathrm{Sp}_{2n}(R) \rightarrow 1.$$

Since the map ρ is R^* -equivariant, the group R^* acts on the exact sequence and hence on the associated Hochschild-Serre spectral sequence

$$E_{p,q}^2 = H_p(\mathrm{Sp}_{2n}(R), H_q(G)) \Rightarrow H_{p+q}(\mathrm{Sp}_{2n+1}(R)).$$

We localise that spectral sequence at $s_m \in \mathbb{Z}[R^*]$ to obtain the spectral sequence

$$(7.3) \quad s_m^{-1}E_{p,q}^2 = H_p(\mathrm{Sp}_{2n}(R), s_m^{-1}H_q(G)) \Rightarrow s_m^{-1}H_{p+q}(\mathrm{Sp}_{2n+1}(R)).$$

Here we used that $s_m^{-1}H_p(\mathrm{Sp}_{2n}(R), H_q(G)) = H_p(\mathrm{Sp}_{2n}(R), s_m^{-1}H_q(G))$ since R^* acts trivially on $\mathrm{Sp}_{2n}(R)$. There is a central extension of groups

$$1 \rightarrow (R, +) \rightarrow G \rightarrow (R^{2n}, +) \rightarrow 1$$

where the second map sends

$$\begin{pmatrix} 1 & c & {}^t u \psi \\ 0 & 1 & 0 \\ 0 & u & 1 \end{pmatrix} \in G$$

to $u \in R^{2n}$. This central extension is R^* -equivariant where $b \in R^*$ acts on $(R, +)$ via multiplication by b^2 and on $(R^{2n}, +)$ via multiplication by b . We have $s_m^{-1}H_0(G) = \mathbb{Z}$ since R^* acts trivially on $H_0(G)$. By Proposition [Sch21,

D.4], we have $s_m^{-1}H_q(G) = 0$ for $m > 2q > 0$. In particular, the localised Hochschild-Serre spectral sequence (7.3) degenerates at E^2 for $m > 2t \geq 2q$ to yield the isomorphism

$$\rho : s_m^{-1}H_t(\mathrm{Sp}_{2n+1} R) \xrightarrow{\cong} H_t(\mathrm{Sp}_{2n} R)$$

for $t < m/2$. Since ρ is a retract of ε , we are done. \square

8. HOMOLOGY STABILITY

As always, R will be a local ring with infinite residue field. In this section we will use the spectral sequence (3.4) with differentials d^r of bidegree $(r - 1, -r)$ to deduce our homology stability results from the Introduction. So far, we have proved the following.

Properties of the spectral sequence (3.4).

- (1) The abutment satisfies $H_{p+q}(\mathrm{Sp}_{2n}, \mathbb{Z}[U_*]_{|\ast| \leq 2n+1}) = 0$ for $p+q \leq 2n$; see Lemma 3.3. In particular, $E_{p,q}^\infty(2n) = 0$ for $p+q \leq 2n$.
- (2)

$$E_{p,q}^1(2n) \cong H_p(\mathrm{Sp}_{2n-q}) \otimes \mathbb{Z}[\mathrm{Skew}_q^+]$$

for $0 \leq q \leq 2n+1$ and any $p \in \mathbb{Z}$; see (3.5).

- (3) Under the isomorphism of (2), the differential in $E^1(2n)$ is

$$d_{p,q}^1 = \varepsilon \otimes d : H_p(\mathrm{Sp}_{2n-q}) \otimes \mathbb{Z}[\mathrm{Skew}_q^+] \rightarrow H_p(\mathrm{Sp}_{2n-q+1}) \otimes \mathbb{Z}[\mathrm{Skew}_{q-1}^+]$$

for $1 \leq q \leq 2n+1$ where ε is the map induced by inclusion of groups and d is the differential of the complex $\mathbb{Z}[\mathrm{Skew}_*^+]$; see Lemma 3.4.

- (4) The complex $(\mathbb{Z}[\mathrm{Skew}_*^+], d)$ is acyclic; see Lemma 2.9. In particular, $E_{0,q}^2(2n) = 0$ for $0 \leq q \leq 2n$.
- (5) For $n \geq 1$ and $p \geq 0$, the differential $d_{p,2}^1 : E_{p,2}^1(R^{2n}) \rightarrow E_{p,1}^1(R^{2n})$ vanishes; see Corollary 3.5.
- (6) For $r \geq 2$ and even $q < 2n$, the differentials in $E^r(2n)$ satisfy $d_{p,q}^r = 0$; see Corollary 4.2.
- (7) For $n = 2$, the composition

$$E_{0,5}^2(4) \xrightarrow{d_{0,5}^2} E_{1,3}^2(4) \subset E_{1,3}^1(4)$$

is surjective; see Proposition 6.21. In particular, $d_{0,5}^2$ is surjective and $E_{1,3}^2(4) = E_{1,3}^1(4)$, that is, $d_{1,3}^1 = 0$ in the spectral sequence $E(4)$.

H_0 stability. As is true for any group, we have

$$H_0(\mathrm{Sp}_n(R)) = \mathbb{Z}, \quad n \geq -1.$$

H_1 stability. It is known that the groups $\mathrm{Sp}_{2n}(R)$ are perfect (R has infinite residue field), in particular, $H_1(\mathrm{Sp}_{2n}(R)) = 0$ for $n \geq 0$. The following theorem reproves this fact and extends it to the groups $\mathrm{Sp}_{2n+1}(R)$.

Theorem 8.1. *Let R be a local ring with infinite residue field. Then*

$$H_1 \mathrm{Sp}_q(R) = \begin{cases} R, & q = 1 \\ 0, & q \geq -1, q \neq 1. \end{cases}$$

Proof. The statement is clear for $q = -1, 0, 1$ since $\mathrm{Sp}_{-1}(R) = \mathrm{Sp}_0(R) = 1$ and $\mathrm{Sp}_1(R) = R$.

For $q = 2n \geq 2$, the group $E_{1,0}^1(2n) = H_1(\mathrm{Sp}_{2n})$ has no outgoing differential and only one incoming differential $d_{1,1}^1 : H_1(\mathrm{Sp}_{2n-1}) \rightarrow H_1(\mathrm{Sp}_{2n})$ since $d_{0,2}^2 = 0$, by the Properties of the spectral sequence (3.4) item (6) above. Since $E_{1,0}^\infty(2n) = 0$, the map $d_{1,1}^1 : H_1(\mathrm{Sp}_{2n-1}) \rightarrow H_1(\mathrm{Sp}_{2n})$ is surjective. By Proposition 7.1, $H_1(\mathrm{Sp}_{2n-2}) \rightarrow H_1(\mathrm{Sp}_{2n})$ is then also surjective for $n \geq 1$. Since $0 = H_1(\mathrm{Sp}_0)$, we find that $H_1(\mathrm{Sp}_{2n}) = 0$ for $n \geq 0$.

For $q = 2n \geq 4$, that is, $n \geq 2$, we have $E_{1,1}^1(2n) = E_{1,1}^2(2n)$ since $E_{1,0}^1(2n) = H_1(\mathrm{Sp}_{2n}) = 0$. The d^1 -sequence $E_{0,4}^1(2n) \rightarrow E_{0,3}^1(2n) \rightarrow E_{0,2}^1(2n)$ is exact, by item (4) above. Therefore, $E_{0,3}^2(2n) = 0$ and $d_{0,3}^2 = 0$. Hence, $H_1(\mathrm{Sp}_{2n-1}) = E_{1,1}^1(2n) = E_{1,1}^2(2n) = E_{1,1}^\infty = 0$, by item (1). \square

H_2 stability. Extending results of Matsumoto [Mat69], van der Kallen [vdK77] shows isomorphisms

$$H_2(\mathrm{Sp}_2(R)) \xrightarrow{\cong} H_2(\mathrm{Sp}_4(R)) \xrightarrow{\cong} H_2(\mathrm{Sp}_6(R)) \xrightarrow{\cong} H_2(\mathrm{Sp}_8(R)) \xrightarrow{\cong} \dots$$

for any local ring with infinite residue field R . The following theorem reproves this fact and extends it to the groups $\mathrm{Sp}_{2n+1}(R)$.

Theorem 8.2. *Let R be a local ring with infinite residue field. Then for $q \geq 2$ inclusion of groups induces an isomorphism $H_2(\mathrm{Sp}_q(R)) \cong H_2(\mathrm{Sp}_{q+1}(R))$:*

$$H_2(\mathrm{Sp}_2(R)) \xrightarrow{\cong} H_2(\mathrm{Sp}_3(R)) \xrightarrow{\cong} H_2(\mathrm{Sp}_4(R)) \xrightarrow{\cong} H_2(\mathrm{Sp}_5(R)) \xrightarrow{\cong} \dots$$

Proof. For $2n \geq 4$, that is, $n \geq 2$, the term $E_{2,0}^1(2n) = H_2 \mathrm{Sp}_{2n}$ has no outgoing differential and only one incoming differential $d^1 : E_{2,1}^1(2n) \rightarrow E_{2,0}^1(2n)$ since $d_{1,2}^2 = 0$ (see Properties of the spectral sequence (3.4) item (6)), and $d_{0,3}^3 = 0$ (item (4)). The term $E_{2,1}^1(2n) = H_2 \mathrm{Sp}_{2n-1}$ has one outgoing differential $d^1 : E_{2,1}^1(2n) \rightarrow E_{2,0}^1(2n)$ and no incoming differential because $d_{2,2}^1 = 0$ (item (5)), $d_{1,3}^2 = 0$ ($E_{1,3}^1 = H_1(\mathrm{Sp}_{2n-3}) \otimes \mathbb{Z}[\mathrm{Skew}_3^+] = 0$ for $n \geq 3$ and item (7) for $n = 2$) and $d_{0,4}^3 = 0$ (because $E_{0,4}^2 = 0$, by item (4)). Since $E_{2,0}^\infty(2n) = E_{2,1}^\infty(2n) = 0$ the differential $d^1 : E_{2,1}^1(2n) \rightarrow E_{2,0}^1(2n)$ is an isomorphism, that is, $H_2(\mathrm{Sp}_{2n-1}) \rightarrow H_2(\mathrm{Sp}_{2n})$ is an isomorphism. By Proposition 7.1, $H_2(\mathrm{Sp}_{2n-2}) \rightarrow H_2(\mathrm{Sp}_{2n})$ is then also an isomorphism for $n \geq 2$. \square

H_3 stability. The following theorem improves results of Essert [Ess13, Theorem 3.9], Sprehn-Wahl [SW20, Theorem A] and answers a question of Hutchinson-Wendt [HW15, Remark 9.6].

Theorem 8.3. *Let R be a local ring with infinite residue field. Then for $q \geq 4$ inclusion of groups induces an isomorphism $H_3(\mathrm{Sp}_q(R)) \cong H_3(\mathrm{Sp}_{q+1}(R))$ and surjections $H_3(\mathrm{Sp}_2(R)) \twoheadrightarrow H_3(\mathrm{Sp}_4(R))$ and $H_3(\mathrm{Sp}_3(R)) \twoheadrightarrow H_3(\mathrm{Sp}_4(R))$:*

$$H_3(\mathrm{Sp}_2) \begin{array}{c} \xrightarrow{\quad} \\ \searrow \quad \nearrow \\ \xrightarrow{\quad} \end{array} H_3(\mathrm{Sp}_3) \xrightarrow{\quad} H_3(\mathrm{Sp}_4) \xrightarrow{\cong} H_3(\mathrm{Sp}_5) \xrightarrow{\cong} H_3(\mathrm{Sp}_6) \xrightarrow{\cong} \cdots$$

Proof. For $2n \geq 4$, the term $E_{3,0}^1 = H_3(\mathrm{Sp}_{2n})$ has no outgoing differential and only one incoming differential $d_{3,1}^1 : H_3(\mathrm{Sp}_{2n-1}) \rightarrow H_3(\mathrm{Sp}_{2n})$ since $d_{2,2}^2 = 0$ (Properties of the spectral sequence (3.4) item (6)), $d_{1,3}^3 = 0$ ($E_{1,3}^3(4) = 0$, by item (7) for $2n = 4$, and Theorem 8.1 for $2n > 4$), and $d_{0,4}^4 = 0$ (item (4)). Since $E_{3,0}^\infty = 0$ (item (1)), the stabilisation map $d_{3,1}^1 : H_3(\mathrm{Sp}_{2n-1}) \rightarrow H_3(\mathrm{Sp}_{2n})$ is surjective. By Proposition 7.1, $H_3(\mathrm{Sp}_{2n-2}) \rightarrow H_3(\mathrm{Sp}_{2n})$ is then also surjective.

For $2n \geq 6$, that is, $n \geq 3$, the term $E_{3,1}^1 = H_3(\mathrm{Sp}_{2n-1})$ has no outgoing differential other than $d_{3,1}^1$ and no incoming differential. Indeed, $d_{2,3}^2 = 0$ as $E_{2,3}^2 = 0$ in view of Theorem 8.2 and items (3) and (4). The differential $d_{1,4}^3 = 0$ because $E_{1,4}^1 = H_1(\mathrm{Sp}_{2n-4}) \otimes \mathbb{Z}[\mathrm{Skew}_4^+] = 0$, by Theorem 8.1. The differential $d_{0,5}^4 = 0$ because $E_{0,5}^2 = 0$, by item (4). Since $E_{3,1}^\infty = 0$ (item (1)), the stabilisation map $d_{3,1}^1 : H_3(\mathrm{Sp}_{2n-1}) \rightarrow H_3(\mathrm{Sp}_{2n})$ is also injective, hence an isomorphism. By Proposition 7.1, $H_3(\mathrm{Sp}_{2n-2}) \rightarrow H_3(\mathrm{Sp}_{2n})$ is then also an isomorphism. \square

Remark 8.4. The homology stability range in Theorem 8.3 is optimal, in general. Indeed, let k be an infinite perfect field of characteristic not 2 which is finitely generated over its prime field, then neither of the two surjective maps

$$(8.1) \quad H_3(\mathrm{Sp}_2(k)) \twoheadrightarrow H_3(\mathrm{Sp}_4(k)), \quad \text{and} \quad H_3(\mathrm{Sp}_3(k)) \twoheadrightarrow H_3(\mathrm{Sp}_4(k))$$

is injective. For the first map, this follows from [HW15, Theorem 7.4] since that map factors through $H_3(B\mathrm{Sp}_2(k[\Delta^\bullet]))$ in view of the isomorphisms

$$H_3(B\mathrm{Sp}_4(k)) \cong H_3(B\mathrm{Sp}(k)) \cong H_3(B\mathrm{Sp}(k[\Delta^\bullet]))$$

of Theorem 8.3 and homotopy invariance of symplectic K -theory for regular rings containing $1/2$. Then the second map cannot be injective either, by Proposition 7.1.

An important consequence of Theorem 8.3 is the following relative homology stability result.

Theorem 8.5. *Let R be a local ring with infinite residue field. Then inclusion of groups induces an isomorphism of relative integral homology groups for $i \leq 3$*

$$H_i(\mathrm{SL}_3(R), \mathrm{Sp}_2(R)) \xrightarrow{\cong} H_i(\mathrm{SL}(R), \mathrm{Sp}(R)) \cong \begin{cases} 0 & i \leq 2 \\ K_3^{MW}(R) & i = 3. \end{cases}$$

Proof. For $i \leq 3$ we have the following commutative diagram

$$\begin{array}{ccccccccc}
H_i(\mathrm{Sp}_2) & \longrightarrow & H_i(\mathrm{SL}_3) & \longrightarrow & H_i(\mathrm{SL}_3, \mathrm{Sp}_2) & \longrightarrow & H_{i-1}(\mathrm{Sp}_2) & \longrightarrow & H_{i-1}(\mathrm{SL}_3) \\
\downarrow \xi_0 & & \cong \downarrow \xi_1 & & \downarrow \xi_2 & & \cong \downarrow \xi_3 & & \cong \downarrow \xi_4 \\
H_i(\mathrm{Sp}_4) & \longrightarrow & H_i(\mathrm{SL}_4) & \longrightarrow & H_i(\mathrm{SL}_4, \mathrm{Sp}_4) & \longrightarrow & H_{i-1}(\mathrm{Sp}_4) & \longrightarrow & H_{i-1}(\mathrm{SL}_4)
\end{array}$$

where the rows are exact, the maps ξ_1 and ξ_4 are isomorphisms, by [Sch17a, Theorem 5.37], the map ξ_0 is surjective, and the map ξ_3 is an isomorphism, by Theorems 8.1, 8.2 and 8.3. By the Five Lemma, ξ_2 is an isomorphism. Similarly we have isomorphisms

$$H_i(\mathrm{SL}_4, \mathrm{Sp}_4) \xrightarrow{\cong} H_i(\mathrm{SL}_6, \mathrm{Sp}_6) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_i(\mathrm{SL}, \mathrm{Sp})$$

for $i \leq 3$. The identification with Milnor-Witt K -theory or 0 follows from [Sch17a, Theorem 5.37] using the identity $\mathrm{Sp}_2(R) = \mathrm{SL}_2(R)$. \square

H_n stability. The following result generalises Theorem 8.3. It is probably not optimal for $k \geq 1$ but it improves on [Ess13, Theorem 3.9], [SW20, Theorem A]. In order to improve stability ranges further with the methods of this paper one would need to study various non-zero differentials in more detail which seems to be quite challenging.

Theorem 8.6. *Let R be a local ring with infinite residue field, and let $k \geq 0$ be an integer. Then for $q \geq 2k + 4$ inclusion of groups induces an isomorphism $H_{k+3}(\mathrm{Sp}_q(R)) \cong H_{k+3}(\mathrm{Sp}_{q+1}(R))$ and surjections $H_{k+3}(\mathrm{Sp}_{2k+2}(R)) \twoheadrightarrow H_{k+3}(\mathrm{Sp}_{2k+4}(R))$ and $H_{k+3}(\mathrm{Sp}_{2k+3}(R)) \twoheadrightarrow H_{k+3}(\mathrm{Sp}_{2k+4}(R))$:*

$$\begin{array}{ccccccc}
H_{k+3}(\mathrm{Sp}_{2k+2}) & & & & & & \\
\downarrow & \searrow & & & & & \\
H_{k+3}(\mathrm{Sp}_{2k+3}) & \twoheadrightarrow & H_{k+3}(\mathrm{Sp}_{2k+4}) & \xrightarrow{\cong} & H_{k+3}(\mathrm{Sp}_{2k+5}) & \xrightarrow{\cong} & H_{k+3}(\mathrm{Sp}_{2k+6}) \xrightarrow{\cong} \cdots
\end{array}$$

Proof. We prove the theorem by induction on k , the case $k = 0$ is Theorem 8.3. Let $n \geq 1$ be an integer and assume the theorem is true for k satisfying $0 \leq k < n$. We consider the Spectral sequence $E_{p,q}^r(2n+4)$ and show that all incoming differentials at $E_{n+3,0}^r(2n+4)$ are zero for $r \geq 2$. Indeed, the incoming differential $d_{n+4-r,r}^r : E_{n+4-r,r}^r(2n+4) \rightarrow E_{n+3,0}^r(2n+4)$ is zero for $r \geq 2$ as $E_{n+4-r,r}^2(2n+4) = 0$. This is because it is the homology of the complex

$$\begin{aligned}
H_{n+4-r}(\mathrm{Sp}_{2n+3-r}) \otimes \mathbb{Z}[\mathrm{Skew}_{r+1}^+] & \xrightarrow{\varepsilon \otimes d} H_{n+4-r}(\mathrm{Sp}_{2n+4-r}) \otimes \mathbb{Z}[\mathrm{Skew}_r^+] \\
& \xrightarrow{\varepsilon \otimes d} H_{n+4-r}(\mathrm{Sp}_{2n+5-r}) \otimes \mathbb{Z}[\mathrm{Skew}_{r-1}^+]
\end{aligned}$$

which is exact, by induction hypothesis and the Properties of the spectral sequence (3.4) item (4). Since $E_{n+3,0}^\infty(2n+4) = 0$, by item (1), the stabilisation map $d_{n+3,1}^1 : H_{n+3}(\mathrm{Sp}_{2n+3}) \rightarrow H_{n+3}(\mathrm{Sp}_{2n+4})$ must be surjective.

By Proposition 7.1, the stabilisation map $H_{n+3}(\mathrm{Sp}_{2n+2}) \rightarrow H_{n+3}(\mathrm{Sp}_{2n+4})$ is then also surjective.

For $m > n$ we consider the spectral sequence $E_{p,q}^r(2m+4)$ and show that all incoming differentials at $E_{n+3,0}^r(2m+4)$ and $E_{n+3,1}^r(2m+4)$ are zero for $r \geq 2$ (their outgoing differentials are zero anyway). The incoming differentials $d_{n+4-r,r}^r : E_{n+4-r,r}^r(2m+4) \rightarrow E_{n+3,0}^r(2m+4)$ and $d_{n+4-r,r+1}^r : E_{n+4-r,r+1}^r(2m+4) \rightarrow E_{n+3,1}^r(2m+4)$ are zero for $r \geq 2$ as $E_{n+4-r,r}^2(2m+4) = 0$ and $E_{n+4-r,r+1}^2(2m+4) = 0$, by induction hypothesis and item (4). Since $E_{n+3,0}^\infty(2m+4) = E_{n+3,1}^\infty(2m+4) = 0$, by item (1), the stabilisation map $d_{n+3,1}^1 : H_{n+3}(\mathrm{Sp}_{2m+3}) \rightarrow H_{n+3}(\mathrm{Sp}_{2m+4})$ must be an isomorphism. By Proposition 7.1, the stabilisation map $H_{n+3}(\mathrm{Sp}_{2m+2}) \rightarrow H_{n+3}(\mathrm{Sp}_{2m+4})$ is then also an isomorphism. \square

9. THE HUREWICZ MAP

Let R be a local ring with infinite residue field. We denote by $K_n^{MW}(R)$ the n -th Milnor-Witt K -group of R [Sch17a, Definition 4.10]. This is the evident generalisation to local rings of the definition for fields given in [Mor12]. We recall the definition of symplectic K -theory $K_n \mathrm{Sp}(R) = \pi_n B \mathrm{Sp}(R)^+$ for $n \geq 1$ where $\mathrm{Sp}(R) = \bigcup_{n \geq 0} \mathrm{Sp}_{2n}(R)$ is the infinite symplectic group over R and "+" is Quillen's plus construction with respect to the perfect (sub-)group $\mathrm{Sp}(R)$.

Theorem 9.1. *Let R be a local ring with infinite residue field. Then the Hurewicz map induces an isomorphism*

$$K_2 \mathrm{Sp}(R) \xrightarrow{\cong} H_2(\mathrm{Sp}(R)) \xleftarrow{\cong} H_2(\mathrm{Sp}_2(R)) \cong K_2^{MW}(R).$$

Proof. The first map is the Hurewicz isomorphism since $\pi_1 B \mathrm{Sp}(R)^+ = H_1 \mathrm{Sp}(R) = 0$. The second isomorphism follows from homology stability (Theorem 8.2). The last isomorphism is [Sch17a, Theorem 5.27] since $\mathrm{Sp}_2(R) = \mathrm{SL}_2(R)$. \square

We denote by $GW_3^{[3]}(R)$ the 2-nd homotopy group of the homotopy fibre of the forgetful map $K \mathrm{Sp}(R) \rightarrow K(R)$ induced by the inclusion of groups $\mathrm{Sp}(R) \subset \mathrm{SL}(R)$. Thus, $GW_3^{[3]}(R)$ sits in an exact sequence

$$K_3 \mathrm{Sp}(R) \rightarrow K_3(R) \rightarrow GW_3^{[3]}(R) \rightarrow K_2 \mathrm{Sp}(R) \rightarrow K_2(R) \rightarrow \dots$$

The definition given here agrees with that in [Sch17b] when $\frac{1}{2} \in R$ in view of [Sch17b, Theorem 6.1].

Theorem 9.2. *Let R be a local ring with infinite residue field. Then the Hurewicz map induces an isomorphism*

$$GW_3^{[3]}(R) \xrightarrow{\cong} H_3(\mathrm{SL}(R), \mathrm{Sp}(R)) \xrightarrow{\cong} K_3^{MW}(R).$$

Proof. By definition, we have $GW_3^{[3]}(R) \cong \pi_3(BSL^+(R), BSp^+(R))$. It follows that $\pi_3(BSL^+, BSp^+) \cong H_3(SL, Sp) \cong K_3^{MW}(R)$, by the relative Hurewicz theorem [GJ99, III Corollary 3.12] and Theorem 8.5, \square

10. THE KO -DEGREE MAP

We write \mathbb{Z}' for the ring $\mathbb{Z}[1/2]$. In this section, all rings are commutative with trivial involution and have 2 as a unit. Let R be such a ring. In [Sch17b, Definition 9.1] we have defined abelian groups $GW_i^{[n]}(R)$ as the homotopy groups $GW_i^{[n]}(R) = \pi_i GW^{[n]}(R)$, $i, n \in \mathbb{Z}$, $i \geq 0$, of a pointed topological space $GW^{[n]}(R)$ associated with the category of bounded chain complexes of finitely generated projective R -modules equipped with the duality $P \mapsto \text{Hom}(P, R[n])$. For $n = 0$, the space $GW^{[0]}(R)$ is the K -theory space of non-degenerate symmetric bilinear forms over R , and for $n = 2$, the space $GW^{[2]}(R)$ is the K -theory space of non-degenerate symplectic forms over R . In particular, the groups $GW_0^{[0]}(R)$ and $GW_0^{[2]}(R)$ are the Grothendieck groups of non-degenerate symmetric and symplectic forms over R . Using a different definition, the groups $GW_m^{[r]}(R)$ have been introduced by Karoubi [Kar73] under the name Hermitian K -theory where $GW^{[1]}(R)$ and $GW^{[-1]}(R)$ are equivalent to Karoubi's spaces ${}_{-1}U(R)$ and ${}_1U(R)$.

There are natural, associative and unital cup product maps [Sch17b, §9.2]

$$\cup : GW_m^{[r]}(R) \otimes GW_n^{[s]}(R) \rightarrow GW_{m+n}^{[r+s]}(R)$$

making $\bigoplus_{s,n} GW_n^{[s]}(S)$ into a bi-graded $\bigoplus_{r,m} GW_m^{[r]}(R)$ -algebra for any R -algebra S . Moreover, there is a natural Bott long exact sequence [Sch17b, Theorem 6.1]

$$\cdots \rightarrow GW_m^{[r]}(R) \xrightarrow{f} K_m(R) \xrightarrow{h} GW_m^{[r+1]}(R) \xrightarrow{\eta} GW_{m-1}^{[r]}(R) \xrightarrow{f} K_{m-1}(R) \rightarrow \cdots$$

where f and h are forgetful and hyperbolic maps and η is cup product with $\eta \in GW_{-1}^{[-1]}(\mathbb{Z}') \cong W(\mathbb{Z}')$ corresponding to the identity element in the Witt ring $W(\mathbb{Z}')$.

The Bott sequence [Sch17b, Theorem 6.1] in low degrees gives an exact sequence

$$(10.1) \quad GW_1^{[0]}(R) \xrightarrow{f} K_1(R) \xrightarrow{h} GW_1^{[1]}(R) \xrightarrow{\eta} GW_0^{[0]}(R) \xrightarrow{f} K_0(R).$$

Assume that $SK_1(R) = 0$, that is, the determinant map $\det : K_1(R) \rightarrow R^*$ is an isomorphism. Then the composition $K_1(R) \xrightarrow{h} GW_1^{[1]}(R) \xrightarrow{f} K_1(R)$ is

multiplication by 2, and we have a map of exact sequences

$$(10.2) \quad \begin{array}{ccccc} K_1(R) & \xrightarrow{h} & GW_1^{[1]}(R) & \xrightarrow{\eta} & I(R) \\ \downarrow 1 & & \downarrow f & & \downarrow \\ K_1(R) & \xrightarrow{2} & K_1(R) & \xrightarrow{\quad} & k_1(R) \end{array}$$

where $k_1(R) = K_1(R)/2$, $I(R) = \ker(f : GW_0^{[0]}(R) \rightarrow K_0(R))$ and the right vertical map is the map on cokernels of the two left horizontal maps.

Lemma 10.1. *Assume that $SK_1(R) = 0$. Then the following map is an isomorphism:*

$$(10.3) \quad (f, \eta) : GW_1^{[1]}(R) \xrightarrow{\cong} K_1(R) \times_{k_1(R)} I(R).$$

Proof. The map is surjective in view of the commutative diagram with exact rows (10.2). For injectivity, let $\xi \in GW_1^{[1]}(R)$ such that $\eta(\xi) = 0$ and $f(\xi) = 0$. Then there is $a \in R^* = K_1(R)$ such that $h(a) = \xi$, by the exact sequence (10.1). Then $a^2 = fh(a) = f(\xi) = 1 \in R^*$, and $a \in K_1(R)$ is in the image of the forgetful map $GW_1^{[0]}(R) \rightarrow K_1(R)$, namely, it is the image of the automorphism of the unit bilinear space $\langle 1 \rangle$ given by multiplication with a . By exactness of (10.1), $\xi = h(a) = 0$. This proves the injectivity claim. \square

Let R be a commutative \mathbb{Z}' -algebra, and R^* its group of units. The integral group ring $\mathbb{Z}[R^*]$ has \mathbb{Z} -basis the elements $\langle a \rangle$, $a \in R^*$, and multiplication $\langle a \rangle \cdot \langle b \rangle = \langle ab \rangle$. If we also denote by $\langle a \rangle$ the rank 1 bilinear space R equipped with the form $(x, y) \mapsto axy$, then the abelian group homomorphism

$$\mathbb{Z}[R^*] \rightarrow GW_0^0(R) : \langle a \rangle \mapsto \langle a \rangle$$

is a ring homomorphism. This makes all groups $GW_n^{[r]}(R)$ into R^* -modules.

The augmentation ideal $I[R^*]$ is the kernel of the ring homomorphism $\mathbb{Z}[R^*] \rightarrow \mathbb{Z} : \langle a \rangle \mapsto 1$. We write $\langle\langle a \rangle\rangle$ for the element $1 - \langle a \rangle$ considered in $\mathbb{Z}[R^*]$, and we write $[a]$ for the same element $1 - \langle a \rangle$ considered in $I[R^*]$. Thus, the inclusion $I[R^*] \rightarrow \mathbb{Z}[R^*]$ sends $[a]$ to $\langle\langle a \rangle\rangle^1$. The augmentation ideal has \mathbb{Z} -basis $[a]$, $a \in R^*$, $a \neq 1$.

The Bass Fundamental Theorem for Grothendieck-Witt groups [Sch17b, Theorem 9.13] together with homotopy invariance [Sch17b, Theorem 9.8] for regular rings gives a split short exact sequence

$$0 \rightarrow GW_1^{[1]}(\mathbb{Z}') \rightarrow GW_1^{[1]}(\mathbb{Z}'[T, T^{-1}]) \xrightarrow{\delta} GW_0^{[0]}(\mathbb{Z}') \rightarrow 0.$$

The left map is split by the map ε evaluating at $T = 1$. As in [KSW16, Paragraph before Lemma 6.9], we denote by $[T] \in GW_1^{[1]}(\mathbb{Z}'[T, T^{-1}])$ the unique element such that $\delta([T]) = 1 \in GW_0^{[0]}(\mathbb{Z}')$ and $\varepsilon([T]) = 0 \in GW_1^{[1]}(\mathbb{Z}')$.

¹In [Sch17a], we used the convention $[a] = \langle a \rangle - 1$.

For $a \in R^*$, denote by $[a] \in GW_1^{[1]}(R)$ the image of $[T]$ under the map $GW_1^{[1]}(\mathbb{Z}'[T, T^{-1}]) \rightarrow GW_1^{[1]}(R)$ induced by $\mathbb{Z}'[T, T^{-1}] \rightarrow R : T \mapsto a$. Note that $[1]$ is 0 in $GW_1^{[1]}(R)$ since $\varepsilon([T]) = 1$. We define the abelian group homomorphism

$$\varphi : I[R^*] \longrightarrow GW_1^{[1]}(R) : [a] \mapsto [a].$$

Lemma 10.2. *For any commutative \mathbb{Z}' -algebra, the following diagram commutes and is R^* -equivariant*

$$\begin{array}{ccc} I[R^*] & \xrightarrow{[a] \mapsto [a]} & GW_1^{[1]}(R) \\ \downarrow & & \downarrow \eta \\ \mathbb{Z}[R^*] & \xrightarrow{\langle a \rangle \mapsto \langle a \rangle} & GW_0^{[0]}(R) \end{array}$$

Proof. For $R = \mathbb{Z}'[T, T^{-1}]$ we have $\eta([T]) = \langle T \rangle$, by [KSW16, Lemma 6.9]. For general R and $a \in R^*$, let $\varepsilon_a : \mathbb{Z}'[T, T^{-1}] \rightarrow R$ be the \mathbb{Z}' -algebra homomorphism sending T to a . Then $\eta([a]) = \eta([\varepsilon_a(T)]) = \eta(\varepsilon_a([T])) = \varepsilon_a \eta([T]) = \varepsilon_a(\langle T \rangle) = \langle a \rangle$, and the diagram commutes.

The map η is a $GW_0^{[0]}(R)$ -module map, in particular, it is R^* -equivariant. The lower horizontal and the left vertical maps are R^* -equivariant by definition. For the top horizontal map we only need to consider products $\langle b \rangle \cdot [a]$ with $a, b \in R^*$. In other words, it suffices to show equivariance for $\mathbb{Z}'[S, T, S^{-1}, T^{-1}]$, then use functoriality under the ring homomorphism $\mathbb{Z}'[S, T, S^{-1}, T^{-1}] \rightarrow R : S \mapsto a, T \mapsto b$. Recall that $SK_1(\mathbb{Z}'[S, T, S^{-1}, T^{-1}]) = 0$. In particular, we only need to check equivariance of the top map for rings with trivial SK_1 . In this case, we can use the R^* -equivariant isomorphism (10.3) and have to show that the composition

$$(\varphi_1, \varphi_2) : I[R^*] \longrightarrow GW_1^{[1]}(R) \xrightarrow{(f, \eta)} K_1(R) \times_{k_1(R)} I(R)$$

is equivariant. Now $\varphi_1([a]) = a \in K_1(R)$ (as $f([T]) = T \in K_1(\mathbb{Z}'[T, T^{-1}])$) and $\varphi_2([a]) = 1 - \langle a \rangle$ (since the diagram commutes). In particular, φ_2 is R^* -equivariant. Since the R^* -action on $K_n(R)$ is via the forgetful map $GW_0^0(R) \rightarrow K_0(R)$, that action is trivial, and we have $\varphi_1(\langle b \rangle \cdot [a]) = \varphi_1([ba] - [b]) = bab^{-1} = a = \langle b \rangle \cdot \varphi_1([a]) \in K_1(R)$. \square

For a connected ring R , let $SK_0(R)$ be the kernel of the rank homomorphism $K_0(R) \rightarrow \mathbb{Z}$.

Lemma 10.3. *Let R be a connected commutative ring with $SK_i(R) = 0$ for $i = 0, 1$. Then the right square of (10.2) is cartesian:*

$$(10.4) \quad \begin{array}{ccc} GW_1^{[1]}(R) & \xrightarrow{\eta} & I(R) \\ f \downarrow & & \downarrow \\ K_1(R) & \longrightarrow & k_1(R). \end{array}$$

The right vertical map $I(R) \rightarrow k_1(R)$ sends $[a]$ to $\{a\}$. If R is local then the vertical maps are surjections and the map $I[R^*] \rightarrow GW_1^{[1]}(R) : [a] \mapsto [a]$ is also surjective.

Proof. The diagram is cartesian, by Lemma 10.1. The composition

$$I[R^*] \rightarrow GW_1^{[1]}(R) \xrightarrow{f} K_1(R) \rightarrow k_1(R)$$

sends $[a]$ to $a \in k_1(R)$ since $f([T]) = T \in K_1(\mathbb{Z}'[T, T^{-1}])$. Therefore, the composition $I[R^*] \rightarrow I(R) \rightarrow k_1(R)$ is $[a] \mapsto [a] \mapsto a$.

If R is local then the map f is surjective because $GW_1^{[1]}(R) \xrightarrow{f} K_1(R) \xrightarrow{h} GW_1^{[2]}(R)$ is exact and $GW_1^{[2]}(R) = K_1 \text{Sp}(R) \cong H_1(\text{Sp}(R)) = 0$ for every local ring. Then the right vertical map is also surjective. Finally, the map $I[R^*] \rightarrow GW_1^{[1]}(R)$ is surjective since the composition with the isomorphism (10.3) is surjective. This follows from the commutative diagram with exact 2nd row and outer vertical maps surjective:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I[R^{2*}] & \longrightarrow & I[R^*] & \longrightarrow & I[R^*/R^{2*}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 2K_1(R) & \longrightarrow & K_1(R) \times_{k_1(R)} I(R) & \longrightarrow & I(R) & \longrightarrow & 0 \end{array}$$

□

Lemma 10.4 (Steinberg Relation). *For any commutative ring R over \mathbb{Z}' and any $a \in R^*$ with $1 - a \in R^*$, we have*

$$[a] \cup [1 - a] = 0 \in GW_2^{[2]}(R).$$

Proof. The proof given in [AF17, Proposition 4.1.10] goes through. In detail, consider the ring $S = R[T]/(T^2 - a)$. Then T and $1 - T$ are units in S , the latter because the matrix in the R -basis $1, T$ of S has determinant $1 - a \in R^*$. Consider the R -linear map $i : S \rightarrow R$ sending 1 to 1 and T to 0 (up to a factor of 2, this is the trace used in [AF17]). It sends a symmetric bilinear form (P, b) over S to the symmetric bilinear form $i_*(P, b) = (P, i \circ b)$ over R and preserves non-degeneracy since the unit form $\langle 1 \rangle$ is sent to the diagonal form $\langle 1 \rangle + \langle a \rangle$. Therefore, i_* defines a map of Grothendieck-Witt groups $GW_n^{[r]}(S) \rightarrow GW_n^{[r]}(R)$ which is $GW_*^{[*]}(R)$ -linear. The Gram matrix of $i_*(1 - T)$ with respect to the R -basis $1, T$ is $\begin{pmatrix} 1 & -a \\ -a & a \end{pmatrix}$ which has determinant $a - a^2$ and contains the non-degenerate subspace $\langle a \rangle$ with basis T . Therefore, the orthogonal subspace is $\langle 1 - a \rangle$ and we have $i_*(1 - T) = \langle a \rangle + \langle 1 - a \rangle \in GW_{[0]}^0(R)$. It follows that $i_*[1 - T] = [1 - a] \in GW_1^{[1]}(R)$, first by considering $R = \mathbb{Z}'[t, t^{-1}, (1 - t)^{-1}]$ using Lemma 10.3, then using functoriality and the map $\mathbb{Z}'[t, t^{-1}, (1 - t)^{-1}] \rightarrow R : t \mapsto a$. By the Steinberg relation in K -theory, we have $0 = \{T\} \cup \{1 - T\} = f([T] \cup [1 - T]) \in K_2(S)$ using that the forgetful map $f : GW_*^{[*]}(S) \rightarrow K_*(S)$ is a map of graded rings. Then

$$0 = hf([T] \cup [1 - T]) = \langle 1, -1 \rangle \cdot [T] \cup [1 - T] = [T^2] \cup [1 - T] = [a] \cup [1 - T] \in GW_2^{[2]}(S)$$

since $\langle 1, -1 \rangle \cdot [T] = [T^2] = [a] \in GW_1^{[1]}(S)$ (it suffices to check the equation $\langle 1, -1 \rangle \cdot [T] = [T^2]$ in $GW_1^{[1]}(\mathbb{Z}[T, T^{-1}])$ using Lemma 10.1). Applying the transfer map i_* and noting that it is $GW_*^*(R)$ -linear, we obtain

$$0 = i_*([a] \cup [1 - T]) = [a] \cup i_*[1 - T] = [a] \cup [1 - a] \in GW_2^{[2]}(R).$$

□

For a commutative \mathbb{Z}' -algebra denote by $\widehat{K}_*^{MW}(R)$ and $K_*^{MW}(R)$ the graded $\mathbb{Z}[R^*]$ and $GW_0^0(R)$ -algebras

$$\widehat{K}_*^{MW}(R) = \text{Tens}_{\mathbb{Z}[R^*]} I[R^*]/[a][1-a], \quad K_*^{MW}(R) = \text{Tens}_{GW_0^0(R)} GW_1^1(R)/[a][1-a]$$

where the relations run over all $a, 1 - a \in R^*$. By Lemma 10.4, the maps $\mathbb{Z}[R^*] \rightarrow GW_0^0(R)$ and $I[R^*] \rightarrow GW_1^1(R)$ extend uniquely to homomorphisms of graded rings, called *symbol map* or *KO-degree map*,

$$(10.5) \quad \widehat{K}_*^{MW}(R) \longrightarrow K_*^{MW}(R) \longrightarrow \bigoplus_{n \geq 0} GW_n^{[n]}(R).$$

When R is local with infinite residue field, the first map is an isomorphism in degrees $* \geq 2$ [Sch17a, Theorem 4.18].

In the following we will use the main result from [GSZ16] which states that if R is a local ring containing an infinite field of characteristic not two, then the following square is cartesian

$$(10.6) \quad \begin{array}{ccc} K_n^{MW}(R) & \xrightarrow{\eta^n} & I^n(R) \\ \downarrow f & & \downarrow \\ K_n^M(R) & \longrightarrow & k_n^M(R) \end{array}$$

where the right vertical map sends $[a_1] \cup \dots \cup [a_n]$ to $\{a_1\} \cup \dots \cup \{a_n\}$ and induces an isomorphism $I^n(R)/I^{n+1}(R) \cong k_n^M(R)$. In other words, the map (10.7)

$$\lambda : K_n^{MW}(R) \xrightarrow{\cong} K_n^M(R) \times_{k_n} I^n(R) : [a_1, \dots, a_n] \mapsto \{a_1, \dots, a_n\}, [a_1] \cdots [a_n]$$

is an isomorphism. In addition to the forgetful map

$$f : K_n^{MW}(R) \rightarrow K_n^M(R) : [a_1] \cup \dots \cup [a_n] \mapsto \{a_1\} \cup \dots \cup \{a_n\}$$

we also have the hyperbolic map

$$h : K_n^M(R) \rightarrow K_n^{MW}(R) : \{a_1\} \cup \dots \cup \{a_n\} \mapsto \langle 1, -1 \rangle [a_1] \cup \dots \cup [a_n]$$

and the map

$$\eta : K_n^{MW}(R) \rightarrow K_{n-1}^{MW}(R) : [a_1] \cup [a_2] \cup \dots \cup [a_n] \mapsto (1 - \langle a_1 \rangle) \cdot [a_2] \cup \dots \cup [a_n].$$

Under the isomorphism (10.7), forgetful, hyperbolic map and η are $K_n(R) \times_{k_n} I^n(R) \rightarrow K_n(R) : (x, y) \mapsto x$, $K_n(R) \rightarrow K_n(R) \times_{k_n} I^n(R) : x \mapsto (2x, 0)$ and

$K_n(R) \times_{k_n} I^n(R) \rightarrow K_{n-1}(R) \times_{k_{n-1}} I^{n-1}(R) : (x, y) \mapsto (0, y)$. In particular, for all integers $n \geq 1$, the following sequence is exact

$$K_n^M(R) \xrightarrow{h} K_n^{MW}(R) \xrightarrow{\eta} K_{n-1}^{MW}(R) \xrightarrow{f} K_{n-1}^M(R) \rightarrow 0$$

as it corresponds under the isomorphism (10.7) to the exact sequence

$$K_n^M \xrightarrow{(2,0)} K_n^M \times_{k_n} I^n \xrightarrow{(0 \times 1)} K_{n-1}^M \times_{k_{n-1}} I^{n-1} \xrightarrow{(1,0)} K_{n-1}^M \rightarrow 0.$$

For the next theorem recall that $GW_2^{[2]}(R) = \pi_2 B \operatorname{Sp}(R)^+$.

Theorem 10.5. *Let R be a local commutative ring containing an infinite field of characteristic $\operatorname{char}(k) \neq 2$. Then the KO -degree map is an isomorphism in degree 2:*

$$K_2^{MW}(R) \xrightarrow{\cong} GW_2^{[2]}(R).$$

Proof. The map in the theorem is injective because the following composition is injective

$$K_2^{MW}(R) \rightarrow GW_2^{[2]}(R) \xrightarrow{(f,\eta)} K_2(R) \times GW_1^{[1]}(R)$$

as that composition can be identified with the injective map

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} : K_2^M(R) \times_{k_2} I^2(R) \rightarrow K_2(R) \times K_1(R) \times_{k_1} I(R)$$

where $i : I^2(R) \subset I(R)$ is the inclusion. The map in the theorem is surjective since it is part of the following map of exact sequences

$$\begin{array}{ccccccc} K_2^M(R) & \xrightarrow{h} & K_2^{MW}(R) & \xrightarrow{\eta} & K_1^{MW}(R) & \xrightarrow{f} & K_1^M(R) \\ \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ K_2(R) & \xrightarrow{h} & GW_2^{[2]}(R) & \xrightarrow{\eta} & GW_1^{[1]}(R) & \xrightarrow{f} & K_1(R). \end{array}$$

□

Consider the diagram (suppressing the local ring R in the notation)
(10.8)

$$\begin{array}{ccccccc} K_3^M & \longrightarrow & K_3 & \longrightarrow & H_3(SL) & \xleftarrow{\cong} & H_3(SL_3) & \longrightarrow & K_3^{MW} \\ \downarrow h & & \downarrow h & & \downarrow & & \downarrow & & \downarrow 1 \\ K_3^{MW} & \longrightarrow & GW_3^{[3]} & \xrightarrow{\cong} & H_3(SL, Sp) & \xleftarrow{\cong} & H_3(SL_3, SL_2) & \xlongequal{\quad} & K_3^{MW} \\ \downarrow \eta & & \downarrow \eta & & \downarrow & & \downarrow & & \downarrow \eta \\ K_2^{MW} & \xrightarrow{\cong} & GW_2^{[2]} & \xrightarrow{\cong} & H_2(Sp) & \xleftarrow{\cong} & H_2(SL_2) & \xlongequal{\quad} & K_2^{MW} \\ \downarrow f & & \downarrow f & & \downarrow & & \downarrow & & \downarrow \\ K_2^M & \xrightarrow{\cong} & K_2 & \xrightarrow{\cong} & H_2(SL) & \xleftarrow{\cong} & H_2(SL_3) & \xrightarrow{\cong} & K_2^M \end{array}$$

(*) (*) (*) (**)

where the horizontal arrows in the first column are the degree maps, in the second column they are the Hurewicz maps (with second and third row isomorphisms, by Theorems 9.1 and 9.2). In the third column, the arrows in the wrong direction are the group homology stability isomorphisms of Theorems 8.5 and 8.2 for the middle two and [Sch17a, Theorem 5.37] for the top and bottom map.

Lemma 10.6. *Diagram (10.8) commutes.*

Proof. Diagrams (*) commute in view of the compatibility of the Hurewicz map with the long exact sequence of homotopy and homology groups for pairs (which, for simplicial sets, is just functoriality of the long exact sequence of homotopy groups for pairs applied to the natural transformation $X \rightarrow \tilde{Z}[X] = \mathbb{Z}[X]/\mathbb{Z}x_0 : x \mapsto x$; see [GJ99, Section III.3]). The square (**) commutes, by [Sch17a, Lemma 5.40(2)]. The rest follows by functoriality and the definition of the KO -degree map. \square

Lemma 10.7. *The composition $\varphi : K_3^M \rightarrow K_3^{MW}$ in the string of maps*

$$K_3^M \longrightarrow K_3 \longrightarrow H_3(SL) \xleftarrow{\cong} H_3(SL_3) \longrightarrow K_3^{MW}$$

that is the top row of diagram (10.8) is the hyperbolic map $h : K_3^M \rightarrow K_3^{MW}$.

Proof. Under the isomorphism (10.7), the map $h : K_n^M(R) \rightarrow K_n^{MW}(R)$ becomes $(2, 0) : K_n^M(R) \times_{k_n} I^n$, and the map $\eta : K_n^{MW}(R) \rightarrow K_{n-1}^{MW}(R)$ becomes $(0, i)$ where $i : I^n \subset I^{n-1}$ is the inclusion. By [NS89, Theorem 4.1(a)], the first factor $K_3^M(R) \rightarrow K_3^M(R)$ of $\lambda\varphi$ is multiplication by 2. The second factor $K_3^M(R) \rightarrow I^3$ of $\lambda\varphi$ is zero since the following subdiagram of (10.8) commutes

$$\begin{array}{ccccccc} K_3 & \longrightarrow & H_3(SL) & \xleftarrow{\cong} & H_3(SL_3) & \longrightarrow & K_3^{MW} \longrightarrow I^3 \\ \downarrow h & & \downarrow & & \downarrow & & \downarrow 1 \quad \downarrow 1 \\ GW_3^{[3]} & \xrightarrow{\cong} & H_3(SL, Sp) & \xleftarrow{\cong} & H_3(SL_3, SL_2) & \xlongequal{\quad} & K_3^{MW} \longrightarrow I^3 \\ \downarrow \eta & & \downarrow & & \downarrow & & \downarrow \eta \quad \downarrow i \\ GW_2^{[2]} & \xrightarrow{\cong} & H_2(Sp) & \xleftarrow{\cong} & H_2(SL_2) & \xlongequal{\quad} & K_2^{MW} \longrightarrow I^2, \end{array}$$

the composition of the left vertical maps is zero whereas the composition of the right vertical maps is injective. \square

Theorem 10.8. *Let R be a local commutative ring containing an infinite field of characteristic $\text{char}(k) \neq 2$. Then the KO -degree map is an isomorphism in degree 3:*

$$K_3^{MW}(R) \xrightarrow{\cong} GW_3^{[3]}(R).$$

Proof. We only need to show that the composition $\phi : K_3^{MW} \rightarrow K_3^M$ of the second top row in diagram (10.8) is an isomorphism. By Lemma 10.7, that map is part of the commutative diagram with exact rows

$$\begin{array}{ccccccc} K_3^M & \xrightarrow{h} & K_3^{MW} & \xrightarrow{\eta} & K_2^{MW} & \xrightarrow{f} & K_2^M \\ \downarrow 1 & & \downarrow \phi & & \downarrow \cong & & \downarrow \cong \\ K_3^M & \xrightarrow{h} & K_3^{MW} & \xrightarrow{\eta} & K_2^{MW} & \xrightarrow{f} & K_2^M. \end{array}$$

The map on the left horizontal kernels is the identity map. Hence, by the Five Lemma, ϕ is an isomorphism. \square

Lemma 10.9. *Let R be a local ring with infinite residue field. Then $GW_2^{[3]}(R) = 0$.*

Proof. By definition, we have $GW_2^{[3]}(R) = \pi_2(BSL^+(R), BSp^+(R))$. By the relative Hurewicz Theorem [GJ99, III Corollary 3.12] that group is $H_2(SL(R), Sp(R))$. By Theorem 8.5, this is $H_2(SL_3(R), SL_2(R))$ which is zero, by [Sch17a, Theorem 5.37]. \square

We recall that $K_3^{ind}(R) = \text{coker}(K_3^M(R) \rightarrow K_3(R))$.

Corollary 10.10. *Let R be a local ring containing an infinite field of characteristic not 2. Then $K_3^{ind}(R) \cong KO_3(R)$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} K_3^{MW}(R) & \twoheadrightarrow & K_3^M(R) & & & & \\ \cong \downarrow & & \downarrow & & & & \\ GW_3^{[3]}(R) & \longrightarrow & K_3(R) & \longrightarrow & GW_3^{[4]}(R) & \longrightarrow & GW_2^{[3]}(R) = 0 \end{array}$$

in which the horizontal arrows in the square are the forgetful maps (in particular the top map is surjective), the vertical maps are the KO and K -theory degree maps, and the bottom row is the Bott exact sequence. The group $GW_2^{[3]}(R)$ is zero, by Lemma 10.9, and the left vertical map is an isomorphism, by Theorem 10.8. By [Sch17b, Remark 5.9 and Corollary A.2], we have $GW_3^{[4]}(R) = KO_3(R)$, and the result follows. \square

APPENDIX A. THE PFAFFIAN OF SOME MATRICES

For space reasons, we will write $[a, b, c]$ for the skew-symmetric matrix

$$[a, b, c] = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}.$$

A.1. The skew-symmetric matrix of Lemma 6.2. The 6×6 skew-symmetric matrix

$$A = \begin{pmatrix} 0 & a & a & a & a & a \\ -a & 0 & b & b & b & b \\ -a & -b & 0 & c & c & c \\ -a & -b & -c & 0 & d & d \\ -a & -b & -c & -d & 0 & e \\ -a & -b & -c & -d & -e & 0 \end{pmatrix}$$

has Pfaffian

$$\text{Pf}(A) = ace,$$

and for $1 \leq i < j < k \leq 6$, the values of $(-1)^{i+j+k} \frac{\text{Pf}(A)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge)}$ and $[A_{ijk}^\wedge]$ are as follows.

(i, j, k)	$(-1)^{i+j+k} \frac{\text{Pf}(A)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge)}$	$[A_{ijk}^\wedge]$
(1, 2, 3)	$1/(be^2)$	$[d, d, e]$
(1, 2, 4)	$(-1)/(be^2)$	$[c, c, e]$
(1, 3, 4)	$c/(b^2e^2)$	$[b, b, e]$
(2, 3, 4)	$(-c)/(a^2e^2)$	$[a, a, e]$
(1, 2, 5)	$1/(bd^2)$	$[c, c, d]$
(1, 3, 5)	$(-c)/(b^2d^2)$	$[b, b, d]$
(2, 3, 5)	$c/(a^2d^2)$	$[a, a, d]$
(1, 4, 5)	$1/(b^2d)$	$[b, b, c]$
(2, 4, 5)	$(-1)/(a^2d)$	$[a, a, c]$
(3, 4, 5)	$1/(a^2d)$	$[a, a, b]$
(1, 2, 6)	$(-1)/(bd^2)$	$[c, c, d]$
(1, 3, 6)	$c/(b^2d^2)$	$[b, b, d]$
(2, 3, 6)	$(-c)/(a^2d^2)$	$[a, a, d]$
(1, 4, 6)	$(-1)/(b^2d)$	$[b, b, c]$
(2, 4, 6)	$1/(a^2d)$	$[a, a, c]$
(3, 4, 6)	$(-1)/(a^2d)$	$[a, a, b]$
(1, 5, 6)	$e/(b^2d^2)$	$[b, b, c]$
(2, 5, 6)	$(-e)/(a^2d^2)$	$[a, a, c]$
(3, 5, 6)	$e/(a^2d^2)$	$[a, a, b]$
(4, 5, 6)	$(-e)/(a^2c^2)$	$[a, a, b]$

A.2. The skew-symmetric matrix of Lemma 6.19. The 6×6 skew-symmetric matrix

$$A = \begin{pmatrix} 0 & d & d & d & d & d \\ -d & 0 & d & d & d & d \\ -d & -d & 0 & d & d & d \\ -d & -d & -d & 0 & a & b \\ -d & -d & -d & -a & 0 & c \\ -d & -d & -d & -b & -c & 0 \end{pmatrix}$$

has Pfaffian

$$\text{Pf}(A) = (a - b + c)d^2,$$

and for $1 \leq i < j < k \leq 6$, the values of $(-1)^{i+j+k} \frac{\text{Pf}(A)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge)}$ and

$[A_{ijk}^\wedge]$ are as follows.

(i, j, k)	$(-1)^{i+j+k} \frac{\text{Pf}(A)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge)}$	$[A_{ijk}^\wedge]$
(1, 2, 3)	$1/d(a - b + c)^2$	$[a, b, c]$
(1, 2, 4)	$(-1)/(c^2d)$	$[d, d, c]$
(1, 3, 4)	$1/(c^2d)$	$[d, d, c]$
(2, 3, 4)	$(-1)/(c^2d)$	$[d, d, c]$
(1, 2, 5)	$1/(b^2d)$	$[d, d, b]$
(1, 3, 5)	$(-1)/(b^2d)$	$[d, d, b]$
(2, 3, 5)	$1/(b^2d)$	$[d, d, b]$
(1, 4, 5)	$(a - b + c)/(bcd^2)$	$[d, d, d]$
(2, 4, 5)	$(-a + b - c)/(bcd^2)$	$[d, d, d]$
(3, 4, 5)	$(a - b + c)/(bcd^2)$	$[d, d, d]$
(1, 2, 6)	$(-1)/(a^2d)$	$[d, d, a]$
(1, 3, 6)	$1/(a^2d)$	$[d, d, a]$
(2, 3, 6)	$(-1)/(a^2d)$	$[d, d, a]$
(1, 4, 6)	$(-a + b - c)/(acd^2)$	$[d, d, d]$
(2, 4, 6)	$(a - b + c)/(acd^2)$	$[d, d, d]$
(3, 4, 6)	$(-a + b - c)/(acd^2)$	$[d, d, d]$
(1, 5, 6)	$(a - b + c)/(abd^2)$	$[d, d, d]$
(2, 5, 6)	$(-a + b - c)/(abd^2)$	$[d, d, d]$
(3, 5, 6)	$(a - b + c)/(abd^2)$	$[d, d, d]$
(4, 5, 6)	$(-a + b - c)/d^4$	$[d, d, d]$

A.3. The skew-symmetric matrix of Lemma 6.20. The 6×6 skew-symmetric matrix

$$A = \begin{pmatrix} 0 & a & b & d & d & d \\ -a & 0 & b & e & e & e \\ -b & -b & 0 & f & f & f \\ -d & -e & -f & 0 & a & b \\ -d & -e & -f & -a & 0 & b \\ -d & -e & -f & -b & -b & 0 \end{pmatrix}$$

has Pfaffian

$$\text{Pf}(A) = a(bd - be + af),$$

and for $1 \leq i < j < k \leq 6$, the values of $(-1)^{i+j+k} \frac{\text{Pf}(A)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge)}$ and $[A_{ijk}^\wedge]$ are as follows.

(i, j, k)	$(-1)^{i+j+k} \frac{\text{Pf}(A)}{\text{Pf}(A_{ij}^\wedge) \text{Pf}(A_{ik}^\wedge) \text{Pf}(A_{jk}^\wedge)}$	$[A_{ijk}^\wedge]$
(1, 2, 3)	$(bd - be + af)/(a^2def)$	$[a, b, b]$
(1, 2, 4)	$(-bd + be - af)/(b^4f)$	$[f, f, b]$
(1, 3, 4)	$(bd - be + af)/(ab^3e)$	$[e, e, b]$
(2, 3, 4)	$(-bd + be - af)/(ab^3d)$	$[d, d, b]$
(1, 2, 5)	$(bd - be + af)/(b^4f)$	$[f, f, b]$
(1, 3, 5)	$(-bd + be - af)/(ab^3e)$	$[e, e, b]$
(2, 3, 5)	$(bd - be + af)/(ab^3d)$	$[d, d, b]$
(1, 4, 5)	a/b^4	$[b, e, f]$
(2, 4, 5)	$(-a)/b^4$	$[b, d, f]$
(3, 4, 5)	$1/(ab^2)$	$[a, d, e]$
(1, 2, 6)	$(-bd + be - af)/(a^2b^2f)$	$[f, f, a]$
(1, 3, 6)	$(bd - be + af)/(a^3be)$	$[e, e, a]$
(2, 3, 6)	$(-bd + be - af)/(a^3bd)$	$[d, d, a]$
(1, 4, 6)	$(-1)/b^3$	$[b, e, f]$
(2, 4, 6)	$1/b^3$	$[b, d, f]$
(3, 4, 6)	$(-1)/(a^2b)$	$[a, d, e]$
(1, 5, 6)	$1/b^3$	$[b, e, f]$
(2, 5, 6)	$(-1)/b^3$	$[b, d, f]$
(3, 5, 6)	$1/(a^2b)$	$[a, d, e]$
(4, 5, 6)	$(-a)/(bd - be + af)^2$	$[a, b, b]$

REFERENCES

- [AF17] Aravind Asok and Jean Fasel. An explicit KO -degree map and applications. *J. Topol.*, 10(1):268–300, 2017.
- [Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.
- [DW95] Andreas W. M. Dress and Walter Wenzel. A simple proof of an identity concerning Pfaffians of skew symmetric matrices. *Adv. Math.*, 112(1):120–134, 1995.
- [Ess13] Jan Essert. Homological stability for classical groups. *Israel J. Math.*, 198(1):169–204, 2013.
- [GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
- [GSZ16] Stefan Gille, Stephen Scully, and Changlong Zhong. Milnor-Witt K -groups of local rings. *Adv. Math.*, 286:729–753, 2016.
- [HT10] Kevin Hutchinson and Liqun Tao. Homology stability for the special linear group of a field and Milnor-Witt K -theory. *Doc. Math.*, (Extra vol.: Andrei A. Suslin sixtieth birthday):267–315, 2010.
- [HW15] Kevin Hutchinson and Matthias Wendt. On the third homology of SL_2 and weak homotopy invariance. *Trans. Amer. Math. Soc.*, 367(10):7481–7513, 2015.
- [Kar73] Max Karoubi. Périodicité de la K -théorie hermitienne. pages 301–411. *Lecture Notes in Math.*, Vol. 343, 1973.
- [KSW16] Max Karoubi, Marco Schlichting, and Charles Weibel. The Witt group of real algebraic varieties. *J. Topol.*, 9(4):1257–1302, 2016.
- [Mat69] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.
- [MH73] John Milnor and Dale Husemoller. *Symmetric bilinear forms*. Springer-Verlag, New York-Heidelberg, 1973. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73*.
- [Mir05] B. Mirzaii. Homology stability for unitary groups. II. *K-Theory*, 36(3-4):305–326 (2006), 2005.
- [Mor12] Fabien Morel. \mathbb{A}^1 -algebraic topology over a field, volume 2052 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.
- [NS89] Yu. P. Nesterenko and A. A. Suslin. Homology of the general linear group over a local ring, and Milnor’s K -theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 53(1):121–146, 1989.
- [Sch17a] Marco Schlichting. Euler class groups and the homology of elementary and special linear groups. *Adv. Math.*, 320:1–81, 2017.
- [Sch17b] Marco Schlichting. Hermitian K -theory, derived equivalences and Karoubi’s fundamental theorem. *J. Pure Appl. Algebra*, 221(7):1729–1844, 2017.
- [Sch21] Marco Schlichting. Higher K -theory of forms I. From rings to exact categories. *J. Inst. Math. Jussieu*, 20(4):1205–1273, 2021.
- [SW20] David Sprehn and Nathalie Wahl. Homological stability for classical groups. *Trans. Amer. Math. Soc.*, 373:4807–4861, 2020.
- [vdK77] Wilberd van der Kallen. The K_2 of rings with many units. *Ann. Sci. École Norm. Sup. (4)*, 10(4):473–515, 1977.

HUSNEY PARVEZ SARWAR, DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR, KHARAGPUR - 721302, WEST BENGAL, INDIA
E-mail address: mathparvez@gmail.com; parvez@maths.iitkgp.ac.in

MARCO SCHLICHTING, MATHEMATICS INSTITUTE, ZEEMAN BUILDING, UNIVERSITY OF WARWICK, COVENTRY CV4 7AL, UK
E-mail address: m.schlichting@warwick.ac.uk