

ARITHMETIC AND GEOMETRIC DEFORMATIONS OF F -PURE AND F -REGULAR SINGULARITIES

KENTA SATO AND SHUNSUKE TAKAGI

Dedicated to Professor Nobuo Hara on the occasion of his sixtieth birthday.

ABSTRACT. Given a flat family \mathcal{X} of \mathbb{Q} -Gorenstein singularities over an open subset U of $\mathrm{Spec} \mathbb{Z}$, our first main result says that if the total space \mathcal{X} is \mathbb{Q} -Gorenstein and a mod- p fiber is F -pure for some $p \in U$, then the generic fiber is log canonical. We also show the analog of this result for log terminal singularities, without assuming that \mathcal{X} is \mathbb{Q} -Gorenstein, as a generalization of the result of Ma-Schwede. Our second main result shows that two-dimensional, strongly F -regular singularities are stable under equal characteristic deformations. Our results provide an affirmative answer to the conjecture of Liedtke-Martin-Matsumoto regarding the deformations of linearly reductive quotient singularities.

1. INTRODUCTION

F -singularities are singularities in positive characteristic defined via the Frobenius morphism. The link between F -singularities and singularities in the minimal model program has been studied intensively over the last 25 years. To explain this link, we establish some notation. Let $(x \in X)$ be a normal \mathbb{Q} -Gorenstein singularity over \mathbb{C} ; that is, X is a normal \mathbb{Q} -Gorenstein affine variety $\mathrm{Spec} \mathbb{C}[X_1, \dots, X_r]/(f_1, \dots, f_s)$ over \mathbb{C} , and x is a closed point of X . We assume for simplicity that x corresponds to the maximal ideal (X_1, \dots, X_r) and that all the coefficients of f_i are rational numbers. We choose an integer $n \geq 1$ such that f_i are polynomials over $\mathbb{Z}[1/n]$ and $\mathcal{X} := \mathrm{Spec} \mathbb{Z}[1/n][X_1, \dots, X_r]/(f_1, \dots, f_s)$ is flat over $U := \mathrm{Spec} \mathbb{Z}[1/n] \subset \mathrm{Spec} \mathbb{Z}$. Let \mathcal{Z} be the closed subscheme of \mathcal{X} defined by (X_1, \dots, X_r) and x_η (resp. x_p) be the unique point of the generic fiber \mathcal{Z}_η (resp. the fiber \mathcal{Z}_p over each closed point $(p) \in U$) of the flat morphism $\mathcal{Z} \subseteq \mathcal{X} \rightarrow U$. Then, $(x \in X)$ is a flat base change of the generic fiber $(x_\eta \in \mathcal{X}_\eta)$, while the closed fiber $(x_p \in \mathcal{X}_p)$ is of characteristic $p > 0$. The link mentioned above is provided by comparing the properties of $(x \in X)$ with those of the closed fibers $(x_p \in \mathcal{X}_p)$. In this paper, pursuing this link, we study how mild the singularity $(x \in X)$ is when some closed fiber $(x_p \in \mathcal{X}_p)$ is a mild F -singularity, such as an F -pure or a strongly F -regular singularity.

Hara-Watanabe [11] proved that $(x \in X)$ is a log terminal (resp. log canonical) singularity if the closed fiber $(x_p \in \mathcal{X}_p)$ is strongly F -regular (resp. F -pure) for infinitely many closed points $(p) \in U$. Recently, using perfectoid techniques, Ma-Schwede [25] developed a new theory of singularities in mixed characteristic, where big Cohen-Macaulay (BCM) test ideals,

a generalization of test ideals to the mixed characteristic case, play a central role. As an application of this theory, under the assumption that the total space \mathcal{X} is \mathbb{Q} -Gorenstein, they proved that $(x \in X)$ is log terminal if the closed fiber $(x_p \in \mathcal{X}_p)$ is strongly F -regular for a single closed point $(p) \in U$. In this paper, we generalize their result to the case where the total space \mathcal{X} is not necessarily \mathbb{Q} -Gorenstein. This generalization is important because \mathbb{Q} -Gorensteinness does not generally lift from Cartier divisors as Gorensteinness does. Perhaps more surprisingly, we also prove an analog of their result for log canonical singularities. A simple form of our first main result is stated (with renewed notation) as follows:

Theorem A (cf. Corollaries 3.10 and 4.3). *Let $(\mathcal{X}, \mathcal{D}) \rightarrow U \subseteq \text{Spec } \mathbb{Z}$ be a flat family of pairs, where \mathcal{X} is a normal integral scheme and \mathcal{D} is an effective \mathbb{Q} -Weil divisor on \mathcal{X} . Let \mathcal{Z} be an irreducible closed subscheme of \mathcal{X} , flat over U , such that the fiber of $\mathcal{Z} \subseteq \mathcal{X} \rightarrow U$ over each closed point $(p) \in U$ is a singleton $\{x_p\}$, and let $x_\eta \in \mathcal{X}_\eta$ be the generic point of \mathcal{Z} .*

- (1) *Suppose that the generic fiber $(\mathcal{X}_\eta, \mathcal{D}_\eta)$ is log \mathbb{Q} -Gorenstein at x_η , that is, $K_{\mathcal{X}_\eta} + \mathcal{D}_\eta$ is \mathbb{Q} -Cartier at x_η . If the fiber $(\mathcal{X}_p, \mathcal{D}_p)$ is log \mathbb{Q} -Gorenstein and strongly F -regular at x_p for a single closed point $(p) \in U$, then $(\mathcal{X}_\eta, \mathcal{D}_\eta)$ is klt at x_η .*
- (2) *If the total space $(\mathcal{X}, \mathcal{D})$ is log \mathbb{Q} -Gorenstein at x_p and the fiber $(\mathcal{X}_p, \mathcal{D}_p)$ is normal and (sharply) F -pure at x_p for a single closed point $(p) \in U$, then the generic fiber $(\mathcal{X}_\eta, \mathcal{D}_\eta)$ is log canonical at x_η .*

This theorem is quite useful for verifying that a given singularity is log terminal or log canonical because strong F -regularity and F -purity can be (relatively) easily checked by a computer algebra system such as Macaulay 2, as opposed to constructing a resolution of singularities (see Remark 3.12).

We briefly explain the idea of the proof of Theorem A. First, we generalize the notion of the BCM test ideal $\tau_{\mathcal{B}}(R, \mathfrak{a}^\lambda)$, which was defined in [25] when \mathfrak{a} is a principal ideal, to the case of an arbitrary ideal \mathfrak{a} . Replacing multiplier ideals by our BCM test ideals, one can use an argument similar to that of Kawakita [18], which shows that log canonical singularities over \mathbb{C} deform to log canonical singularities if the total space is \mathbb{Q} -Gorenstein, to obtain assertion (2). Similarly, one can use an argument analogous to that of Esnault-Viehweg [6], which proves that two-dimensional log terminal singularities over \mathbb{C} deform to log terminal singularities, to obtain assertion (1).

Theorem A concerns arithmetic deformations of F -singularities, and next, we discuss geometric deformations of F -singularities. In general, log terminal singularities over \mathbb{C} are not stable under small deformations unless the total space is \mathbb{Q} -Gorenstein. A notable exception is the two-dimensional case, which was described by [6] as mentioned above. In fact, their proof tells us that higher-dimensional log terminal singularities over \mathbb{C} are also stable under small deformations if the nearby fibers are \mathbb{Q} -Gorenstein. Since strongly F -regular singularities can be viewed as theoretical F -singularity counterparts of log terminal singularities, it is natural to ask how strongly F -regular singularities behave under equal characteristic deformations. Strong

F -regularity does not deform in general (see [37]), but we prove that the analog of the result of Esnault-Viehweg holds for strongly F -regular singularities. A simple form of our second main result is stated as follows:

Theorem B (cf. Corollaries 4.9 and 4.11). *Let $(\mathcal{X}, \mathcal{D}) \rightarrow T$ be a proper flat family of pairs, where \mathcal{X} is a normal variety over a perfect field k of characteristic $p > 0$, \mathcal{D} is an effective \mathbb{Q} -Weil divisor on \mathcal{X} , and T is a smooth curve over k .*

- (1) *Suppose that the generic fiber $(\mathcal{X}_\eta, \mathcal{D}_\eta)$ and some closed fiber $(\mathcal{X}_{t_0}, \mathcal{D}_{t_0})$ are log \mathbb{Q} -Gorenstein. If $(\mathcal{X}_{t_0}, \mathcal{D}_{t_0})$ is strongly F -regular, then so is the geometric generic fiber $(\mathcal{X}_{\bar{\eta}}, \mathcal{D}_{\bar{\eta}})$.*
- (2) *Suppose that a general closed fiber $(\mathcal{X}_t, \mathcal{D}_t)$ and some closed fiber $(\mathcal{X}_{t_0}, \mathcal{D}_{t_0})$ are log \mathbb{Q} -Gorenstein. If k is an uncountable algebraically closed field and $(\mathcal{X}_{t_0}, \mathcal{D}_{t_0})$ is strongly F -regular, then so is the general closed fiber $(\mathcal{X}_t, \mathcal{D}_t)$.*

As a corollary, we see that two-dimensional strongly F -regular singularities are stable under equal characteristic deformations. The proof of Theorem B is similar to that of Theorem A (1), but we use classical test ideals instead of BCM test ideals.

Finally, Theorems A and B enable us to provide an affirmative answer to a conjecture of Liedtke-Martin-Matsumoto [22, Conjecture 12.1 (1)], which states that isolated linearly reductive quotient singularities deform to linearly reductive quotient singularities in arbitrary characteristic.

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Notation. Throughout this paper, all rings are assumed to be commutative and have unit elements, and all schemes are assumed to be Noetherian and separated.

2. PRELIMINARIES

2.1. Singularities in the MMP. In this subsection, we recall the definition and basic properties of singularities in the MMP (minimal model program).

Throughout this subsection, we assume that X is an excellent normal integral scheme with a dualizing complex ω_X^\bullet . The *canonical sheaf* ω_X associated to ω_X^\bullet is the coherent \mathcal{O}_X -module defined as the first nonzero cohomology module of ω_X^\bullet . It is well-known that the canonical sheaf ω_X

is compatible with localization ([40, Lemma 0A7G]), rank one ([40, Lemma 0AWX]) and satisfies the Serre's second condition (S_2) ([40, Lemma 0AWK], see also [40, Section 0DW3] and [2, (1.10)]). A *canonical divisor* of X associated to ω_X^\bullet is any Weil divisor K_X on X such that $\mathcal{O}_X(K_X) \cong \omega_X$. We fix a canonical divisor K_X of X associated to ω_X^\bullet . Given a proper birational morphism $\pi : Y \rightarrow X$ from a normal integral scheme Y , we always choose a canonical divisor K_Y of Y that is associated to $\pi^!\omega_X^\bullet$ and whose pushforward f_*K_Y agrees with K_X .

Definition 2.1. A proper birational morphism $f : Y \rightarrow X$ between integral schemes is said to be a *resolution of singularities* of X if Y is regular. When Δ is a \mathbb{Q} -Weil divisor on X and $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ are nonzero coherent ideal sheaves, a resolution $f : Y \rightarrow X$ is said to be a *log resolution* of $(X, \Delta, \mathfrak{a}, \mathfrak{b})$ if $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ and $\mathfrak{b}\mathcal{O}_Y = \mathcal{O}_Y(-G)$ are invertible and if the union of the exceptional locus $\text{Exc}(f)$ of f and the supports of F, G and the strict transform $f_*^{-1}\Delta$ of Δ is a simple normal crossing divisor.

First, we give the definition of singularities in the MMP that makes sense in arbitrary characteristic.

Definition 2.2. Suppose that Δ is an effective \mathbb{Q} -Weil divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier, $\mathfrak{a} \subseteq \mathcal{O}_X$ is a nonzero coherent ideal sheaf, and $\lambda > 0$ is a real number.

- (i) Given a proper birational morphism $f : Y \rightarrow X$ from a normal integral scheme Y , we write

$$\Delta_Y := f^*(K_X + \Delta) - K_Y.$$

When $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is invertible, for each prime divisor E on Y , the *discrepancy* $a_E(X, \Delta, \mathfrak{a}^\lambda)$ of the triple $(X, \Delta, \mathfrak{a}^\lambda)$ at E is defined as

$$a_E(X, \Delta, \mathfrak{a}^\lambda) := -\text{ord}_E(\Delta_Y + \lambda F).$$

- (ii) The triple $(X, \Delta, \mathfrak{a}^\lambda)$ is said to be *log canonical* (resp. *klt*) at a point $x \in X$ if $a_E(\text{Spec } \mathcal{O}_{X,x}, \Delta_x, \mathfrak{a}_x^\lambda) \geq -1$ (resp. > -1) for every proper birational morphism $f : Y \rightarrow \text{Spec } \mathcal{O}_{X,x}$ from a normal integral scheme Y with $\mathfrak{a}\mathcal{O}_Y$ invertible and for every prime divisor E on Y , where Δ_x is the flat pullback of Δ by the canonical morphism $\text{Spec } \mathcal{O}_{X,x} \rightarrow X$ and $\mathfrak{a}_x := \mathfrak{a}\mathcal{O}_{X,x}$. We say that $(X, \Delta, \mathfrak{a}^\lambda)$ is *log canonical* (resp. *klt*) if it is log canonical (resp. klt) for every $x \in X$.
- (iii) ([24]) X is said to be *pseudorational* at a point $x \in X$ if X is Cohen-Macaulay and if for every projective birational morphism $f : Y \rightarrow \text{Spec } \mathcal{O}_{X,x}$ from a normal integral scheme Y , the natural morphism $f_*\omega_Y \rightarrow \omega_{X,x}$ is an isomorphism.

Remark 2.3.

- (i) ([23, Proposition 17.1]) If X is pseudorational at x and $\dim \mathcal{O}_{X,x} = 2$, then X is \mathbb{Q} -factorial at x .
- (ii) (cf. [19, Corollary 11.14], [28]) Let Δ be an effective \mathbb{Q} -Weil divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. If X is an excellent \mathbb{Q} -scheme and (X, Δ) is klt at x , then X is pseudorational at x . We note

that [19, Corollary 11.14] is formulated for varieties, but the same statement regarding excellent \mathbb{Q} -schemes can be obtained by using [28, Theorems A and B] instead of the local vanishing theorem and the Grauert-Riemenschneider vanishing theorem.

Remark 2.4. Let $(X, \Delta, \mathfrak{a}^\lambda)$ be as in Definition 2.2. Then the set of points $x \in X$ such that $(X, \Delta, \mathfrak{a}^\lambda)$ is log canonical (resp. klt) at x is closed under generalization.

To see this, suppose that $(X, \Delta, \mathfrak{a}^\lambda)$ is log canonical (resp. klt) at $x \in X$ and take a point $y \in X$ that is a generalization of x . Let $f : Y \rightarrow \text{Spec } \mathcal{O}_{X,y}$ be a proper birational morphism from a normal integral scheme Y and E be a prime divisor on Y . Since E defines a discrete valuation $\text{ord}_E : K(X)^* \rightarrow \mathbb{Z}$, it follows from [20, Lemma 2.45] that there exist a proper birational morphism $g : Z \rightarrow \text{Spec } \mathcal{O}_{X,x}$ from a normal integral scheme Z and a prime divisor F on Z such that the valuation $\text{ord}_F : K(Z)^* \rightarrow \mathbb{Z}$ coincides with ord_E . Then $Z_y := Z \times_{\text{Spec } \mathcal{O}_{X,x}} \text{Spec } \mathcal{O}_{X,y}$ is a normal integral scheme, the base change $g_y : Z_y \rightarrow \text{Spec } \mathcal{O}_{X,y}$ of g is a proper birational morphism, and the pullback F' of F to Z_y is a prime divisor such that $\text{ord}_{F'} = \text{ord}_E$. Take a proper birational morphism $h : W \rightarrow \text{Spec } \mathcal{O}_{X,y}$ from a normal integral scheme W that factors through both f and g_y . The strict transform of E to W coincides with that of F' , leading to

$$\begin{aligned} a_E(\text{Spec } \mathcal{O}_{X,y}, \Delta_y, \mathfrak{a}_y^\lambda) &= a_{F'}(\text{Spec } \mathcal{O}_{X,y}, \Delta_y, \mathfrak{a}_y^\lambda) \\ &= a_F(\text{Spec } \mathcal{O}_{X,x}, \Delta_x, \mathfrak{a}_x^\lambda) \geq -1 \quad (\text{resp. } > -1). \end{aligned}$$

Thus, $(X, \Delta, \mathfrak{a}^\lambda)$ is log canonical (resp. klt) at y .

Definition 2.5. Let $(X, \Delta, \mathfrak{a}^\lambda)$ be as in Definition 2.2. The *multiplier ideal sheaf* $\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda)$ associated to $(X, \Delta, \mathfrak{a}^\lambda)$ is defined as

$$\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda) := \bigcap_{f:Y \rightarrow X} f_* \mathcal{O}_Y(-[\Delta_Y + \lambda F]),$$

where $f : Y \rightarrow X$ runs through all proper birational morphisms from a normal integral scheme Y with $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ invertible and $\Delta_Y := f^*(K_X + \Delta) - K_Y$.

Remark 2.6. Let $(X, \Delta, \mathfrak{a}^\lambda)$ be as in Definition 2.2.

- (i) For any point $x \in X$, we have $\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda)_x \subseteq \mathcal{J}(\text{Spec } \mathcal{O}_{X,x}, \Delta_x, \mathfrak{a}_x^\lambda)$. In particular, if $\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda)_x = \mathcal{O}_{X,x}$, then $(X, \Delta, \mathfrak{a}^\lambda)$ is klt at x .
- (ii) If $f : Y \rightarrow X$ is a log resolution of $(X, \Delta, \mathfrak{a})$, then

$$\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda) = f_* \mathcal{O}_Y(-[\Delta_Y + \lambda F]).$$

In particular, if X is defined over a field of characteristic zero, or if X is defined over a field of positive characteristic with $\dim X \leq 3$, or if $\dim X \leq 2$, then the following statements hold:

- (a) The converse of (i) is true.
- (b) $\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda)$ is coherent.
- (c) It is sufficient to check the condition in Definition 2.2 (ii) for only one f , namely, for a log resolution of $(X, \Delta, \mathfrak{a}^\lambda)$.

The following lemma is probably well known to experts, but we include its proof for the reader's convenience.

Lemma 2.7. *Suppose that $(R, \mathfrak{m}, \kappa) \rightarrow (R', \mathfrak{m}', \kappa')$ is a flat local homomorphism of excellent local rings with dualizing complexes. Assume that we have $\mathfrak{m}R' = \mathfrak{m}'$ and that κ' is separable over κ . Let $f : X' := \text{Spec } R' \rightarrow X := \text{Spec } R$ be the induced morphism sending a point $x' \in X'$ to a point $x \in X$, Δ be an effective \mathbb{Q} -Weil divisor on $X := \text{Spec } R$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier at x , $\mathfrak{a} \subseteq R$ be a nonzero ideal sheaf and $\lambda > 0$ be a real number.*

- (1) *The flat pullback $K_{X'} := f^*K_X$ of K_X by the induced morphism $f : X' := \text{Spec } R' \rightarrow X$ is a canonical divisor of X' . In particular, $K_{X'} + \Delta'$ is \mathbb{Q} -Cartier at x' , where $\Delta' := f^*\Delta$ is the flat pullback of Δ .*
- (2) *If $(X', \Delta', (\mathfrak{a}\mathcal{O}_{X'})^\lambda)$ is log canonical (resp. klt) at x' , then so is $(X, \Delta, \mathfrak{a}^\lambda)$ at x .*
- (3) *If X is defined over a field of characteristic zero, then the converse of (2) also holds.*

Proof. (1) First, note by [40, Lemma 0AWD] that $\omega_{X'}^\bullet := f^*\omega_X^\bullet$ is a dualizing complex on X' . Since $f^*\mathcal{O}_X(K_X) \cong \mathcal{O}_{X'}(f^*K_X)$, we see that f^*K_X is a canonical divisor of X' associated to $\omega_{X'}^\bullet$.

(2) Since the closed fiber $\text{Spec } \kappa'$ of f is formally smooth over $\text{Spec } \kappa$, it follows from [27, Theorem 28.10] and [1] that f is a regular morphism; that is, all fibers are geometrically regular. We set $V := \text{Spec } \mathcal{O}_{X,x}$ and $V' := \text{Spec } \mathcal{O}_{X',x'}$. Then, the induced morphism $h : V' \rightarrow V$ is regular, $\omega_V^\bullet := \omega_X^\bullet|_V$ is a dualizing complex of V and $\omega_{V'}^\bullet := \omega_{X'}^\bullet|_{V'} \cong h^*\omega_V^\bullet$ is a dualizing complex of V' .

Let $\pi : Y \rightarrow V$ be a proper birational morphism from a normal integral scheme Y , where $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is invertible. We set $Y' := V' \times_V Y$ and let $\pi' : Y' \rightarrow V'$ and $g : Y' \rightarrow Y$ be the first and second projections, respectively.

Claim. Y' is a normal integral scheme.

Proof. Let K be a function field of V and Y . First note that

$$Y' \times_Y \text{Spec } K \cong V' \times_V \text{Spec } K$$

because π is birational. Since $V' \times_V \text{Spec } K$ is irreducible, $Y' \times_Y \text{Spec } K$ is also irreducible. Considering that any generic point of Y' lies in the generic fiber $Y' \times_Y \text{Spec } K$ of g , we see that Y' is irreducible. It then follows from [27, p.184 Corollary] that Y' is a normal integral scheme. \square

We fix a canonical divisor K_Y of Y associated to the dualizing complex $\omega_Y^\bullet := \pi^!\omega_V^\bullet$ such that $\pi_*K_Y = K_V$. We write $\Delta_Y := \pi^*(K_V + \Delta|_V) - K_Y$, and we let $F' := g^*F$, $\Delta_{Y'} := g^*\Delta_Y$ and $K_{Y'} := g^*K_Y$ be the flat pullbacks of F , Δ_Y and K_Y , respectively. Then, $\mathfrak{a}\mathcal{O}_{Y'} = \mathcal{O}_{Y'}(-F')$, and $\Delta_{Y'} = \pi'^*(K_V' + \Delta') - K_{Y'}$. Noting that $K_{Y'}$ is a canonical divisor of Y' associated to $\pi'^!(\omega_{V'}^\bullet) \cong g^*\omega_Y^\bullet$ (see [40, Lemma 0AA8]) such that $\pi'_*K_{Y'} = K_{X'}$, we deduce that

$$-\text{ord}_G(g^*(\Delta_Y + \lambda F)) = -\text{ord}_G(\Delta_{Y'} + \lambda F') = a_G(V', \Delta'|_{V'}, (\mathfrak{a}\mathcal{O}_{V'})^\lambda)$$

for every prime divisor G on Y' . Since g is regular, the flat pullback g^*E of a prime divisor E on Y is a reduced divisor on Y' (see [27, p.184 Corollary] again). Therefore, for any irreducible component E' of $g^{-1}(E)$, we have

$$\begin{aligned} a_E(V, \Delta|_V, (\mathfrak{a}\mathcal{O}_V)^\lambda) &= -\text{ord}_E(\Delta_Y + \lambda F) = -\text{ord}_{E'}(g^*(\Delta_Y + \lambda F)) \\ &= a_{E'}(V', \Delta'|_{V'}, (\mathfrak{a}\mathcal{O}_{V'})^\lambda), \end{aligned}$$

which proves (2).

(3) We choose a log resolution of $(V, \Delta|_V, \mathfrak{a}\mathcal{O}_V)$ to be π in the proof of (2). It then follows from [27, Theorem 23.7] that π' is also a log resolution of $(V', \Delta'|_{V'}, \mathfrak{a}\mathcal{O}_{V'})$. Therefore, the assertion is immediate from Remark 2.6. \square

2.2. F -singularities. We briefly review the theory of F -singularities, particularly focusing on strongly F -regular and F -pure singularities and test ideals.

Definition 2.8 ([11], [42], [32], [34]). Let x be a point of an F -finite normal integral scheme X and Δ be an effective \mathbb{Q} -Weil divisor on X . Given an integer $e \geq 1$, let

$$\varphi_\Delta^{(e)} : \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X \hookrightarrow F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$$

be the composite of the e -th iterated Frobenius map $\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$ and the pushforward of the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)$ by F^e . Let $\mathfrak{a}_1, \dots, \mathfrak{a}_\ell \subseteq \mathcal{O}_X$ be nonzero coherent ideal sheaves and $\lambda_1, \dots, \lambda_\ell \geq 0$ be real numbers.

- (i) $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ is said to be *sharply F -pure* at x if there exist an integer $e \geq 1$ and a nonzero element $d \in \mathfrak{a}_1^{\lceil \lambda_1(p^e - 1) \rceil} \cdots \mathfrak{a}_\ell^{\lceil \lambda_\ell(p^e - 1) \rceil} \mathcal{O}_{X,x}$ such that the composite

$$\mathcal{O}_{X,x} \xrightarrow{\varphi_{\Delta,x}^{(e)}} F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)_x \xrightarrow{\times F_*^e d} F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)_x$$

of the $\mathcal{O}_{X,x}$ -linear map $\varphi_{\Delta,x}^{(e)}$ induced by $\varphi_\Delta^{(e)}$ and the multiplication map induced by $F_*^e d$ splits as an $\mathcal{O}_{X,x}$ -module homomorphism.

- (ii) $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ is said to be *strongly F -regular* at x if for every nonzero element $c \in \mathcal{O}_{X,x}$, there exist $e \geq 1$ and $0 \neq d \in \mathfrak{a}_1^{\lceil \lambda_1(p^e - 1) \rceil} \cdots \mathfrak{a}_\ell^{\lceil \lambda_\ell(p^e - 1) \rceil} \mathcal{O}_{X,x}$ such that the composite

$$\mathcal{O}_{X,x} \xrightarrow{\varphi_{\Delta,x}^{(e)}} F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)_x \xrightarrow{\times F_*^e (cd)} F_*^e \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil)_x$$

of the $\mathcal{O}_{X,x}$ -linear map $\varphi_{\Delta,x}^{(e)}$ induced by $\varphi_\Delta^{(e)}$ and the multiplication map induced by $F_*^e (cd)$ splits as an $\mathcal{O}_{X,x}$ -module homomorphism.

We say that $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ is *sharply F -pure* (resp. *strongly F -regular*) if it is sharply F -pure (resp. strongly F -regular) at all points of X .

Remark 2.9. It is known by [34, Lemma 2.8] that if $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ is sharply F -pure at x , then there exist infinitely many integers $e \geq 1$ satisfying the condition in Definition 2.8 (i).

Remark 2.10 ([10], [38, Theorem 3.1]). We have the following hierarchy of F -finite normal singularity properties (see [10] for the definition of F -rational singularities):

$$\text{strongly } F\text{-regular} \implies F\text{-rational} \implies \text{pseudorational.}$$

Definition 2.11 ([3, Definition-Proposition 3.3], cf. [13], [41]). Let (R, \mathfrak{m}) be an F -finite normal local ring of characteristic $p > 0$ and Δ be an effective \mathbb{Q} -Weil divisor on $X := \text{Spec } R$. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_\ell \subseteq R$ be nonzero ideals and $\lambda_1, \dots, \lambda_\ell \geq 0$ be real numbers. We fix a big sharp test element $d \in R$ for $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ (see [34, Definition 2.16] for the definition of big sharp test elements). Then, the *test ideal* $\tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ for the triple $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ is defined by

$$\tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = \sum_{e \geq 0} \sum_{\psi} \psi(F_*^e(d \mathfrak{a}_1^{\lceil \lambda_1(p^e-1) \rceil} \cdots \mathfrak{a}_\ell^{\lceil \lambda_\ell(p^e-1) \rceil})),$$

where e runs through all nonnegative integers and ψ runs through all elements of $\text{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R)$.

Remark 2.12. We do not define big sharp test elements in this paper, but such elements always exist according to [34, Lemma 2.17]. It follows from an argument analogous to the proof of the equivalence of (4) and (5) in [3, Definition-Proposition 3.3] that if d' is a big sharp test element for $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_{\ell-1}^{\lambda_{\ell-1}})$, then

$$\tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = \sum_{e \geq 0} \sum_{\psi} \psi(F_*^e(d' \mathfrak{a}_1^{\lceil \lambda_1(p^e-1) \rceil} \cdots \mathfrak{a}_{\ell-1}^{\lceil \lambda_{\ell-1}(p^e-1) \rceil} \mathfrak{a}_\ell^{\lceil \lambda_\ell p^e \rceil})),$$

where ψ runs through all elements of $\text{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R)$. Similarly, if c is a big sharp test element for (X, Δ) , then

$$\tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = \sum_{e \geq 0} \sum_{\psi} \psi(F_*^e(c \mathfrak{a}_1^{\lceil \lambda_1 p^e \rceil} \cdots \mathfrak{a}_\ell^{\lceil \lambda_\ell p^e \rceil})),$$

where ψ runs through all elements of $\text{Hom}_R(F_*^e R(\lceil (p^e-1)\Delta \rceil), R)$.

Test ideals for triples can be described as sums of test ideals for pairs.

Lemma 2.13. *Let the notation be the same as that in Definition 2.11. Then,*

$$\tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = \sum_{m_1, \dots, m_\ell \geq 1} \sum_{f_i \in \mathfrak{a}_i^{\lceil m_i \lambda_i \rceil}} \tau(X, \Delta + \frac{\text{div}_X(f_1)}{m_1} + \cdots + \frac{\text{div}_X(f_\ell)}{m_\ell}),$$

where the first summation is taken over all integers $m_1, \dots, m_\ell \geq 1$ and the second summation is taken over all nonzero elements $f_i \in \mathfrak{a}_i^{\lceil m_i \lambda_i \rceil}$ for each $i = 1, \dots, \ell$.

Proof. $\tau'(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ denotes the ideal on the right-hand side. We first show that $\tau'(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ is contained in $\tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$. When m_i

is a positive integer and $f_i \in \mathfrak{a}_i^{\lceil m_i \lambda_i \rceil}$ for each $i = 1, \dots, \ell$,

$$\begin{aligned} \tau(X, \Delta + \frac{\operatorname{div}_X(f_1)}{m_1} + \dots + \frac{\operatorname{div}_X(f_\ell)}{m_\ell}) &= \tau(X, \Delta, (f_1)^{1/m_1} \dots (f_\ell)^{1/m_\ell}) \\ &\subseteq \tau(X, \Delta, \mathfrak{a}_1^{\lceil m_1 \lambda_1 \rceil / m_1} \dots \mathfrak{a}_\ell^{\lceil m_\ell \lambda_\ell \rceil / m_\ell}) \\ &\subseteq \tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \dots \mathfrak{a}_\ell^{\lambda_\ell}). \end{aligned}$$

Thus, $\tau'(X, \Delta, \mathfrak{a}_1^{\lambda_1} \dots \mathfrak{a}_\ell^{\lambda_\ell}) \subseteq \tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \dots \mathfrak{a}_\ell^{\lambda_\ell})$.

We next prove the reverse inclusion. Let $c \in R$ be a big sharp test element for (R, Δ) . Then, by Remark 2.12,

$$\begin{aligned} \tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \dots \mathfrak{a}_\ell^{\lambda_\ell}) &= \sum_{e \geq 0} \sum_{\psi} \psi(F_*^e(c \mathfrak{a}_1^{\lceil \lambda_1 p^e \rceil} \dots \mathfrak{a}_\ell^{\lceil \lambda_\ell p^e \rceil})) \\ &= \sum_{e \geq 0} \sum_{g_i \in \mathfrak{a}_i^{\lceil \lambda_i p^e \rceil}} \sum_{\psi} \psi(F_*^e(c g_1 \dots g_\ell)) \\ &\subseteq \sum_{e \geq 0} \sum_{g_i \in \mathfrak{a}_i^{\lceil \lambda_i p^e \rceil}} \tau(X, \Delta, g_1^{1/p^e} \dots g_\ell^{1/p^e}) \\ &= \sum_{e \geq 0} \sum_{g_i \in \mathfrak{a}_i^{\lceil \lambda_i p^e \rceil}} \tau(X, \Delta + \frac{\operatorname{div}_X(g_1)}{p^e} + \dots + \frac{\operatorname{div}_X(g_\ell)}{p^e}), \end{aligned}$$

where ψ ranges over all elements of $\operatorname{Hom}_R(F_*^e R(\lceil (p^e - 1)\Delta \rceil), R)$ and g_i ranges over all nonzero elements of $\mathfrak{a}_i^{\lceil \lambda_i p^e \rceil}$ for each $i = 1, \dots, \ell$. Therefore, $\tau(X, \Delta, \mathfrak{a}_1^{\lambda_1} \dots \mathfrak{a}_\ell^{\lambda_\ell}) \subseteq \tau'(X, \Delta, \mathfrak{a}_1^{\lambda_1} \dots \mathfrak{a}_\ell^{\lambda_\ell})$. \square

2.3. Deformations. In this subsection, we recall some basic terminology from the theory of deformations.

Definition 2.14. Let X be an algebraic scheme over a field k . Suppose that T is a scheme and $t \in T$ is a k -rational point. A *deformation* of X over T with a reference point t is a pair (\mathcal{X}, i) consisting of a scheme \mathcal{X} that is flat and of finite type over T and an isomorphism $i : X \xrightarrow{\sim} \mathcal{X} \times_T \operatorname{Spec} \kappa(t)$ of k -schemes.

In the subsequent sections, we consider several problems regarding deformations of singularities with the following setup.

Setting 2.15. Suppose that X is a normal integral scheme over a field k , T is a regular integral scheme with a generic point η and $t \in T$ is a k -rational point. Let (\mathcal{X}, i) be a deformation of X over T with a reference point t such that \mathcal{X} is an excellent normal integral scheme with a dualizing complex. Let \mathcal{D} be an effective \mathbb{Q} -Weil divisor on \mathcal{X} whose support does not contain the closed fiber X . Let $\mathfrak{a} \subseteq \mathcal{O}_{\mathcal{X}}$ be a coherent ideal sheaf such that $\mathfrak{a} \mathcal{O}_X$ is nonzero and $\lambda > 0$ be a real number.

Remark 2.16. We use the notation in Setting 2.15.

- (i) In Section 3, we mainly focus on the case where the following condition holds.

(A) The pair $(\mathcal{X}, \mathcal{D})$ on the total space \mathcal{X} is log \mathbb{Q} -Gorenstein; that is, $K_{\mathcal{X}} + \mathcal{D}$ is \mathbb{Q} -Cartier.

Condition (A) implies the following two conditions:

(B) The pair $(\mathcal{X}_\eta, \mathcal{D}_\eta)$ on the generic fiber \mathcal{X}_η is log \mathbb{Q} -Gorenstein.

(C) The pair $(X, \mathcal{D}|_X)$ on the closed fiber X is log \mathbb{Q} -Gorenstein.

We note that there is no relation between (B) and (C) (see [17, Example 9.1.8] or Example 4.4 below for a counterexample to the implication (C) \Rightarrow (B)). On the other hand, in Sections 4 and 5, we discuss the case where conditions (B) and (C) are satisfied, although condition (A) is not necessarily satisfied.

(ii) If k is an uncountable algebraically closed field, T is of finite type over k , and a general closed fiber of $\mathcal{X} \rightarrow T$ is normal, then condition (B) is equivalent to the following condition:

(D) For a general closed point $s \in T$, the pair $(\mathcal{X}_s, \mathcal{D}_s)$ on the fiber \mathcal{X}_s is log \mathbb{Q} -Gorenstein.

Indeed, it is obvious that (B) implies (D). For the converse implication, we fix an integer $m > 0$ such that $m\mathcal{D}$ is an integral Weil divisor. For every integer $n > 0$, we consider the coherent sheaf $\mathcal{F}_n := \mathcal{O}_{\mathcal{X}}(nm(K_{\mathcal{X}} + \mathcal{D}))$. Since \mathcal{F}_n satisfies the (S_2) -condition, so does the restriction of \mathcal{F}_n to the generic fiber. It then follows from [5, (9.9.3)] that there exists a nonempty open subset $V_n \subseteq T$ such that for every point $s \in V_n$, the restriction $\mathcal{F}_n|_{\mathcal{X}_s}$ to the fiber \mathcal{X}_s satisfies the (S_2) -condition. Therefore, for such an s , we have

$$\mathcal{F}_n|_{\mathcal{X}_s} \cong \mathcal{O}_{\mathcal{X}_s}(nm(K_{\mathcal{X}_s} + \mathcal{D}_s)).$$

In particular, if the $\mathcal{O}_{\mathcal{X}_s}$ -module $\mathcal{O}_{\mathcal{X}_s}(nm(K_{\mathcal{X}_s} + \mathcal{D}_s))$ is invertible, then so is \mathcal{F}_n along \mathcal{X}_s by Nakayama's lemma. In other words, if $nm(K_{\mathcal{X}_s} + \mathcal{D}_s)$ is Cartier, then so is $nm(K_{\mathcal{X}} + \mathcal{D})$ along \mathcal{X}_s .

Since k is uncountable, we find a closed point $s \in \bigcap_n V_n$ such that $(\mathcal{X}_s, \mathcal{D}_s)$ is log \mathbb{Q} -Gorenstein. Now, we pick an integer $l > 0$ such that $lm(K_{\mathcal{X}_s} + \mathcal{D}_s)$ is Cartier. Considering that $s \in V_l$, we conclude that $lm(K_{\mathcal{X}} + \mathcal{D})$ is Cartier along \mathcal{X}_s , which implies condition (B).

(iii) If k is perfect and \mathcal{X} is proper over T , then it follows from [5, (12.2.4)] that a general fiber of $\mathcal{X} \rightarrow T$ is geometrically normal, and in particular, the third assumption in (ii) is satisfied.

Similarly, if k is perfect, then it follows from [5, (12.1.6)] that after shrinking \mathcal{X} around X , we may assume that $\mathcal{X} \rightarrow T$ is a normal morphism; that is, all fibers are geometrically normal.

3. DEFORMATIONS WITH A \mathbb{Q} -GORENSTEIN TOTAL SPACE

In this section, we study arithmetic deformations of F -pure singularities when the total space is \mathbb{Q} -Gorenstein.

3.1. BCM test ideals. First, we recall the definition of BCM test ideals for pairs introduced by Ma-Schwede [25].

Definition 3.1 ([25, Definition 6.2, Definition 6.9]). Let (R, \mathfrak{m}) be a d -dimensional complete normal local ring of mixed characteristic $(0, p)$, and

fix an effective canonical divisor K_X of $X := \text{Spec } R$. Let Δ be an effective \mathbb{Q} -Weil divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Then, there exist an integer $n \geq 1$ and a nonzero element $f \in R$ such that $n(K_X + \Delta) = \text{div}_X(f)$. We set

$$(0)_{H_m^d(R)}^{\mathcal{B}, K_X + \Delta} := \bigcup_B \ker \left(H_m^d(R) \xrightarrow{\times f^{1/n}} H_m^d(B) \right),$$

where B runs through all integral perfectoid big Cohen-Macaulay R^+ -algebras. Then, the *BCM test ideal* for (X, Δ) is defined as

$$\tau_{\mathcal{B}}(X, \Delta) := \text{Ann}_{\omega_R}(0)_{H_m^d(R)}^{\mathcal{B}, K_X + \Delta} \subseteq R.$$

We generalize the definitions of the BCM test ideals and some of their properties to the case of triples.

Definition 3.2. Let R be a complete normal local ring of mixed characteristic $(0, p)$ and Δ be an effective \mathbb{Q} -Weil divisor on $X := \text{Spec } R$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_\ell \subseteq R$ be nonzero ideals and $\lambda_1, \dots, \lambda_\ell \geq 0$ be real numbers.

(i) We define the BCM test ideal for the triple $(R, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ as

$$\tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) := \sum_{m_i} \sum_{f_i} \tau_{\mathcal{B}}(X, \Delta + \frac{\text{div}_X(f_1)}{m_1} + \cdots + \frac{\text{div}_X(f_\ell)}{m_\ell}),$$

where the first summation is taken over all positive integers m_i and the second summation is taken over all nonzero elements $f_i \in \mathfrak{a}_i^{\lceil m_i \lambda_i \rceil}$ for each $i = 1, \dots, \ell$.

(ii) We say that $(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$ is *BCM-regular* if we have

$$\tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = R.$$

Remark 3.3. Let the notation be the same as that in Definition 3.2. If $\mathfrak{a}_i = (r)$ is a principal ideal and λ_i is a rational number, then it follows from [25, Lemma 6.11] that

$$\tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) = \tau_{\mathcal{B}}(X, \Delta + \lambda_i \text{div}_X(r), \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_{\ell-1}^{\lambda_{\ell-1}}).$$

Lemma 3.4. *Let the notation be the same as that in Definition 3.2. Then, we have*

$$\tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) \subseteq \mathcal{J}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}).$$

Proof. Let $\pi : Y \rightarrow X$ be a proper birational morphism from a normal integral scheme Y such that $\mathfrak{a}_i \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$ is invertible for every $i = 1, \dots, \ell$. When m_i is a positive integer and $f_i \in \mathfrak{a}_i^{\lceil m_i \lambda_i \rceil}$ for each $i = 1, \dots, \ell$, it follows from [25, Theorem 6.21] that

$$\begin{aligned} & \tau_{\mathcal{B}}(X, \Delta + \frac{1}{m_1} \text{div}_X(f_1) + \cdots + \frac{1}{m_\ell} \text{div}_X(f_\ell)) \\ & \subseteq \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + \frac{1}{m_1} \text{div}_X(f_1) + \cdots + \frac{1}{m_\ell} \text{div}_X(f_\ell)) \rceil) \\ & \subseteq \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) - \lambda_1 F_1 - \cdots - \lambda_\ell F_\ell \rceil). \end{aligned}$$

Thus, $\tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell}) \subseteq \mathcal{J}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})$. \square

Lemma 3.5. *Let the notation be the same as that in Definition 3.2. Let $h_1, \dots, h_r \in R$ be a regular sequence such that $S := R/(h_1, \dots, h_r)$ is an F -finite normal local ring of characteristic p . In addition, we assume that \mathfrak{a}_i is not contained in the ideal (h_1, \dots, h_r) for all i and $Z := \text{Spec } S$ is not contained in the support of Δ . Then, we have*

$$\tau(Z, \Delta|_Z, (\mathfrak{a}_1 S)^{\lambda_1} \cdots (\mathfrak{a}_\ell S)^{\lambda_\ell}) \subseteq \tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})S.$$

Proof. If the Cartier index of $K_X + \Delta$ is not divisible by p , then the above assertion follows from a combination of Lemma 2.13 and [25, Theorem 6.27]. Therefore, we assume that the Cartier index n of $K_X + \Delta$ is divisible by p . We choose an effective \mathbb{Q} -Weil divisor D on X that is linearly equivalent to $K_X + \Delta$ and does not contain Z in its support. By [30, Proposition 2.14 (2)], we can take a sufficiently large integer s such that

$$\tau(Z, \Delta|_Z, (\mathfrak{a}_1 S)^{\lambda_1} \cdots (\mathfrak{a}_\ell S)^{\lambda_\ell}) = \tau(Z, (\Delta + \Delta')|_Z, (\mathfrak{a}_1 S)^{\lambda_1} \cdots (\mathfrak{a}_\ell S)^{\lambda_\ell}),$$

where $\Delta' := \frac{n-1}{ns+1}D$. Since $(ns+1)(K_X + \Delta + \Delta') \sim n(s+1)(K_X + \Delta)$, the Cartier index of $K_X + \Delta + \Delta'$ is not divisible by p . It then follows from Lemma 2.13 and [25, Theorem 6.27, Lemma 6.11] that

$$\begin{aligned} \tau(Z, \Delta|_Z, (\mathfrak{a}_1 S)^{\lambda_1} \cdots (\mathfrak{a}_\ell S)^{\lambda_\ell}) &= \tau(Z, (\Delta + \Delta')|_Z, (\mathfrak{a}_1 S)^{\lambda_1} \cdots (\mathfrak{a}_\ell S)^{\lambda_\ell}) \\ &\subseteq \tau_{\mathcal{B}}(X, \Delta + \Delta', \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})S \\ &\subseteq \tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_\ell^{\lambda_\ell})S. \end{aligned}$$

□

3.2. Deformations of F -pure singularities. We start with two auxiliary lemmas on log canonical and F -pure singularities.

Lemma 3.6. *Let (R, \mathfrak{m}) be a complete normal local ring and Δ be an effective \mathbb{Q} -Weil divisor on $X := \text{Spec } R$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $\lambda > 0$ be a real number. If $(X, \Delta, \mathfrak{a}^\lambda)$ is not log canonical, then there exist a descending chain of nonzero ideals of R*

$$R = \mathfrak{b}_0 \supseteq \mathfrak{b}_1 \supseteq \cdots \supseteq \mathfrak{b}_n \supseteq \cdots$$

and a decreasing sequence of positive real numbers

$$1 = \varepsilon_0 \geq \varepsilon_1 \geq \cdots \geq \varepsilon_n \geq \cdots$$

with the following properties:

(i) For every integer $n \geq 0$, we have

$$\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda \mathfrak{b}_n^{1-\varepsilon_n}) \subseteq \mathfrak{b}_{n+1}.$$

(ii) For every integer $\ell \geq 1$, there exists an integer $n(\ell) \geq 0$ such that $\mathfrak{b}_{n(\ell)} \subseteq \mathfrak{m}^\ell$.

Proof. Since $(X, \Delta, \mathfrak{a}^\lambda)$ is not log canonical, there exist a proper birational morphism $f : Y \rightarrow X$ (where Y is normal) and a prime divisor E on Y such that $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-F)$ is invertible and $\text{ord}_E(\Delta_Y + \lambda F) > 1$, where $\Delta_Y := f^*(K_X + \Delta) - K_Y$. We set $\varepsilon_0 := 1$, $\varepsilon_n := \min\{1, (\text{ord}_E(\Delta_Y + \lambda F) - 1)/n\}$ for $n \geq 1$, and $\mathfrak{b}_n := f_*\mathcal{O}_Y(-nE)$ for $n \geq 0$.

We verify that the above $\{\varepsilon_n\}_{n \geq 0}$ and $\{\mathfrak{b}_n\}_{n \geq 0}$ satisfy properties (i) and (ii). Since $\bigcap_{n \geq 0} \mathfrak{b}_n = (0)$, property (ii) follows from Chevalley's theorem (see, for example, [27, Exercise 8.7]). For (i), we first observe that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda \mathfrak{b}_n^{1-\varepsilon_n}) \subseteq f_* \mathcal{O}_Y(-[\Delta_Y + \lambda F + (1 - \varepsilon_n)nE]),$$

because $\mathfrak{b}_n \mathcal{O}_Y \subseteq \mathcal{O}_Y(-nE)$. We take any nonzero element r of the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^\lambda \mathfrak{b}_n^{1-\varepsilon_n})$. Then,

$$\begin{aligned} \text{ord}_E r &\geq \text{ord}_E([\Delta_Y + \lambda F + (1 - \varepsilon_n)nE]) \\ &= \lfloor \text{ord}_E(\Delta_Y + \lambda F) + n - \varepsilon_n n \rfloor \\ &\geq n + 1. \end{aligned}$$

Combining this with the inclusion $r \in R$, we have that $r \in \mathfrak{b}_{n+1}$ as desired. \square

Lemma 3.7. *Let (R, \mathfrak{m}) be an F -finite normal local ring of characteristic $p > 0$ and Δ be an effective \mathbb{Q} -Weil divisor on $X := \text{Spec } R$. Let $\mathfrak{a} \subseteq R$ be a nonzero ideal and $\lambda > 0$ be a real number. If $(X, \Delta, \mathfrak{a}^\lambda)$ is sharply F -pure, then there exists a nonzero ideal $J \subseteq R$ such that*

$$J \subseteq \tau(X, \Delta, \mathfrak{a}^\lambda J^{1-\varepsilon})$$

for every real number $0 < \varepsilon \leq 1$.

Proof. We fix any real number $0 < \varepsilon \leq 1$. Since $(X, \Delta, \mathfrak{a}^\lambda)$ is sharply F -pure, there exist an integer $e \geq 1$, a nonzero element $d \in \mathfrak{a}^{\lceil \lambda(p^e - 1) \rceil}$ and an R -module homomorphism $\psi : F_*^e R(\lceil (p^e - 1)\Delta \rceil) \rightarrow R$ sending $F_*^e d$ to 1. By Remark 2.9, we may assume that e is sufficiently large so that $\lceil (1 - \varepsilon)p^e \rceil \leq p^e - 1$.

Let $c \in R$ be a big sharp test element for the triple $(X, \Delta, \mathfrak{a}^\lambda)$ (see Remark 2.12). We take J as the principal ideal of R generated by c . Then,

$$c = \psi(F_*^e(c^{p^e} d)) \in \psi(F_*^e(c \mathfrak{a}^{\lceil \lambda(p^e - 1) \rceil} J^{\lceil (1-\varepsilon)p^e \rceil})) \subseteq \tau(X, \Delta, \mathfrak{a}^\lambda J^{1-\varepsilon}),$$

where the last containment follows from Remark 2.12. Thus, we have that $J = cR \subseteq \tau(X, \Delta, \mathfrak{a}^\lambda J^{1-\varepsilon})$. \square

The following is the main result of this section.

Theorem 3.8. *Let (R, \mathfrak{m}) be a complete normal local ring of mixed characteristic $(0, p)$ and Δ be an effective \mathbb{Q} -Weil divisor on $X := \text{Spec } R$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Suppose that h_1, \dots, h_r is a regular sequence in R such that $S := R/(h_1, \dots, h_r)$ is an F -finite normal local ring of characteristic p and $Z := \text{Spec } S$ is not contained in the support of Δ . Let $\lambda > 0$ be a real number and $\mathfrak{a} \subseteq R$ be an ideal not contained in the ideal (h_1, \dots, h_r) . If $(Z, \Delta|_Z, (\mathfrak{a}S)^\lambda)$ is sharply F -pure, then $(X, \Delta, \mathfrak{a}^\lambda)$ is log canonical.*

Proof. Assume on the contrary that $(X, \Delta, \mathfrak{a}^\lambda)$ is not log canonical. Let $\{\mathfrak{b}_n\}_{n \geq 0}$ and $\{\varepsilon_n\}_{n \geq 0}$ be as in Lemma 3.6. By Lemma 3.7, there exists a nonzero ideal $J \subseteq S$ such that $J \subseteq \tau(Z, \Delta|_Z, (\mathfrak{a}S)^\lambda J^{1-\varepsilon})$ for all $0 < \varepsilon \leq 1$.

We show by induction that $J \subseteq \mathfrak{b}_n S$ for every integer $n \geq 0$. The $n = 0$ case is trivial because $\mathfrak{b}_0 = R$. Therefore, suppose that the inclusion holds

for some $n \geq 0$. Then,

$$\begin{aligned} J &\subseteq \tau(Z, \Delta|_Z, (\mathfrak{a}S)^\lambda J^{1-\varepsilon_n}) \subseteq \tau(Z, \Delta|_Z, (\mathfrak{a}S)^\lambda (\mathfrak{b}_n S)^{1-\varepsilon_n}) \\ &\subseteq \tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}^\lambda \mathfrak{b}_n^{1-\varepsilon_n})S \\ &\subseteq \mathcal{J}(X, \Delta, \mathfrak{a}^\lambda \mathfrak{b}_n^{1-\varepsilon_n})S \\ &\subseteq \mathfrak{b}_{n+1}S, \end{aligned}$$

where the third containment follows from Lemma 3.5, the fourth containment follows from Lemma 3.4 and the fifth containment follows from Lemma 3.6 (i). Thus, we have that $J \subseteq \mathfrak{b}_n S$ for every $n \geq 0$.

Combining the above inclusion with Lemma 3.6 (ii), we see that

$$J \subseteq \bigcap_{n \geq 0} \mathfrak{b}_n S \subseteq \bigcap_{\ell \geq 1} \mathfrak{m}_S^\ell = (0),$$

which contradicts the fact that J is a nonzero ideal. \square

An inversion of adjunction-type result also follows from a similar argument to that above. We would like to thank Linquan Ma for pointing out this fact.

Theorem 3.9. *Let (R, \mathfrak{m}) be a complete normal local ring of mixed characteristic $(0, p)$ and Δ be an effective \mathbb{Q} -Weil divisor on $X := \text{Spec } R$ such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Suppose that Z is a \mathbb{Q} -Cartier prime divisor on X that is F -finite with characteristic p and is not contained in the support of Δ . Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal not contained in $\mathcal{O}_X(-Z)$ and $\lambda > 0$ be a real number. Z^N denotes the normalization of Z , and $\text{diff}_{Z^N}(Z + \Delta)$ denotes the different of $Z + \Delta$ on Z^N (see [26, Subsection 2.1] for its definition). If $(Z^N, \text{diff}_{Z^N}(\Delta), (\mathfrak{a}\mathcal{O}_{Z^N})^\lambda)$ is sharply F -pure, then $(X, \Delta + Z, \mathfrak{a}^\lambda)$ is log canonical.*

Proof. First, note that for any real numbers $\delta, \mu > 0$ and any ideal $\mathfrak{b} \subseteq \mathcal{O}_X$, we have the inclusion

$$\tau(Z^N, \text{diff}_{Z^N}(Z + \Delta), (\mathfrak{a}\mathcal{O}_{Z^N})^\lambda (\mathfrak{b}\mathcal{O}_{Z^N})^\mu) \subseteq \tau_{\mathcal{B}}(X, \Delta + (1 - \delta)Z, \mathfrak{a}^\lambda \mathfrak{b}^\mu) \mathcal{O}_{Z^N}.$$

Indeed, by Lemma 2.13, it is sufficient to show that for any effective \mathbb{Q} -Cartier divisor E on X having no common component with Z , we have

$$\tau(Z^N, \text{diff}_{Z^N}(Z + \Delta + E)) \subseteq \tau_{\mathcal{B}}(X, \Delta + (1 - \delta)Z + E) \mathcal{O}_{Z^N}.$$

We take a big Cohen-Macaulay R^+ -algebra B , with an R^+ -algebra homomorphism $B \rightarrow C$ to a big Cohen-Macaulay S^+ -algebra C , such that

$$\tau_{\mathcal{B}}(X, \Delta + (1 - \delta)Z + E) = \tau_B(X, \Delta + (1 - \delta)Z + E).$$

The assertion then follows from [25, Definition-Proposition 2.7] and [26, Proposition 2.10 and Theorem 3.1].

Now, we assume to the contrary that $(X, \Delta + Z, \mathfrak{a}^\lambda)$ is not log canonical. Then, there exists a rational number $\delta > 0$ such that $(X, \Delta + (1 - \delta)Z, \mathfrak{a}^\lambda)$ is not log canonical. We apply Lemma 3.6 to the triple $(X, \Delta + (1 - \delta)Z, \mathfrak{a}^\lambda)$ to obtain a descending chain $\{\mathfrak{b}_n\}_{n \geq 0}$ of nonzero ideals of \mathcal{O}_X and a descending

sequence $\{\varepsilon_n\}_{n \geq 0}$ of positive real numbers. We also apply Lemma 3.7 to $(Z^N, \text{diff}_{Z^N}(\Delta), (\mathfrak{a}\mathcal{O}_{Z^N})^\lambda)$ to obtain a nonzero ideal $J \subseteq \mathcal{O}_{Z^N}$ such that

$$J \subseteq \tau(Z^N, \text{diff}_{Z^N}(Z + \Delta), (\mathfrak{a}\mathcal{O}_{Z^N})^\lambda J^{1-\varepsilon})$$

for all $0 < \varepsilon \leq 1$. It then follows from an argument similar to the proof of Theorem 3.8 that $J \subseteq \mathfrak{b}_n \mathcal{O}_{Z^N}$ for all integers $n \geq 0$, which contradicts the fact that J is a nonzero ideal. \square

Corollary 3.10. *With the same notation as that in Setting 2.15, let $x \in X$ be a closed point and $\mathcal{Z} \subseteq \mathcal{X}$ be an irreducible closed subset that dominates T and contains x . Let y be the generic point of \mathcal{Z} , which lies in the generic fiber \mathcal{X}_η . We further assume that the following three conditions hold:*

- (i) $K_{\mathcal{X}} + \mathcal{D}$ is \mathbb{Q} -Cartier at x .
- (ii) The generic fiber \mathcal{X}_η has characteristic zero.
- (iii) X is defined over an F -finite field k of characteristic $p > 0$, and the triple $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is sharply F -pure at x .

Then the triple $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is log canonical at y .

Proof. $R := \widehat{\mathcal{O}_{\mathcal{X},x}}$ denotes the completion of the stalk $\mathcal{O}_{\mathcal{X},x}$, and $S := \widehat{\mathcal{O}_{X,x}}$ denotes the completion of $\mathcal{O}_{X,x}$. Since T is regular and \mathcal{X} is flat over T , the kernel of the surjection $\mathcal{O}_{\mathcal{X},x} \rightarrow \mathcal{O}_{X,x}$ is generated by a regular sequence, and therefore, the same holds for the surjection $R \rightarrow S$. Note that sharp F -purity is preserved under completion. It then follows from Theorem 3.8 that $(R, f^*\Delta, (\mathfrak{a}R)^\lambda)$ is log canonical, where $f^*\Delta$ is the flat pullback of Δ under the canonical morphism $f : \text{Spec } R \rightarrow \mathcal{X}$. By Lemma 2.7, the triple $(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda)$ is log canonical at x . Taking into account that y is a generalization of x , we see from Remark 2.4 that $(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda)$ is log canonical at y . \square

Remark 3.11. Using the theory of jet schemes, Zhu [44, Corollary 4.2] proved Corollary 3.10 for the case in which the total space \mathcal{X} is the affine space $\mathbb{A}_{\mathbb{Z}}^n$ over \mathbb{Z} .

Remark 3.12. Corollary 3.10 implies that one can use a method analogous to [25, Algorithm 8.1] utilizing [4] to verify that a ring of finite type over \mathbb{Q} has log canonical singularities.

Corollary 3.13. *With the same notation as that in Setting 2.15, assume that the following four conditions hold:*

- (i) $K_{\mathcal{X}} + \mathcal{D}$ is \mathbb{Q} -Cartier.
- (ii) The generic fiber \mathcal{X}_η has characteristic zero.
- (iii) X is defined over an F -finite field k of characteristic $p > 0$, and the triple $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is sharply F -pure.
- (iv) \mathcal{X} is proper over T .

Then, $(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda)$ is log canonical near \mathcal{X}_η , and in particular, the triple $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is log canonical.

Proof. We take any point $y \in \mathcal{X}_\eta$. Since the structure map $\mathcal{X} \rightarrow T$ is a closed map, there exists a point $x \in X$ that is a specialization of y . It then follows from Corollary 3.10 that $(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda)$ is log canonical at y . \square

4. DEFORMATIONS WITH A NON- \mathbb{Q} -GORENSTEIN TOTAL SPACE

In this section, we study deformations of strongly F -regular/klt singularities when the total space is not necessarily \mathbb{Q} -Gorenstein.

Throughout this section, we say that $(R, \Delta, \mathfrak{a}^\lambda)$ is a *triple* if (R, \mathfrak{m}) is an excellent normal local ring with a dualizing complex, Δ is an effective \mathbb{Q} -Weil divisor on $\text{Spec } R$, \mathfrak{a} is a nonzero ideal of R , and $\lambda > 0$ is a real number.

Proposition 4.1. *Suppose that $(R, \Delta, \mathfrak{a}^\lambda)$ is a triple and A is an effective Weil divisor on $X := \text{Spec } R$ that is linearly equivalent to $-K_X$ such that $B := A - \Delta$ is also effective. We fix an integer $m \geq 1$ such that $m\Delta$ is an integral Weil divisor, and we let $\mathfrak{b} \subseteq R$ be a nonzero ideal contained in $\mathcal{O}_X(-mB)$. We further assume that one of the following three cases occurs.*

- (a) (R, \mathfrak{m}) is a complete local domain of mixed characteristic $(0, p)$. In this case, we set

$$I := \tau_{\mathcal{B}}(X, A, \mathfrak{a}^\lambda \mathfrak{b}^{1-1/m}) \subseteq R.$$

- (b) (R, \mathfrak{m}) is a F -finite local ring of characteristic $p > 0$. In this case, we set

$$I := \tau(X, A, \mathfrak{a}^\lambda \mathfrak{b}^{1-1/m}) \subseteq R.$$

- (c) (R, \mathfrak{m}) is a local ring of equal characteristic zero. In this case, we set

$$I := \mathcal{J}(X, A, \mathfrak{a}^\lambda \mathfrak{b}^{1-1/m}) \subseteq R.$$

Then, the following statements hold.

- (1) The ideal I is contained in $\mathcal{O}_X(-mB)$.
- (2) Let $U \subseteq X$ be the locus where $m(K_X + \Delta)$ is Cartier. If $I|_U = \mathcal{O}_X(-mB)|_U$, then $(U, \Delta|_U, \mathfrak{a}|_U^\lambda)$ is klt in case (a) or (c) and is strongly F -regular in case (b).
- (3) Assume that $m(K_X + \Delta)$ is Cartier. If $(X, \Delta, \mathfrak{a}^\lambda)$ is BCM-regular (resp. strongly F -regular, klt) in case (a) (resp. (b), (c)), then \mathfrak{b} is contained in I .

Proof. First, we consider case (a). For (1), since $\mathcal{O}_X(-mB)$ is reflexive and U is an open subset of X whose complement has codimension at least two, it is sufficient to show that $I|_U \subseteq \mathcal{O}_U(-mB|_U)$. Noting that $\mathcal{O}_U(-mB|_U)$ is invertible, we have

$$\begin{aligned} I|_U &\subseteq \mathcal{J}(U, A|_U, \mathfrak{a}|_U^\lambda \mathfrak{b}|_U^{1-1/m}) \\ &\subseteq \mathcal{J}(U, A|_U, \mathfrak{a}|_U^\lambda \mathcal{O}_U(-mB|_U)^{1-1/m}) \\ &= \mathcal{J}(U, A|_U + \frac{m-1}{m}(mB|_U), \mathfrak{a}|_U^\lambda) \\ &= \mathcal{J}(U, \Delta|_U + mB|_U, \mathfrak{a}|_U^\lambda) \\ &= \mathcal{J}(U, \Delta|_U, \mathfrak{a}|_U^\lambda) \otimes_{\mathcal{O}_U} \mathcal{O}_U(-mB|_U), \end{aligned}$$

where the first inclusion follows from Lemma 3.4 and the last equality follows from essentially the same argument as that in the proof of [21, Proposition 9.2.31]. Therefore, we have that $I|_U \subseteq \mathcal{O}_U(-mB|_U)$ as desired.

If the equality $I|_U = \mathcal{O}_X(-mB)|_U$ holds, then we see from the above inclusions that $\mathcal{J}(U, \Delta|_U, \mathfrak{a}|_U^\lambda) = \mathcal{O}_U$, which proves (2).

For (3), we set $\mathfrak{q} := \mathfrak{b}\mathcal{O}_X(mB) \subseteq R$. Noting that mB is a principal divisor by assumption, we have

$$\begin{aligned} I &= \tau_{\mathcal{B}}(X, A, \mathfrak{a}^\lambda \mathfrak{q}^{1-1/m} \mathcal{O}_X(-mB)^{1-1/m}) \\ &= \tau_{\mathcal{B}}(X, A + (m-1)B, \mathfrak{a}^\lambda \mathfrak{q}^{1-1/m}) \\ &= \tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}^\lambda \mathfrak{q}^{1-1/m}) \mathcal{O}_X(-mB), \end{aligned}$$

where the second equality follows from Remark 3.3 and the third equality follows from [25, Lemma 6.6]. On the other hand, by Definition 3.2 and [25, Lemma 6.6],

$$\begin{aligned} \tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}^\lambda \mathfrak{q}^{1-1/m}) &\supseteq \tau_{\mathcal{B}}(X, \Delta + \operatorname{div}_X(f), \mathfrak{a}^\lambda) \\ &= f \tau_{\mathcal{B}}(X, \Delta, \mathfrak{a}^\lambda) \\ &= fR \end{aligned}$$

for all nonzero elements $f \in \mathfrak{q}$, where the last equality follows from the assumption that $(X, \Delta, \mathfrak{a}^\lambda)$ is BCM-regular. Therefore, $I \supseteq \mathfrak{q}\mathcal{O}_X(-mB) = \mathfrak{b}$, which completes the proof of (3).

In case (b) (resp. (c)), the assertion follows from a similar argument by replacing [25, Lemma 6.6] with [41, p.402 Basic Properties (ii)] (resp. [21, Proposition 9.2.31]). \square

The following theorem, the main result of this section, should be compared with Theorem 3.8.

Theorem 4.2. *Suppose that $(R, \Delta, \mathfrak{a}^\lambda)$ is a triple and that h is a nonzero element in R such that $S := R/(h)$ is normal. In addition, we assume that $Z := \operatorname{Spec} S$ is not contained in the support of Δ , $K_Z + \Delta|_Z$ is \mathbb{Q} -Cartier, and \mathfrak{a} is not contained in the ideal (h) . Let $U \subseteq X$ be the locus where $K_X + \Delta$ is \mathbb{Q} -Cartier.*

- (1) *Suppose that (R, \mathfrak{m}) is a complete local domain of mixed characteristic $(0, p)$ and S is an F -finite local domain of characteristic $p > 0$. If $(Z, \Delta|_Z, (\mathfrak{a}S)^\lambda)$ is strongly F -regular, then $(U, \Delta|_U, \mathfrak{a}|_U^\lambda)$ is klt.*
- (2) *Suppose that (R, \mathfrak{m}) is an F -finite local domain of characteristic $p > 0$. If $(Z, \Delta|_Z, (\mathfrak{a}S)^\lambda)$ is strongly F -regular, then so is $(U, \Delta|_U, \mathfrak{a}|_U^\lambda)$.*
- (3) *(cf. [6]) Suppose that (R, \mathfrak{m}) is a local ring of equal characteristic zero. If $(Z, \Delta|_Z, (\mathfrak{a}S)^\lambda)$ is klt, then so is $(U, \Delta|_U, \mathfrak{a}|_U^\lambda)$.*

Proof. (1) Since X is affine and Gorenstein at the generic point of Z , we can take an effective Weil divisor A on X that is linearly equivalent to $-K_X$ such that $B := A - \Delta$ is effective and $\operatorname{Supp} A$ does not contain Z . We take an integer $m \geq 1$ such that $m(K_Z + \Delta|_Z)$ is Cartier. We set

$$\begin{aligned} I &:= \tau_{\mathcal{B}}(X, A, \mathfrak{a}^\lambda \mathfrak{b}^{1-1/m}) \subseteq R, \\ J &:= \tau(Z, A|_Z, (\mathfrak{a}S)^\lambda (\mathfrak{b}S)^{1-1/m}) \subseteq S, \end{aligned}$$

where we write $\mathfrak{b} := \mathcal{O}_X(-mB) \subseteq R$.

Since $A|_Z$ is linearly equivalent to $-K_Z$, $B|_Z = A|_Z - \Delta|_Z$ and $\mathfrak{b}S \subseteq \mathcal{O}_Z(-mB|_Z)$, we apply Proposition 4.1 (3) with $X = Z$ and Lemma 3.5 to deduce that $\mathfrak{b}S \subseteq J \subseteq IS$. It follows from a combination of the inclusion $\mathfrak{b}S \subseteq IS$ and Proposition 4.1 (1) that

$$I \subseteq \mathfrak{b} \subseteq I + \mathfrak{b} \cap (h).$$

By assumption, $\operatorname{div}_X(h) = Z$ is a prime divisor on X , which is not an irreducible component of B . Thus, $\mathfrak{b} \cap (h) = h(\mathfrak{b} :_R (h)) = h\mathfrak{b} \subseteq m\mathfrak{b}$, so $\mathfrak{b} = I + m\mathfrak{b}$. By Nakayama's lemma, we have that $I = \mathfrak{b}$, which implies assertion (1) by Proposition 4.1 (2).

For (2), we set

$$\begin{aligned} I &:= \tau(X, A, \mathfrak{a}^\lambda \mathfrak{b}^{1-1/m}) \subseteq R, \\ J &:= \tau(Z, A|_Z, (\mathfrak{a}S)^\lambda (\mathfrak{b}S)^{1-1/m}) \subseteq S. \end{aligned}$$

The proof then follows from an argument similar to that in the proof of (1) by replacing Lemma 3.5 with [13, Theorem 6.10 (1)].

For (3), we set

$$\begin{aligned} I &:= \mathcal{J}(X, A, \mathfrak{a}^\lambda \mathfrak{b}^{1-1/m}) \subseteq R \text{ and} \\ J &:= \mathcal{J}(Z, A|_Z, (\mathfrak{a}S)^\lambda (\mathfrak{b}S)^{1-1/m}) \subseteq S. \end{aligned}$$

The proof then follows from an argument similar to that in the proof of (1) by replacing Lemma 3.5 with [21, Theorem 9.5.13]. (We note that [21, Theorem 9.5.13] is formulated for varieties, but the same statement regarding excellent \mathbb{Q} -schemes is obtained by using [28, Theorem A] instead of the local vanishing theorem.) \square

Corollary 4.3. *With the same notation as that in Setting 2.15, let $x \in X$ be a closed point and $\mathcal{Z} \subseteq \mathcal{X}$ be an irreducible closed subset that dominates T and contains x . Let y be a generic point of \mathcal{Z} , which lies in the generic fiber \mathcal{X}_η . We further assume that the following conditions are all satisfied.*

- (i) T is a Dedekind scheme; that is, $\dim T = 1$.
- (ii) One of the following holds.
 - (a) $K_X + \mathcal{D}|_X$ is \mathbb{Q} -Cartier at x and $K_{\mathcal{X}_\eta} + \mathcal{D}_\eta$ is \mathbb{Q} -Cartier at y ,
or
 - (b) $\dim \mathcal{O}_{X,x} \leq 2$.
- (iii) One of the following cases occurs.
 - (a) $\mathcal{O}_{T,t}$ is of mixed characteristic $(0, p)$, the residue field $\kappa(t)$ is F -finite and of characteristic p , and $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is strongly F -regular at x .
 - (b) $\mathcal{O}_{T,t}$ is F -finite and of positive characteristic $p > 0$ and the triple $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is strongly F -regular at x , or
 - (c) $\mathcal{O}_{T,t}$ is of equal characteristic zero and $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is klt at x .

Then, $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is klt at y in case (iii-a) or (iii-c) and is strongly F -regular at y in case (iii-b).

Proof. First, we assume that (ii-a) holds. For case (iii-a), let $R := \widehat{\mathcal{O}_{\mathcal{X},x}}$ denote the completion of stalk $\mathcal{O}_{\mathcal{X},x}$ and $S := \widehat{\mathcal{O}_{X,x}}$ denote the completion of $\mathcal{O}_{X,x}$. Note that S is F -finite because $\mathcal{O}_{X,x}$ is essentially of finite type over an F -finite field $\kappa(t)$. Since the morphism $\mathrm{Spec} R \rightarrow \mathrm{Spec} \mathcal{O}_{\mathcal{X},x}$ induced by the completion $\mathcal{O}_{\mathcal{X},x} \rightarrow R$ is surjective and y is a generalization of x , there exists a point $y' \in \mathrm{Spec} R$ such that $f(y') = y$, where $f : \mathrm{Spec} R \rightarrow \mathcal{X}$ is a canonical morphism. When $f^*\mathcal{D}$ denotes the flat pullback of \mathcal{D} by f , it follows from Lemma 2.7 and the assumption (ii-a) that $K_{\mathrm{Spec} R} + f^*\mathcal{D}$ is \mathbb{Q} -Cartier at y' . Similarly, it also follows from Lemma 2.7 that $K_{\mathrm{Spec} S} + (f^*\mathcal{D})|_{\mathrm{Spec} S}$ is \mathbb{Q} -Cartier because this \mathbb{Q} -Weil divisor is the pullback of the \mathbb{Q} -Cartier divisor $K_X + \mathcal{D}|_X$ via the natural morphism $\mathrm{Spec} S \rightarrow X$. We see from the fact that T is a Dedekind scheme and \mathcal{X} is flat over T that the kernel of the surjection $R \rightarrow S$ is a principal ideal. Since strong F -regularity is preserved under completion, we apply Theorem 4.2 to deduce that $(\mathrm{Spec} R, f^*\mathcal{D}, (\mathfrak{a}R)^\lambda)$ is klt at y' . Therefore, by Lemma 2.7 again, $(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda)$ is klt at y . Cases (iii-b) and (iii-c) follow similarly by replacing R with $\mathcal{O}_{\mathcal{X},x}$ and S with $\mathcal{O}_{X,x}$.

Next, we assume that (ii-b) holds. It suffices to show that condition (iii) implies condition (ii-a). First we consider case (iii-a). The closed fiber X is F -rational at x by Remark 2.10. Therefore, the log \mathbb{Q} -Gorensteinness of $(X, \mathcal{D}|_X)$ is an immediate consequence of Remark 2.3 (ii). Additionally, it follows from [25, Theorem 3.8] that \mathcal{X} is pseudorational at x and, in particular, at y because y is a generalization of x . Thus, \mathcal{X}_η is pseudorational at y , and by Remark 2.3 (ii) again, $K_{\mathcal{X}_\eta} + \mathcal{D}_\eta$ is \mathbb{Q} -Cartier at y . Case (iii-b) follows similarly by replacing [25, Theorem 3.8] with [16, Theorem 4.2 (h)] and Remark 2.10. Case (iii-c) follows similarly by replacing Remark 2.10 with Remark 2.3 (iii) and [25, Theorem 3.8] with [9]. \square

In Corollary 4.3, the log \mathbb{Q} -Gorenstein assumption regarding the generic fiber is essential.

Example 4.4 ([37, Theorem 1.1], cf. [8, Remark 6.5]). We give an example of

$$(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda, \mathcal{Z}, T, x, y, t),$$

which satisfies all the assumptions of Corollary 4.3, except for the condition that $K_{\mathcal{X}_\eta} + \mathcal{D}_\eta$ is \mathbb{Q} -Cartier at y , where $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is not klt at y in the sense of de Fernex-Hacon (cf. [7, Section 7]). The latter condition means that there does not exist an effective \mathbb{Q} -Weil divisor Δ on \mathcal{X}_η such that $K_{\mathcal{X}_\eta} + \mathcal{D}_\eta + \Delta$ is \mathbb{Q} -Cartier at y and $(\mathcal{X}_\eta, \mathcal{D}_\eta + \Delta, \mathfrak{a}_\eta^\lambda)$ is klt at y .

We fix integers $m, n \geq 1$ such that $m - m/n > 2$. Let I be the ideal of a polynomial ring $\mathbb{Z}[A, B, C, D, E]$ generated by the size-two minors of the following matrix:

$$\begin{pmatrix} A^2 + (3E)^m & B & D \\ C & A^2 & B^n - D \end{pmatrix}$$

and we write $R := \mathbb{Z}[A, B, C, D, E]/I$. We set

$$\mathcal{X} := \mathrm{Spec} R, \quad \mathcal{D} := 0, \quad \mathfrak{a} := \mathcal{O}_{\mathcal{X}}, \quad \lambda := 1, \quad T := \mathrm{Spec} \mathbb{Z},$$

and let $f : \mathcal{X} \rightarrow T$ be a canonical morphism and \mathcal{Z} be the closed subscheme of \mathcal{X} defined by the ideal $(A, B, C, D, E) \subseteq R$. Then, all the fibers of $f|_{\mathcal{Z}}$:

$\mathcal{Z} \rightarrow T$ are singletons. We choose $t \in T$ as the closed point corresponding to the prime number 3, and we let $x \in \mathcal{X}$ be the unique element of the fiber \mathcal{Z}_t over t and $y \in \mathcal{X}$ be the unique element of the generic fiber \mathcal{Z}_η .

First, we note that $X := \mathcal{X}_t = \text{Spec } R/(3) = \text{Spec } S \times_{\mathbb{F}_3} \mathbb{A}_{\mathbb{F}_3}^1$, where S is the quotient ring of a polynomial ring $\mathbb{F}_3[A, B, C, D]$ modulo the ideal generated by the size-two minors of the following matrix:

$$\begin{pmatrix} A^2 & B & D \\ C & A^2 & B^n - D \end{pmatrix}.$$

Since S is strongly F -regular and \mathbb{Q} -Gorenstein by the proof of [37, Proposition 4.3], so is X .

Next, we show that \mathcal{X}_η is not klt at y in the sense of de Fernex-Hacon. Assume to the contrary that there exists an effective \mathbb{Q} -Weil divisor Δ on \mathcal{X}_η such that $K_{\mathcal{X}_\eta} + \Delta$ is \mathbb{Q} -Cartier and $(\mathcal{X}_\eta, \Delta)$ is klt at y . It then follows from [41, Corollary 3.4] that for general prime numbers p , the fiber \mathcal{X}_p of f over the closed point $(p) \in T$ is strongly F -regular at x_p , where $x_p \in \mathcal{X}$ is the unique element of the fiber \mathcal{Z}_p over (p) . However, by [37, Theorem 1.1], \mathcal{X}_p is not strongly F -regular at x_p if $p > 3$, which is a contradiction. Therefore, \mathcal{X}_η is not klt in the sense of de Fernex-Hacon, as desired.

To discuss the singularities of geometric generic fibers, we introduce the notion of geometrically strongly F -regular triples.

Definition 4.5. Suppose that X is a scheme that is essentially of finite type over an F -finite field k of characteristic $p > 0$, and Δ is an effective \mathbb{Q} -Weil divisor on X . Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a nonzero coherent ideal sheaf and $\lambda > 0$ be a real number. $X_l := X \times_{\text{Spec } k} \text{Spec } l$ denotes the base change of X to the perfect closure $l := k^{1/p^\infty}$ of k , and Δ_l (resp. \mathfrak{a}_l) denotes the flat pullback of Δ (resp. \mathfrak{a}) to X_l .

- (i) We say that $(X, \Delta, \mathfrak{a}^\lambda)$ is *geometrically strongly F -regular* over k at a point $x \in X$ if $(X_l, \Delta_l, \mathfrak{a}_l^\lambda)$ is strongly F -regular at the (unique) point $x_l \in X_l$ lying over x .
- (ii) We say that $(X, \Delta, \mathfrak{a}^\lambda)$ is *geometrically strongly F -regular* over k if it is geometrically strongly F -regular at every point $x \in X$.

Using the theory of relative test ideals (see Appendix A), we establish the following equivalent criteria for a triple to be geometrically strongly F -regular.

Proposition 4.6. *Suppose that X is a geometrically normal scheme that is essentially of finite type over an F -finite field k of characteristic $p > 0$, and Δ is an effective \mathbb{Q} -Weil divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier with an index not divisible by p . Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a nonzero coherent ideal sheaf and λ be a rational number. For a field extension $k \subseteq l$, X_l denotes the fiber product $X \times_{\text{Spec } k} \text{Spec } l$, and Δ_l (resp. \mathfrak{a}_l) denotes the flat pullback of Δ (resp. \mathfrak{a}) to X_l . Given a point $x \in X$, the following conditions are equivalent to each other.*

- (i) $(X, \Delta, \mathfrak{a}^\lambda)$ is *geometrically strongly F -regular* over k at x .

- (ii) For any perfect field $l \supseteq k$, $(X_l, \Delta_l, \mathfrak{a}_l^\lambda)$ is strongly F -regular at every point of X_l lying over x .
- (iii) For any F -finite field $l \supseteq k$, $(X_l, \Delta_l, \mathfrak{a}_l^\lambda)$ is strongly F -regular at every point of X_l lying over x .
- (iv) For a sufficiently divisible integer $n \geq 1$, $(X_{k^{1/p^n}}, \Delta_{k^{1/p^n}}, \mathfrak{a}_{k^{1/p^n}}^\lambda)$ is strongly F -regular at a (unique) point $X_{k^{1/p^n}}$ lying over x .

Proof. Conditions (i), (ii), and (iv) are equivalent according to Theorem A.14. Since the implication (iii) \Rightarrow (iv) is obvious, we show that (ii) implies (iii).

Given an F -finite field $l \supseteq k$, let l' be the perfect closure of l . By (ii), $(X_{l'}, \Delta_{l'}, \mathfrak{a}_{l'}^\lambda)$ is strongly F -regular at every point of $X_{l'}$ lying over x . Applying an argument similar to that in the proof of [15, Theorem 3.1] to the faithfully flat morphism $X_{l'} \rightarrow X_l$, we see that $(X_l, \Delta_l, \mathfrak{a}_l^\lambda)$ is strongly F -regular at every point of X_l lying over x . \square

Theorem 4.7. *With the same notation as that in Setting 2.15, assume that conditions (i), (ii), and (iii-b) in Corollary 4.3 hold. If the residue field $\kappa(t)$ is perfect, then $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is geometrically strongly F -regular over the function field $\kappa(\eta)$ at y .*

Proof. First, note that the fiber X is geometrically normal over $\kappa(t)$ at x because $\kappa(t)$ is perfect. Shrinking \mathcal{X} and T if necessary, we may assume that \mathcal{X} is affine and $g : \mathcal{X} \rightarrow T$ is a normal morphism; that is, its fibers are geometrically normal. We take an effective \mathbb{Q} -Weil divisor A on \mathcal{X} that has no common component with X and is linearly equivalent to $K_{\mathcal{X}} + \mathcal{D}$. Replacing \mathcal{D} by $\mathcal{D} + \varepsilon A$ with a sufficiently small $\varepsilon > 0$, we may assume that the index of $K_{\mathcal{X}_\eta} + \mathcal{D}_\eta$ is not divisible by p . Similarly, we may assume that λ is a rational number.

We fix an integer $n \geq 1$ and set $L := \kappa(\eta)^{1/p^n}$. Let T' be the normalization of T in L and $\mathcal{X}' := \mathcal{X} \times_T T'$ be the base change of \mathcal{X} to T' . We note that T' is a Dedekind scheme and that \mathcal{X}' is normal. Let η' be the generic point of T' and $y' \in \mathcal{X}'_{\eta'} \cong \mathcal{X}_\eta \times_{\text{Spec } \kappa(\eta)} \text{Spec } L$ be the unique point lying over $y \in \mathcal{X}_\eta$. By Proposition 4.6, it is sufficient to show that $(\mathcal{X}'_{\eta'}, \mathcal{D}'_{\eta'}, \mathfrak{a}'_{\eta'}^\lambda)$ is strongly F -regular at y' , where $\mathfrak{a}' := \mathfrak{a}_{\mathcal{X}'}$ and $\mathcal{D}' := \mu^* \mathcal{D}$ is the pullback of \mathcal{D} by the finite flat morphism $\mu : \mathcal{X}' \rightarrow \mathcal{X}$.

We verify that $K_{\mathcal{X}'_{\eta'}} + \mathcal{D}'_{\eta'}$ is \mathbb{Q} -Cartier at y' . Let $\omega_{\mathcal{X}_\eta/\kappa(\eta)}$ and $\omega_{\mathcal{X}'_{\eta'}/L}$ be the relative canonical sheaves of $\mathcal{X}_\eta \rightarrow \text{Spec } \kappa(\eta)$ and $\mathcal{X}'_{\eta'} \rightarrow \text{Spec } L$, respectively. Since $\text{Spec } \kappa(\eta)$ and $\text{Spec } L$ are Gorenstein, the relative canonical sheaves are simply canonical sheaves according to [40, Lemma 0BZL]. Moreover, by [40, Lemma 0BZV], we have an isomorphism $\mu_{\eta'}^* \omega_{\mathcal{X}_\eta/\kappa(\eta)} \cong \omega_{\mathcal{X}'_{\eta'}/L}$, where $\mu_{\eta'} : \mathcal{X}'_{\eta'} \rightarrow \mathcal{X}_\eta$ is the morphism induced by μ . Considering that a canonical divisor is unique up to adding a Cartier divisor ([14, V. Theorem 3.1]), we conclude that $K_{\mathcal{X}'_{\eta'}} - \mu_{\eta'}^* K_{\mathcal{X}_\eta}$ is a Cartier divisor. Thus, $K_{\mathcal{X}'_{\eta'}} + \mathcal{D}'_{\eta'}$ is \mathbb{Q} -Cartier at y' .

Finally, we show that $(\mathcal{X}'_{\eta'}, \mathcal{D}'_{\eta'}, \mathfrak{a}'_{\eta'}^\lambda)$ is strongly F -regular at y' . We take a point $t' \in T'$ lying over $t \in T$ and a point $x' \in X' := \mathcal{X}'_{t'}$ lying over

$x \in X = \mathcal{X}_t$. Since $\kappa(t)$ is perfect and $X' \cong X \times_{\mathrm{Spec} \kappa(t)} \mathrm{Spec} \kappa(t')$, it follows from Proposition 4.6 that $(X', \mathcal{D}'|_{X'}, (\mathfrak{a}'\mathcal{O}_{X'})^\lambda)$ is strongly F -regular at x' . Moreover, $K_{X'} + \mathcal{D}'|_{X'}$ is \mathbb{Q} -Cartier at x' , and $x' \in \mathcal{X}'$ is a specialization of y' . It then follows from Corollary 4.3 that $(\mathcal{X}'_{\eta'}, \mathcal{D}'_{\eta'}, \mathfrak{a}'_{\eta'}{}^\lambda)$ is strongly F -regular at y' , as desired. \square

As a corollary of Corollary 4.3 and Theorem 4.7, we provide an affirmative answer to a conjecture of Liedtke-Martin-Matsumoto [22] regarding the deformations of isolated lrq singularities. An *isolated lrq singularity* over a field k is the spectrum $\mathrm{Spec} R$ of a normal local k -algebra (R, \mathfrak{m}) such that

$$\widehat{R} \cong k[[x_1, \dots, x_d]]^G,$$

where \widehat{R} is the \mathfrak{m} -adic completion of R and G is a finite linearly reductive group scheme that acts on a formal power series ring $k[[x_1, \dots, x_d]]$ over k , whose action fixes the closed point and is free away from it (see [22, Definition 6.4]).

Corollary 4.8 ([22, Conjecture 12.1 (1)]). *Let B be the spectrum of a DVR with an algebraically closed residue field k and $\mathcal{X} \rightarrow B$ be a flat morphism of finite type with special and geometric generic fibers \mathcal{X}_0 and $\mathcal{X}_{\bar{\eta}}$, respectively. Let $x \in \mathcal{X}_0$ and $y \in \mathcal{X}_{\bar{\eta}}$ be points such that $x \in \mathcal{X}$ is a specialization of the image $u(y) \in \mathcal{X}$ of y by the morphism $u : \mathcal{X}_{\bar{\eta}} \rightarrow \mathcal{X}$. If the special fiber \mathcal{X}_0 has an isolated lrq singularity at x , then so does the geometric generic fiber $\mathcal{X}_{\bar{\eta}}$ at y .*

Proof. By [22, Lemma 12.5], the problem is reduced to showing that if B is of mixed characteristic $(0, p)$ (resp. characteristic $p > 0$) and \mathcal{X}_0 is two-dimensional and strongly F -regular at x , then $\mathcal{X}_{\bar{\eta}}$ is klt (resp. strongly F -regular) at y . The mixed characteristic case follows from case (iii-a) of Corollary 4.3 (ii-b), and therefore, we consider the case where B is of characteristic p . Since strong F -regularity descends under faithfully flat morphisms according to [15, Theorem 3.1 b)], by passing to the completion, we may assume that B is the spectrum of a complete DVR. Then, B is F -finite, and the above assertion follows from Theorem 4.7. \square

Next, we study the behaviors of singularities in proper flat families.

Corollary 4.9. *With the same notation as that in Setting 2.15, assume that the following conditions are all satisfied.*

- (i) T is a Dedekind scheme; that is, $\dim T = 1$.
- (ii) One of the following holds.
 - (a) $K_X + \mathcal{D}|_X$ and $K_{\mathcal{X}_\eta} + \mathcal{D}_\eta$ are \mathbb{Q} -Cartier, or
 - (b) $\dim X \leq 2$.
- (iii) One of the following cases occurs.
 - (a) $\mathcal{O}_{T,t}$ is of mixed characteristic $(0, p)$, the residue field $\kappa(t)$ is F -finite and of characteristic p , and $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is strongly F -regular.
 - (b) $\mathcal{O}_{T,t}$ is F -finite and of positive characteristic $p > 0$, and the triple $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is strongly F -regular.

- (c) $\mathcal{O}_{T,t}$ is of equal characteristic zero, and $(X, \mathcal{D}|_X, (\mathfrak{a}\mathcal{O}_X)^\lambda)$ is klt.
 (iv) \mathcal{X} is proper over T .

Then, $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is klt in case (iii-a) or (iii-c) and is strongly F -regular in case (iii-b). Moreover, in case (iii-b), if the residue field $\kappa(t)$ is perfect, then $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is geometrically strongly F -regular over the function field $\kappa(\eta)$.

Proof. We take any point $y \in \mathcal{X}_\eta$. Since the structure map $\mathcal{X} \rightarrow T$ is a closed map, there exists a point $x \in X$ that is a specialization of y . It then follows from Corollary 4.3 that $(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda)$ is klt (resp. strongly F -regular) at y in case (a) or (c) (resp. case (b)). The assertion regarding geometrically strong F -regularity follows from Theorem 4.7. \square

Corollary 4.10. *With the same notation as that in Setting 2.15, assume that conditions (i), (ii), and (iv) in Corollary 4.9 hold. If T is of finite type over $\text{Spec } \mathbb{Z}$, then the following conditions are equivalent to each other:*

- (i) $(\mathcal{X}_p, \mathcal{D}_p, \mathfrak{a}_p^\lambda)$ is strongly F -regular for some closed point $p \in T$.
- (ii) $(\mathcal{X}_p, \mathcal{D}_p, \mathfrak{a}_p^\lambda)$ is strongly F -regular for a general closed point $p \in T$.
- (iii) $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is klt.

Proof. It is obvious that (ii) implies (i). The implication (i) \Rightarrow (iii) is a consequence of Corollary 4.9. The implication (iii) \Rightarrow (ii) follows from the fact that the mod- p reduction of a klt singularity is strongly F -regular for a general p (see [13] and [41]). \square

Corollary 4.11. *Let T be a smooth curve over a perfect field k of characteristic $p > 0$ and $(\mathcal{X}, \mathcal{D}, \mathfrak{a}^\lambda) \rightarrow T$ be a proper flat family of triples over T , where \mathcal{D} is an effective \mathbb{Q} -Weil divisor on a normal variety \mathcal{X} over k , $\mathfrak{a} \subseteq \mathcal{O}_X$ is a nonzero coherent ideal sheaf, and $\lambda > 0$ is a real number.*

- (1) *Suppose that k is an uncountable algebraically closed field. If some closed fiber $(\mathcal{X}_{t_0}, \mathcal{D}_{t_0}, (\mathfrak{a}\mathcal{O}_{\mathcal{X}_{t_0}})^\lambda)$ is log \mathbb{Q} -Gorenstein and strongly F -regular and if a general closed fiber $(\mathcal{X}_t, \mathcal{D}_t, (\mathfrak{a}\mathcal{O}_{\mathcal{X}_t})^\lambda)$ is log \mathbb{Q} -Gorenstein, then it is also strongly F -regular.*
- (2) *If some closed fiber $(\mathcal{X}_{t_0}, \mathcal{D}_{t_0}, (\mathfrak{a}\mathcal{O}_{\mathcal{X}_{t_0}})^\lambda)$ is two-dimensional and strongly F -regular, then so is a general fiber $(\mathcal{X}_t, \mathcal{D}_t, (\mathfrak{a}\mathcal{O}_{\mathcal{X}_t})^\lambda)$.*

Proof. (1) First, we note that the generic fiber $(\mathcal{X}_\eta, \mathcal{D}_\eta, (\mathfrak{a}\mathcal{O}_{\mathcal{X}_\eta})^\lambda)$ is log \mathbb{Q} -Gorenstein by Remark 2.16. Since the closed fiber \mathcal{X}_{t_0} is geometrically normal over $\kappa(t_0) = k$, by shrinking T if necessary, we may assume that all fibers of $\mathcal{X} \rightarrow T$ are geometrically normal. On the other hand, it follows from Corollary 4.9 that $(\mathcal{X}_\eta, \mathcal{D}_\eta, \mathfrak{a}_\eta^\lambda)$ is geometrically strongly F -regular over the function field $\kappa(\eta)$. Since $K_{\mathcal{X}} + \mathcal{D}$ is \mathbb{Q} -Cartier along \mathcal{X}_η , there exists a nonempty open subset $V \subseteq T$ such that $K_{\mathcal{X}} + \mathcal{D}$ is \mathbb{Q} -Cartier on $\mathcal{X}_V := \mathcal{X} \times_T V$. Applying Proposition 4.13 to \mathcal{X}_V , we see that the fiber $(\mathcal{X}_s, \mathcal{D}_s, \mathfrak{a}_s^\lambda)$ over a general closed point $s \in V$ is geometrically strongly F -regular over $\kappa(s)$. In particular, $(\mathcal{X}_s, \mathcal{D}_s, \mathfrak{a}_s^\lambda)$ is strongly F -regular by Proposition 4.6.

(2) The proof is essentially the same as that of (1), but the log \mathbb{Q} -Gorensteinness of the generic fiber $(\mathcal{X}_\eta, \mathcal{D}_\eta)$ follows from Corollary 4.3, and therefore, the extra assumption regarding the base field k is unnecessary. \square

Remark 4.12. When $\mathcal{D} = 0$, we do not need to assume the normality of \mathcal{X} in Corollaries 4.3, 4.9, and 4.11. In this case, since normality lifts from Cartier divisors, by shrinking \mathcal{X} and T in Corollary 4.3 (resp. by shrinking T in Corollaries 4.9 and 4.11), we can reduce the problem to the case where \mathcal{X} is normal.

In the proof of Corollary 4.11, we use the following proposition, which shows that the strong F -regularity of general fibers is deduced from that of geometric generic fibers.

Proposition 4.13 (cf. [29, Corollary 4.21]). *Suppose that V is an F -finite regular integral scheme of characteristic $p > 0$, $f : X \rightarrow V$ is a flat morphism of finite type from a normal integral scheme X , and Δ is an effective \mathbb{Q} -Weil divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier. Let \mathfrak{a} be a nonzero coherent ideal sheaf, λ be a real number, and $\eta \in V$ denote a generic point. If the generic fiber $(X_\eta, \Delta_\eta, \mathfrak{a}_\eta^\lambda)$ is geometrically strongly F -regular over the function field $\kappa(\eta)$ of V , then $(X_s, \Delta_s, \mathfrak{a}_s^\lambda)$ is geometrically strongly F -regular over $\kappa(s)$ for a general point $s \in V$.*

Proof. Taking an affine open covering of X , we can assume that X is affine, and then there exists an effective \mathbb{Q} -Weil divisor D on X such that $D \sim K_X + \Delta$. Since $(X_\eta, \Delta_\eta + \varepsilon D_\eta, \mathfrak{a}_\eta^\lambda)$ is geometrically strongly F -regular over $\kappa(\eta)$ for any sufficiently small $\varepsilon > 0$, by replacing Δ by $\Delta + \varepsilon D$, we may assume that the index of $K_X + \Delta$ is not divisible by p . Similarly, we may assume that λ is a rational number.

It follows from Theorem A.14 that, possibly after shrinking V , there exists an integer n such that $(X_s, \Delta_s, \mathfrak{a}_s^\lambda)$ is geometrically strongly F -regular over $\kappa(s)$ for every point $s \in V$ if and only if $\tau(X_{V^n}, h^*\Delta, (\mathfrak{a}\mathcal{O}_{X_{V^n}})^\lambda) = \mathcal{O}_{X_{V^n}}$ near $h^{-1}(X_s)$, where $h : X_{V^n} = X \times_V V^n \rightarrow X$ is the first projection. Let $Z \subseteq X_{V^n}$ be the closed subscheme defined by the test ideal $\tau(X_{V^n}, h^*\Delta, (\mathfrak{a}\mathcal{O}_{X_{V^n}})^\lambda)$. By Chevalley's theorem on constructible sets, $(f \circ h)(Z) \subseteq V$ is a constructible set. Since the complement $V \setminus (f \circ h)(Z)$ is a constructible set containing the generic point η , it contains a dense open subset $U \subseteq V$. By the definition of Z , we see that $(X_s, \Delta_s, \mathfrak{a}_s^\lambda)$ is geometrically strongly F -regular over $\kappa(s)$ for every point $s \in U$. \square

We close this section with an example showing that the global analog of Corollary 4.9 does not hold.

Definition 4.14. Let X be a normal projective variety over a perfect field k and Δ be an effective \mathbb{Q} -Weil divisor on X .

- (1) The pair (X, Δ) is said to be *log Fano* if (X, Δ) is klt and $-(K_X + \Delta)$ is ample. We say that X is of *Fano type* if there exists an effective Weil divisor B on X such that (X, B) is a log Fano pair.
- (2) ([12], [39]) Suppose that k is of characteristic $p > 0$. Then, X is said to be *globally F -regular* if for every effective Weil divisor D on X , there exists an integer $e \geq 1$ such that the composite

$$\mathcal{O}_X \xrightarrow{\varphi_0^{(e)}} F_*^e \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(D)$$

splits as an \mathcal{O}_X -module homomorphism, where $\varphi_0^{(e)}$ is defined as in Definition 2.8 and $F_*^e \iota$ is the pushforward of the natural inclusion $\iota : \mathcal{O}_X \rightarrow \mathcal{O}_X(D)$ by the e -th iterated Frobenius morphism $F^e : X \rightarrow X$.

Proposition 4.15 (cf. [36, Proposition 5.4]). *Let (X, Δ) be a log Fano pair over an algebraically closed field k of characteristic zero and H be an ample Cartier divisor on X . Let $S = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nH))$ be the section ring of X with respect to H and Δ_S be the \mathbb{Q} -Weil divisor on $\text{Spec } S$ corresponding to Δ . Then, $(\text{Spec } S, \Delta_S)$ is klt in the sense of de Fernex-Hacon.*

Proof. We take a sufficiently small $\varepsilon \ll 1$ so that $-(K_X + \Delta) - \varepsilon H$ is ample. By Bertini, we can take $\Delta' \in |-(K_X + \Delta) - \varepsilon H|_{\mathbb{Q}}$ so that $(X, \Delta + \Delta')$ is klt. It then follows from an argument similar to the proof of [36, Proposition 5.4] that $(\text{Spec } S, \Delta_S + \Delta'_S)$ is klt, where Δ'_S is the \mathbb{Q} -Weil divisor on $\text{Spec } S$ corresponding to Δ' . In particular, $(\text{Spec } S, \Delta_S)$ is klt in the sense of de Fernex-Hacon. \square

Since globally F -regular varieties (resp. varieties of Fano type) can be viewed as global analogs of strongly F -regular singularities (resp. klt singularities) (see [36, Proposition 5.3, 5.4]), it is natural to ask whether globally F -regular varieties deform to varieties of Fano type. In the following example, we provide a negative answer to this question even if we assume that the fibers are \mathbb{Q} -Gorenstein.

Example 4.16. Let m, n, I, R, t and T be as in Example 4.4. First, we observe that R is an \mathbb{N} -graded \mathbb{Z} -algebra with respect to the grading

$$\deg A = nm, \deg B = 2m, \deg C = 2mn, \deg D = 2mn, \deg E = 2n.$$

We show that the generic fiber \mathcal{Y}_η of a flat projective morphism $\mathcal{Y} := \text{Proj } R \rightarrow T$ is not of Fano type, while \mathcal{Y}_η is \mathbb{Q} -Gorenstein and the fiber \mathcal{Y}_t over the closed point $t \in T$ is \mathbb{Q} -Gorenstein globally F -regular.

Since the graded ring $R/(3)$ is strongly F -regular (see Example 4.4), every Veronese subring of $R/(3)$ is also strongly F -regular by [15, Theorem 3.1 (e)]. Noting that $\mathcal{Y}_t = \text{Proj } R/(3)$, one can pick an ample invertible sheaf L on \mathcal{Y}_t such that the section ring $\bigoplus_{\ell \geq 0} H^0(\mathcal{Y}_t, L^{\otimes \ell})$ is isomorphic to a Veronese subring of $R/(3)$. Therefore, it follows from [36, Proposition 5.3] that \mathcal{Y}_t is globally F -regular. Moreover, \mathcal{Y}_t is \mathbb{Q} -Gorenstein by Remark 2.3 (ii) and Remark 2.10 because \mathcal{Y}_t has only strongly F -regular singularities and $\dim \mathcal{Y}_t = 2$. As we have seen in the proof of Corollary 4.3, the generic fiber \mathcal{Y}_η is also \mathbb{Q} -Gorenstein.

We finally show that \mathcal{Y}_η is not of Fano type. Assume to the contrary that \mathcal{Y}_η is of Fano type; then, so is $\mathcal{Y}_\eta \times_{\text{Spec } \mathbb{Q}} \text{Spec } \mathbb{C}$. Since $\mathcal{Y}_\eta \times_{\text{Spec } \mathbb{Q}} \text{Spec } \mathbb{C} = \text{Proj}(R \otimes_{\mathbb{Z}} \mathbb{C})$, by Proposition 4.15, there exists an integer $u \geq 1$ such that the spectrum of the u -th Veronese subring of $R \otimes_{\mathbb{Z}} \mathbb{C}$ is klt in the sense of de Fernex-Hacon. It follows from an argument used in Example 4.4 that the factor ring $R^{(u)}/(p)$ of the u -th Veronese subring $R^{(u)}$ of R modulo a general prime p is strongly F -regular. We may assume that p is sufficiently large so that $p > 3$, and u is not divisible by p . Then, the extension $R^{(u)}/(p) \subseteq R/(p)$

is étale in codimension one, and therefore, $R/(p)$ is strongly F -regular by [43, Theorem 2.7], which contradicts [37, Theorem 1.1]. Thus, \mathcal{Y}_η is not of Fano type.

APPENDIX A. RELATIVE TEST IDEALS FOR TRIPLES

In this appendix, we generalize the theory of relative test ideals, introduced in [29] for pairs, to the case of triples. To obtain a stabilization result (Proposition A.11) in this case, which is slightly more subtle than the case involving pairs ([29, Lemma 4.2]), we use an argument similar to that in the proof of [31, Proposition 3.8].

Let R be an integral domain of characteristic $p > 0$ and $q = p^e$ be a power of p . We fix an algebraic closure $\overline{\text{Frac}(R)}$ of the fractional field $\text{Frac}(R)$ of R , and we let $R^{1/q}$ be the ring of the q -th roots of the elements in R ; that is,

$$R^{1/q} = \{x \in \overline{\text{Frac}(R)} \mid x^q \in R\}.$$

Given an R -module M , the ring isomorphism $R^{1/q} \xrightarrow{\sim} R$ sending x to x^q induces an $R^{1/q}$ -module structure on M , which is denoted by $M^{1/q}$. $R^{1/q}$ -modules are considered R -modules via the natural inclusion $R \hookrightarrow R^{1/q}$. When $M^{1/q}$ is regarded as an R -module in this way, it is simply the push-forward $F_*^e M$ of M by the e -th iterated Frobenius morphism $F^e : \text{Spec } R \rightarrow \text{Spec } R$.

Lemma A.1. *Let M be a module over an integral domain R of characteristic $p > 0$.*

- (1) *For an invertible R -module L , we have an isomorphism*

$$L \otimes_R M^{1/q} \cong (L^{\otimes q} \otimes_R M)^{1/q}$$

of $R^{1/q}$ -modules.

- (2) *For an ideal $\mathfrak{a} \subseteq R$, we have the equality*

$$\mathfrak{a} \cdot M^{1/q} = (\mathfrak{a}^{[q]} M)^{1/q},$$

where $\mathfrak{a}^{[q]} \subseteq R$ is the ideal generated by the q -th powers of all elements of \mathfrak{a} .

Proof. Since $(F^e)^* L \cong L^{\otimes q}$ as R -modules, it follows from the projection formula that there exists an R -module isomorphism

$$f : L \otimes_R M^{1/q} \xrightarrow{\sim} ((F^e)^* L \otimes_R M)^{1/q} \xrightarrow{\sim} (L^{\otimes q} \otimes_R M)^{1/q}.$$

It is straightforward to verify that f is an $R^{1/q}$ -module homomorphism, which shows that (1) holds. Assertion (2) is obvious. \square

Let N be an invertible R -module and $\gamma : N^{1/q} \rightarrow R$ be an R -module homomorphism. For each integer $n \geq 0$, we set $N^{(n)} := N^{\otimes \frac{q^n - 1}{q - 1}}$, and $\gamma^n : (N^{(n)})^{1/q^n} \rightarrow R$ denotes the R -module homomorphism defined inductively

by the following composite:

$$\begin{array}{ccc}
 \gamma^n : (N^{(n)})^{1/q^n} & \xrightarrow{\sim} & (((N^{(n-1)})^{\otimes q} \otimes_R N)^{1/q})^{1/q^{n-1}} \\
 & \xrightarrow{\sim} & (N^{(n-1)} \otimes_R N^{1/q})^{1/q^{n-1}} \\
 & \xrightarrow{(\text{id} \otimes \gamma)^{1/q^{n-1}}} & (N^{(n-1)})^{1/q^{n-1}} \\
 & \xrightarrow{\gamma^{n-1}} & R.
 \end{array}$$

From now on, by abuse of notation, the map $(N^{(n)})^{1/q^n} \rightarrow (N^{(n-1)})^{1/q^{n-1}}$ is also denoted by γ .

Definition A.2. Suppose that R is an integral domain of characteristic $p > 0$ and that $\gamma : N^{1/q} \rightarrow R$ is an R -module homomorphism, where N is an invertible R -module and $q = p^e$ is a power of p . Let I and \mathfrak{a} be nonzero ideals of R and $\lambda > 0$ be a real number. The *test ideal* $\tau(X, \gamma I, \mathfrak{a}^\lambda)$ of $(X := \text{Spec } R, \gamma, \mathfrak{a}^\lambda)$ with respect to I is then defined as

$$\tau(X, \gamma I, \mathfrak{a}^\lambda) = \sum_{i \geq 0} \gamma^i ((\mathfrak{a}^{\lceil q^i \lambda \rceil} I N^{(i)})^{1/q^i}) \subseteq R.$$

Example A.3. With the same notation as above, we suppose in addition that R is an F -finite normal domain with a dualizing complex ω_R^\bullet such that $F^! \omega_R^\bullet \cong \omega_R^\bullet$, and there exists an effective \mathbb{Q} -Weil divisor Δ on $X := \text{Spec } R$ such that $(q-1)(K_X + \Delta)$ is Cartier and $I \subseteq \tau(X, \Delta)$. If $N = \mathcal{O}_X((1-q)(K_X + \Delta))$ and γ is obtained from the equivalence relation (\star) in [3, Paragraphs after Definition 2.4] (see also [33, Theorem 3.11]), then

$$\tau(X, \gamma I, \mathfrak{a}^\lambda) = \tau(X, \Delta, \mathfrak{a}^\lambda).$$

Indeed, after localization, we may assume that R is local and that $N = R$. Since γ^n is a generator of the free $F^{en}R$ -module $\text{Hom}_R(F_*^{en} \mathcal{O}_X((q^n - 1)\Delta), R)$ of rank one, the above assertion follows from Remark 2.12 and [35, Proposition 4.6].

Let $f : A \rightarrow R$ be a ring homomorphism of integral domains of characteristic $p > 0$. We write $R_{A^{1/q}} := R \otimes_A A^{1/q}$ and note that the inclusion $R \rightarrow R^{1/q}$ and the natural morphism $f^{1/q} : A^{1/q} \rightarrow R^{1/q}$ induce a natural morphism $R_{A^{1/q}} \rightarrow R^{1/q}$.

$$\begin{array}{ccccc}
 R & \longrightarrow & R_{A^{1/q}} & \longrightarrow & R^{1/q} \\
 \uparrow f & & \uparrow & \nearrow f^{1/q} & \\
 A & \longrightarrow & A^{1/q} & &
 \end{array}$$

Given an $R^{1/q}$ -module M , we view M as an $R_{A^{1/q}}$ -module via this ring homomorphism.

Setting A.4. Suppose that A is a Noetherian integral domain of characteristic $p > 0$, R is an integral domain that is flat and essentially of finite type over A , $\lambda > 0$ is a real number, and $I, \mathfrak{a} \subseteq R$ are nonzero ideals. Let

$\varphi : L^{1/q} \rightarrow R_{A^{1/q}}$ be an $R_{A^{1/q}}$ -module homomorphism, where L is an invertible R -module and $q = p^e$ is a power of p . For each integer $n \geq 0$, we set $B_n := R_{A^{1/q^n}}$, $L^{(n)} := L^{\otimes \frac{q^n - 1}{q - 1}}$, $X := \text{Spec } R$, and $V := \text{Spec } A$.

For each integer $n > 0$, we define a B_n -homomorphism $\tilde{\varphi}_n : (L \otimes_R B_{n-1})^{1/q} \rightarrow B_n$ by

$$\tilde{\varphi}_n : (L \otimes_R B_{n-1})^{1/q} \cong L^{1/q} \otimes_{A^{1/q}} A^{1/q^n} \xrightarrow{\varphi \otimes \text{id}} B_1 \otimes_{A^{1/q}} A^{1/q^n} \cong B_n.$$

Moreover, we define a B_n -homomorphism $\varphi^n : (L^{(n)})^{1/q^n} \rightarrow B_n$ inductively as follows:

$$\begin{aligned} \varphi^n : (L^{(n)})^{1/q^n} &\cong (L \otimes_R (L^{(n-1)})^{1/q^{n-1}})^{1/q} \xrightarrow{(\text{id} \otimes \varphi^{n-1})^{1/q}} (L \otimes_R B_{n-1})^{1/q} \\ &\xrightarrow{\tilde{\varphi}_n} B_n. \end{aligned}$$

For integers $n \geq i \geq 0$, let $a_{i,n} : B_i \rightarrow B_n$ be the ring homomorphism induced by the natural inclusion $A^{1/q^i} \hookrightarrow A^{1/q^n}$.

Definition A.5 (cf. [29, Definition 4.3]). With the same notation as that in Setting A.4, the n -th limiting relative test ideal $\tau_n(X/V, \varphi I, \mathfrak{a}^\lambda)$ of $(X/V, \varphi, \mathfrak{a}^\lambda)$ with respect to I is defined as

$$\tau_n(X/V, \varphi I, \mathfrak{a}^\lambda) := \sum_{i=0}^n \varphi^i((\mathfrak{a}^{\lceil q^i \lambda \rceil} IL^{(i)})^{1/q^i}) B_n \subseteq B_n.$$

Lemma A.6. *With the same notation as that in Setting A.4, we have*

$$\tilde{\varphi}_{n+1}((L \otimes_R \tau_n(X/V, \varphi I, \mathfrak{a}^\lambda))^{1/q}) + (\mathfrak{a}^{\lceil \lambda/q \rceil} I) B_{n+1} = \tau_{n+1}(X/V, \varphi I, \mathfrak{a}^{\lambda/q}).$$

Proof. Note that we have the following commutative diagram:

$$\begin{array}{ccc} (L \otimes_R B_i)^{1/q} & \xrightarrow{\tilde{\varphi}_{i+1}} & B_{i+1} \\ (\text{id} \otimes a_{i,n})^{1/q} \downarrow & & \downarrow a_{i+1,n+1} \\ (L \otimes_R B_n)^{1/q} & \xrightarrow{\tilde{\varphi}_{n+1}} & B_{n+1}. \end{array}$$

Therefore,

$$\begin{aligned} &\tilde{\varphi}_{n+1}((L \otimes_R \varphi^i((\mathfrak{a}^{\lceil q^i \lambda \rceil} IL^{(i)})^{1/q^i}) B_n)^{1/q}) \\ &= \tilde{\varphi}_{i+1}((L \otimes_R \varphi^i((\mathfrak{a}^{\lceil q^i \lambda \rceil} IL^{(i)})^{1/q^i}))^{1/q}) B_{n+1} \\ &= \varphi^{i+1}((\mathfrak{a}^{\lceil q^i \lambda \rceil} IL^{(i+1)})^{1/q^{i+1}}) B_{n+1} \\ &= \varphi^{i+1}((\mathfrak{a}^{\lceil q^{i+1}(\lambda/q) \rceil} IL^{(i+1)})^{1/q^{i+1}}) B_{n+1}, \end{aligned}$$

which implies the assertion. \square

The notation $\mu(\mathfrak{a})$ denotes the minimal number of generators for the ideal \mathfrak{a} .

Lemma A.7. *With the same notation as that in Setting A.4, we assume that $\lambda > \mu(\mathfrak{a}) - 1$. Then,*

$$\tau_n(X/V, \varphi I, \mathfrak{a}^\lambda) \mathfrak{a} = \tau_n(X/V, \varphi I, \mathfrak{a}^{\lambda+1}).$$

Proof. Since $a_{i,n} : B_i \rightarrow B_n$ and φ^i are R -homomorphisms,

$$\begin{aligned} \tau_n(X/V, \varphi I, \mathfrak{a}^\lambda) \mathfrak{a} &= \sum_{i=0}^n \varphi^i(\mathfrak{a}^{[q^i \lambda]} IL^{(i)})^{1/q^i} B_n \\ &= \sum_{i=0}^n \varphi^i((\mathfrak{a}^{[q^i]} \mathfrak{a}^{[q^i \lambda]} IL^{(i)})^{1/q^i}) B_n \\ &= \sum_{i=0}^n \varphi^i((\mathfrak{a}^{q^i + [q^i \lambda]} IL^{(i)})^{1/q^i}) B_n \\ &= \tau_n(X/V, \varphi I, \mathfrak{a}^{\lambda+1}), \end{aligned}$$

where the third equality follows from [31, Lemma 2.11] because $\lambda > \mu(\mathfrak{a}) - 1$. \square

Proposition A.8. *With the same notation as that in Setting A.4, we assume that $\lambda > \mu(\mathfrak{a}) - 1$ and that $(q-1)\lambda$ is an integer. If the equality*

$$\tau_{n-1}(X/V, \varphi I, \mathfrak{a}^\lambda) B_n = \tau_n(X/V, \varphi I, \mathfrak{a}^\lambda)$$

holds for some integer $n \geq 1$, then

$$\tau_{m-1}(X/V, \varphi I, \mathfrak{a}^\lambda) B_m = \tau_m(X/V, \varphi I, \mathfrak{a}^\lambda)$$

holds for every integer $m > n$.

Proof. It is sufficient to check the equality when $m = n + 1$. Since we have $\tau_{n-1}(X/V, \varphi I, \mathfrak{a}^\lambda) B_n = \tau_n(X/V, \varphi I, \mathfrak{a}^\lambda)$, the commutative diagram in the proof of Lemma A.6 yields

$$\begin{aligned} &\tilde{\varphi}_n((L \otimes_R \tau_{n-1}(X/V, \varphi I, \mathfrak{a}^\lambda) \mathfrak{a}^{(q-1)\lambda})^{1/q}) B_{n+1} \\ &= \tilde{\varphi}_{n+1}((L \otimes_R (\tau_n(X/V, \varphi I, \mathfrak{a}^\lambda) \mathfrak{a}^{(q-1)\lambda})^{1/q}). \end{aligned}$$

Combining this with Lemmas A.6 and A.7, we complete the proof. \square

Setting A.9. With the same notation as that in Setting A.4, we suppose that $V' := \text{Spec } A' \rightarrow V$ is a morphism from a Noetherian integral affine scheme such that $X' := X \times_V V'$ is integral. We set $R' := R \otimes_A A'$, $L' := L \otimes_R R'$, $I' := IR'$, $\mathfrak{a}' := \mathfrak{a}R'$, and $B'_n := R' \otimes_{A'} A'^{1/q^n}$. Let $\varphi' : (L')^{1/q} \rightarrow B'_1$ be the morphism induced by φ (see [29, Subsection 2.16] for details).

Lemma A.10 (cf. [29, Theorem 4.5]). *With the same notation as that in Setting A.9, we have*

$$\tau_n(X'/V', \varphi' I', \mathfrak{a}'^\lambda) = \tau_n(X/V, \varphi I, \mathfrak{a}^\lambda) B'_n.$$

Proof. The assertion follows from an argument similar to that in the proof of [29, Theorem 4.5]. \square

Proposition A.8, with the aid of Lemma A.10, yields the following stabilization result. The strategy of this proof refers to [31, Proposition 3.8].

Proposition A.11 (cf. [29, Lemma 4.2]). *With the same notation as that in Setting A.4, we assume that $\lambda > \mu(\mathfrak{a}) - 1$ and that $(q-1)\lambda$ is an integer. There exist a dense open subset $U \subseteq V$ and an integer n_0 such that for any*

integer $n \geq n_0$ and any morphism $V' = \text{Spec } A' \rightarrow V$ as in Setting A.9t whose image is contained in U , we have

$$\tau_{n-1}(X'/V', \varphi'I', \mathfrak{a}'^\lambda)B'_n = \tau_n(X'/V', \varphi'I', \mathfrak{a}'^\lambda).$$

Proof. By an argument similar to that in the proof of [29, Proposition 3.3], there exist a dense open subset $U \subseteq V$ and an integer n_0 such that

$$(\tau_{n_0-1}(X/V, \varphi I, \mathfrak{a}^\lambda)B_n)|_U = \tau_{n_0}(X/V, \varphi I, \mathfrak{a}^\lambda)|_U.$$

The assertion then follows from Lemma A.10 and Proposition A.8, \square

With the same notation as that in Setting A.4, we assume that the morphism $X \rightarrow V = \text{Spec } A$ is normal and that A is an F -finite regular integral domain with a dualizing complex ω_A^\bullet such that $F^!\omega_A^\bullet \cong \omega_A^\bullet$. We note that in this case, $B_m = R_{A^{1/q^m}} = R \otimes_A A^{1/q^m}$ is normal for every m according to [27, p.184 Corollary].

We fix a canonical divisor K_V of $V = \text{Spec } A$ and write $M := A((1 - q)K_V)$. Let $\Psi : M^{1/q} \rightarrow A$ be a generator of $\text{Hom}_A(M^{1/q}, A)$, which is a free $F_*^e A$ -module of rank one. For each integer $m \geq 1$, we define the B_m -module $N_m := L \otimes_A M^{1/q^m}$ and the B_m -module homomorphism $\gamma_m : N_m^{1/q} \rightarrow B_m$ as the composite map

$$\gamma_m : N_m^{1/q} \cong L^{1/q} \otimes_{A^{1/q}} M^{1/q^{m+1}} \xrightarrow{\varphi \otimes \Psi^{1/q^m}} B_1 \otimes_{A^{1/q}} A^{1/q^m} \cong B_m.$$

Given a scheme T of characteristic $p > 0$ and an integer $n \geq 0$, the n -th iterated Frobenius morphism $F : T \rightarrow T$ induces a T -scheme structure on T , which is denoted by T^n . In our setting, $V^{em} \cong \text{Spec } A^{1/q^m}$ as V -schemes, and $X_{V^{em}} := X \times_V V^{em} \cong \text{Spec } B_m$.

Lemma A.12. *With the same notation as that above, we set*

$$\mathfrak{c}_m := \tau(X_{V^{em}}, \gamma_m(IB_m), (\mathfrak{a}B_m)^\lambda) \subseteq B_m.$$

Then,

$$\begin{aligned} & \tilde{\varphi}_{m+1}((L \otimes_R \mathfrak{c}_m)^{1/q}) + (\mathfrak{a}^{\lceil \lambda/q \rceil} I)B_{m+1} \\ &= \tau(X_{V^{e(m+1)}}, \gamma_{m+1}(IB_{m+1}), (\mathfrak{a}B_{m+1})^{\lambda/q}). \end{aligned}$$

Proof. First, we note that for any integer $n \geq 0$,

$$\begin{aligned} N_{m+1}^{(n+1)} &\cong L^{\otimes q^n} \otimes_R L^{(n)} \otimes_A (M^{1/q^m})^{(n)} \otimes_{A^{1/q^m}} M^{1/q^{m+1}} \\ &\cong L^{\otimes q^n} \otimes_R N_m^{(n)} \otimes_{A^{1/q^m}} M^{1/q^{m+1}}, \end{aligned}$$

where $(M^{1/q^m})^{(n)} := (M^{1/q^m})^{\otimes_{A^{1/q^m}} \frac{q^n - 1}{q - 1}}$. We define the $B_m^{1/q^{n+1}}$ -module homomorphism $f_n : (N_{m+1}^{(n+1)})^{1/q^{n+1}} \rightarrow (L \otimes_R (N_m^{(n)})^{1/q^n})^{1/q}$ as the composite map

$$\begin{aligned} f_n : (N_{m+1}^{(n+1)})^{1/q^{n+1}} &\xrightarrow{(\text{id} \otimes \text{id} \otimes \Psi^{1/q^m})^{1/q^{n+1}}} (L^{\otimes q^n} \otimes_R N_m^{(n)})^{1/q^{n+1}} \\ &\xrightarrow{\sim} (L \otimes_R (N_m^{(n)})^{1/q^n})^{1/q}. \end{aligned}$$

Then, the following diagram commutes.

$$\begin{array}{ccc}
 (N_{m+1}^{(n+1)})^{1/q^{n+1}} & \xrightarrow{f_n} & (L \otimes_R (N_m^{(n)})^{1/q^n})^{1/q} \\
 \downarrow \gamma_{m+1} & & \downarrow (\text{id} \otimes \gamma_m)^{1/q} \\
 (N_{m+1}^{(n)})^{1/q^n} & \xrightarrow{f_{n-1}} & (L \otimes_R (N_m^{(n-1)})^{1/q^{n-1}})^{1/q} \\
 \downarrow \gamma_{m+1} & & \downarrow (\text{id} \otimes \gamma_m)^{1/q} \\
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 (N_{m+1})^{1/q} & \xrightarrow{f_0} & (L \otimes_R B_m)^{1/q} \\
 \downarrow \gamma_{m+1} & \swarrow \tilde{\varphi}_{m+1} & \\
 B_{m+1} & &
 \end{array}$$

Since A is regular, the morphism Ψ is surjective, as is f_n . Therefore,

$$\tilde{\varphi}_{m+1}((L \otimes_R \gamma_m^n((\mathfrak{a}^{[q^n \lambda]} IN_m^{(n)})^{1/q^n}))^{1/q}) = \gamma_{m+1}^{n+1}((\mathfrak{a}^{[q^n \lambda]} IN_{m+1}^{(n+1)})^{1/q^{n+1}}),$$

which proves the assertion. \square

Proposition A.13 (cf. [29, Theorem 4.13]). *With the same notation as above, we assume further that there exists an integer $l > 0$ such that $q^l(q-1)\lambda$ is an integer. Then, there exist a dense open subset $U \subseteq V$ and an integer $n_1 \geq 1$ satisfying the following: for any integer $m \geq n_1$ and any morphism $V' \rightarrow V$ with an image in U , where $V' = \text{Spec } A'$ is an F -finite regular integral affine scheme with a dualizing complex $\omega_{V'}^\bullet$, such that $F^1 \omega_{V'}^\bullet \cong \omega_{V'}^\bullet$, and $X' := X \times_V V'$ is an integral scheme, we have*

$$\tau_m(X'/V', \varphi' I', \mathfrak{a}'^\lambda) = \tau(X'_{V' \text{ em}}, \gamma'_m(I' B'_m), (\mathfrak{a}' B'_m)^\lambda).$$

Here, $\gamma'_m : (N'_m)^{1/q} := (L' \otimes_{A'} A'((1-q)K_{V'})^{1/q^m})^{1/q} \rightarrow B'_m$ denotes the B'_m -module homomorphism induced by $\varphi' : (L')^{1/q} \rightarrow B'_1$.

Proof. By Lemmas A.6 and A.12, the problem is reduced to the case where $(q-1)\lambda$ is an integer and $\lambda > \mu(\mathfrak{a}) - 1$. In this case, by taking U and n_0 as in Proposition A.11, we see that $n_1 := n_0$ satisfies the assertion by an argument similar to that in the proof of Proposition A.8. The reader is referred to the proof of [29, Theorem 4.13] for more details. \square

Theorem A.14 (cf. [29, Corollary 4.15]). *Suppose that V is an F -finite regular integral scheme of characteristic $p > 0$, $f : X \rightarrow V$ is a normal morphism that is essentially of finite type from a normal integral scheme X , and Δ is an effective \mathbb{Q} -Weil divisor on X such that $K_X + \Delta$ is \mathbb{Q} -Cartier with an index that is not divisible by p . Let $\mathfrak{a} \subseteq \mathcal{O}_X$ be a nonzero coherent ideal sheaf and $\lambda > 0$ be a rational number. Then, there exist a dense open subset $U \subseteq V$ and an integer $n_2 \geq 1$ such that for every positive multiple n of n_2 and every perfect point $u : \text{Spec } k \rightarrow U$ of U , one has*

$$\tau(X_{V^n}, h^* \Delta, (\mathfrak{a} \mathcal{O}_{X_{V^n}})^\lambda) \mathcal{O}_{X_u} = \tau(X_u, \Delta|_{X_u}, (\mathfrak{a} \mathcal{O}_{X_u})^\lambda),$$

where $h^*\Delta$ is the pullback of Δ by the projection $h : X_{V^n} := X \times_V V^n \rightarrow X$.

Proof. After shrinking V if necessary, we may assume that Δ does not contain any fiber of f in its support and that V is affine with a dualizing complex ω_V^\bullet such that $F^!\omega_V^\bullet \cong \omega_V^\bullet$. Taking an affine open covering of X , we may assume that X is also affine. We fix a relative canonical divisor $K_{X/V}$ on X . Since V is Gorenstein, it follows from [40, Lemma 0BZL] that $K_{X/V}$ is a canonical divisor of X , and in particular, by [14, V. Theorem 3.1], the \mathbb{Q} -Weil divisor $K_{X/V} + \Delta$ is \mathbb{Q} -Cartier with an index that is not divisible by p . We choose a power q of p such that $(q-1)(K_{X/V} + \Delta)$ is Cartier and $q^l(q-1)\lambda$ is an integer for some $l \geq 1$. The assertion then follows from Proposition A.13, Lemma A.10, and Example A.3. The reader should compare this proof with that of [29, Corollary 4.15]. \square

REFERENCES

- [1] M. André, Localisation de la lissité formelle, *Manuscripta Math.* **13** (1974), 297–307.
- [2] Y. Aoyama, Some basic results on canonical modules, *J. Math. Kyoto. Univ.* **23** (1983), 85–94.
- [3] M. Blickle, K. Schwede, S. Takagi and W. Zhang, Discreteness and rationality of F -jumping numbers on singular varieties, *Math. Ann.* **347** (2010), 917–949.
- [4] A. F. Boix, D. J. Hernández, Z. Kadyrsizova, M. Katzman, S. Malec, M. Robinson, K. Schwede, D. Smolkin, P. Teixeira, E. E. Witt, The Testideals package for Macaulay2, *J. Softw. Algebra Geom.* **9** (2019), no. 2, 89–110.
- [5] A. Grothendieck, J.A. Dieudonné, *Eléments de Géométrie Algébrique IV*, *Publ. Math. IHÉS* **28** (1966), 5–255.
- [6] H. Esnault and E. Viehweg, Two dimensional quotient singularities deform to quotient singularities, *Math. Ann.* **271** (1985), no. 3, 439–449.
- [7] T. de Fernex and C. Hacon, Singularities on normal varieties, *Compos. Math.* **145** (2009), no. 2, 393–414.
- [8] A. De Stefani and I. Smirnov, Stability and deformation of F -singularities, arXiv: 2002.00242, preprint (2020).
- [9] R. Elkik, Singularités rationnelles et déformations, *Invent. Math.* **47** (1978), no. 2, 139–147.
- [10] R. Fedder and K.-i. Watanabe, A characterization of F -regularity in terms of F -purity, *Commutative algebra (Berkeley, CA, 1987)*, 227–245, *Math. Sci. Res. Inst. Publ.*, 15, Springer, New York, 1989.
- [11] N. Hara and K.-i. Watanabe, F -regular and F -pure rings vs. log terminal and log canonical singularities, *J. Algebraic Geom.* **11** (2002), no. 2, 363–392.
- [12] N. Hara, K.-i. Watanabe and K.-i. Yoshida, Rees algebras of F -regular type, *J. Algebra* **247** (2002), no. 1, 191–218.
- [13] N. Hara and K.-i. Yoshida, A generalization of tight closure and multiplier ideals, *Trans. Amer. Math. Soc.* **355** (2003), no. 8, 3143–3174.
- [14] R. Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, With an appendix by P. Deligne, *Lecture Notes in Mathematics*, No.20 Springer-Verlag, Berlin-New York 1966.
- [15] M. Hochster and C. Huneke, Tight closure and strong F -regularity, *Mém. Soc. Math. France (N.S.)* (1989), no. 38, 119–133, *Colloque en l’honneur de Pierre Samuel (Orsay, 1987)*.
- [16] M. Hochster and C. Huneke, F -Regularity, Test Elements, and Smooth Base Change, *Trans. Amer. Math. Soc.* **346** (1994), no. 1, 1–62.
- [17] S. Ishii, *Introduction to singularities*, Springer, Japan, 2018.
- [18] M. Kawakita, Inversion of adjunction on log canonicity, *Invent. Math.* **167** (2006) no. 1, 129–133.

- [19] J. Kollár, Singularities of pairs, in *Algebraic Geometry, Santa Cruz 1995*, Proceedings of Symposia in Pure Mathematics, Vol. **62**, Amer. Math. Soc., 1997, 221–286.
- [20] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, 134, Cambridge University Press, Cambridge, 1998.
- [21] R. Lazarsfeld, *Positivity in Algebraic Geometry II*, *Ergeb. Math. Grenzgeb. 3. Folge, A Series of Modern Surveys in Mathematics*, 49, Springer-Verlag, Berlin, 2004.
- [22] C. Liedtke, G. Martin and Y. Matsumoto, Linearly reductive quotient singularities, arXiv:2102.01067, preprint (2021).
- [23] J. Lipman, Rational singularities with applications to algebraic surfaces and unique factorization, *Publ. Math. IHÉS*, **36** (1969), 195–279.
- [24] J. Lipman and B. Teissier, Pseudorational local rings and a theorem of Briançon-Skoda about integral closures of ideals, *Michigan Math. J.* **28** (1981), no. 1, 97–116.
- [25] L. Ma and K. Schwede, Singularities in mixed characteristic via perfectoid big Cohen-Macaulay algebras, *Duke Math. J.* **170** (2021), no. 13, 2815–2890.
- [26] L. Ma, K. Schwede, K. Tucker, J. Waldron and J. Witaszek, An analog of adjoint ideals and plt singularities in mixed characteristic, *J. Algebr. Geom.* **31** (2022), no. 3, 497–559.
- [27] H. Matsumura, *Commutative ring theory*, Translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics, **8**, Cambridge University Press, Cambridge, 1989.
- [28] T. Murayama, Relative vanishing theorems for \mathbf{Q} -schemes, arXiv:2101.10397, preprint (2021).
- [29] Z. Patakfalvi, K. Schwede and W. Zhang, F -singularities in families, *Algebr. Geom.* **5** (2018), no. 3, 264–327.
- [30] K. Sato, Stability of test ideals of divisors with small multiplicity, *Math. Z.* **288** (2018), 783–802.
- [31] K. Sato, Ascending chain condition for F -pure thresholds on a fixed strongly F -regular germ, *Compos. Math.* **155** (2019), no. 6, 1194–1223.
- [32] K. Schwede, Generalized test ideals, sharp F -purity, and sharp test elements, *Math. Res. Lett.* **15** (2008), no. 6, 1251–1261.
- [33] K. Schwede, F -adjunction, *Algebra Number Theory* **3** (2009), no. 8, 907–950.
- [34] K. Schwede, Centers of F -purity, *Math. Z.* **265** (2010), 687–714.
- [35] K. Schwede, Test ideals in non- \mathbf{Q} -Gorenstein rings, *Trans. Amer. Math. Soc.* **363** (2011), no. 11, 5925–5941.
- [36] K. Schwede and K. E. Smith, Globally F -regular and log Fano varieties, *Adv. Math.* **224** (2010), no. 3, 863–894.
- [37] A. K. Singh, F -regularity does not deform, *Amer. J. Math.* **121** (1999), no. 4, 919–929.
- [38] K. E. Smith, F -rational rings have rational singularities, *Amer. J. Math.* **119** (1997), no. 1, 159–180.
- [39] K. E. Smith, Globally F -regular varieties: applications to vanishing theorems for quotients of Fano varieties, *Michigan Math. J.* **48** (2000), 553–572.
- [40] T. Stacks Project Authors, The Stacks Project.
- [41] S. Takagi, An interpretation of multiplier ideals via tight closure, *J. Algebraic Geom.* **13** (2004), no. 2, 393–415.
- [42] S. Takagi, F -singularities of pairs and Inversion of Adjunction of arbitrary codimension, *Invent. Math.* **157** (2004), no. 1, 123–146.
- [43] K.-i. Watanabe, F -regular and F -pure normal graded rings, *J. Pure Appl. Algebra* **71** (1991), no. 2–3, 341–350.
- [44] Z. Zhu, Log canonical thresholds in positive characteristic, *Math. Z.* **287** (2017), no. 3–4, 1235–1253.

FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, 744 MOTOOKA, NISHI-KU, FUKUOKA 819-0395, JAPAN

Email address: `ksato@math.kyushu-u.ac.jp`

GRADUATE SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF TOKYO, 3-8-1 KOMABA, MEGURO-KU, TOKYO 153-8914, JAPAN

Email address: `stakagi@ms.u-tokyo.ac.jp`