SOBOLEV $\frac{1}{2}$ ESTIMATE FOR $\overline{\partial}$ EQUATION ON STRICTLY PSEUDOCONVEX DOMAINS WITH C^2 BOUNDARY

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ABSTRACT. We construct a solution operator for $\overline{\partial}$ equation that gains $\frac{1}{2}$ derivative in the fractional Sobolev space $H^{s,p}$ on bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with C^2 boundary, for all $1 < p < \infty$ and $s > \frac{1}{p}$.

CONTENTS

1. INTRODUCTION

The main result of the paper is the following:

Theorem 1.1. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with C^2 boundary, for $n \geq 2$. Suppose φ is a $\overline{\partial}$-closed $(0,q)$ form in Ω where $q \geq 1$. If $\varphi \in H^{s,p}(\Omega)$, for $1 < p < \infty$ and $s > \frac{1}{p}$, then there exists a $(0,q-1)$ form u that solves the equation $\overline{\partial} u = \varphi$, such that $u \in H^{s+\frac{1}{2},p}(\Omega)$. Here $H^{s,p}(\Omega)$ is the fractional Sobolev space on Ω (see Definition 2.6).*

We also prove an analogous result when φ is in Hölder-Zygmund space $\Lambda^r(\Omega)$ which improves an earlier result of Gong [Gon19].

Theorem 1.2. *Keep the assumptions of the above theorem. Suppose $\varphi \in \Lambda^r(\Omega)$, where $r > 0$. Then there exists a solution u for $\overline{\partial} u = \varphi$ such that $u \in \Lambda^{r+\frac{1}{2}}(\Omega)$. Here $\Lambda^r(\Omega)$ is the Hölder-Zygmund space on Ω.*

A domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is called strictly pseudoconvex if it admits a C^2 real-valued defining function ρ whose Levi-form along $b\Omega$ is positive definite in the complex tangent space, i.e. there is a $c > 0$ such that $\sum \frac{\partial^2 \rho}{\partial z_\alpha \partial \overline{z}_\beta}(p) t_\alpha \overline{t_\beta} \geq c|t|^2$ for all $t \in \mathbb{C}^n$ satisfying $\partial\rho(t) = 0$.

It is well-known that on a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with sufficiently smooth boundary, there exist solutions u to the equation $\overline{\partial} u = \varphi$ which gains $1/2$ derivative up to boundary if φ belongs to some suitable space. By restricting to C^2 boundary, our results establish the

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1/2 estimate for $\overline{\partial}$ equation with the *minimum* smoothness requirement in the context of above definition.

In the category of the L^2 Sobolev space (denoted as $H^{s,2}$), one can obtain a solution in the form $\overline{\partial}^* \mathcal{N}\varphi$ where $\mathcal{N}$ is the solution operator for the associated $\overline{\partial}$-Neumann boundary value problem, and $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ in the L^2 Hilbert space. When the boundary $b\Omega$ is C^∞, Kohn in his famous work [Koh63] showed that the solution $\overline{\partial}^* \mathcal{N}\varphi$ is in $H^{s+\frac{1}{2},2}(\Omega)$ if $\varphi \in H^{s,2}(\Omega)$ for any $s \geq 0$. See [CS01, Cor.4.4.2, Thm 5.2.6].

Later on Greiner and Stein [GS77, p. 174], proved that for any $(0,1)$ form $\varphi \in H^{k,p}(\Omega)$ where k is a non-negative integer and $1 < p < \infty$, $\overline{\partial}^* \mathcal{N}\varphi \in H^{k+\frac{1}{2},p}(\Omega)$. Their results were later extended by Chang [Cha89] to any $(0,q)$ forms φ for $1 \leq q \leq n$. Similarly one can obtain a gain of $\frac{1}{2}$ derivative for the operator $\overline{\partial}^* \mathcal{N}$ in the Hölder-Zygmund space Λ^r, for all $r > 0$. See [GS77, p. 174]. All of these results require that $b\Omega \in C^\infty$.

Besides the $\overline{\partial}$-Neumann approach, one can also solve the $\overline{\partial}$ equation on strictly pseudoconvex domains using integral formula with certain "holomorphic like" kernels. The solutions obtained through this method are no longer L^2 canonical, but have the advantage that boundary no longer needs to be C^∞. In this direction, Henkin and Ramanov in [RH71] first constructed a solution which is in $C^{\frac{1}{2}}(\overline{\Omega})$ if φ is a $(0,1)$ form in the class $C^0(\overline{\Omega})$ and $b\Omega \in C^2$. Later Siu [Siu74] and Lieb-Range [LR80] found solutions that are in $C^{k+\frac{1}{2}}(\overline{\Omega})$ for φ in $C^k(\overline{\Omega})$, if the boundary is C^{k+2} and k is a positive integer. The requirement on the smoothess of boundary is a result of using integration by parts on certain boundary integral. It is also important to point out that in both papers, the estimates for the solution operators rely on the fact that φ is $\overline{\partial}$ closed.

More recently, Gong [Gon19] used the integral formula method to construct a $\overline{\partial}$ solution operator for any C^2 strictly pseudoconvex domains, and the solution u lies in $\Lambda^{r+\frac{1}{2}}(\Omega)$ if φ is any $(0,q)$ form $(q \geq 1)$ in the class $\Lambda^r(\Omega)$, for all $r > 1$.

In our paper we give a variant of Gong's solution operator which allows one to work on Sobolev spaces when the boundary is C^2. Furthermore our operator allows us to obtain $\frac{1}{2}$ estimate when the right-hand side is $\Lambda^r(\Omega)$, for all $r > 0$, which improves the above result of Gong. See also [Shi21] for estimates on a certain class of weighted Sobolev space.

Here is the outline of the paper: In Section 2 we review the definition and properties of the function spaces we are using. To do estimates we need a characterization of the Sobolev space by Littlewood-Paley theory. We also include some results on interpolation which will be used extensively in our proofs. In Section 3 we recall Rychkov's universal extension operator E_ω on a special Lipschitz domain ω, whose boundary is the graph of a Lipschitz function. Section 4 contains the most technical part of the paper. We show that the commutator $[D, E_\omega] = DE_\omega - E_\omega D$ maps $H^{s,p}(\omega)$ into $L^p(\omega, \lambda)$, where the weight λ is some power of the distance-to-boundary function. In Section 5 we prove various results on the embedding of weighted Sobolev spaces $W^{k,p}(\Omega, \lambda)$ to $H^{s,p}(\Omega)$ spaces. Much of the results in this section are probably not new and the procedures are quite routine, although we are unable to find references for the actual results. Section 6 and Section 7 contain the estimates for the homotopy operators which lead to the proof of Theorem 1.1 and Theorem 1.2. The main novelty here is the introduction of a weight factor which seems necessary to prove the relevant estimates. We mention that the commutator was first introduced by Peters [Pet91] and have been used by Michel [Mic91], Michel-Shaw [MS99] among others.

Throughout the paper we assume that all the domains are in $\mathbb{C}^n$ for $n \geq 2$. We denote the set of non-negative integers by $\mathbb{N}$, and the set of positive integers by $\mathbb{Z}^+$. For a set $\Omega \subset \mathbb{R}^N$ we denote $\Omega^c = \mathbb{R}^N \backslash \Omega$. We will use the notation $x \lesssim y$ to mean that $x \leq Cy$ where C is a constant independent of x, y, and $x \approx y$ for "$x \lesssim y$ and $y \lesssim x$". For the unit ball in $\mathbb{R}^N$ we use $\mathbb{B}^N$.

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2. Function Spaces and Interpolation

In this section we review some basic facts about function spaces.

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, and let $k \in \mathbb{N}$, $1 \leq p < \infty$. We denote by $W^{k,p}(\Omega)$ the space of (complex-valued) functions $f \in L^p(\Omega)$ such that $D^\alpha f \in L^p(\Omega)$ for all $|\alpha| \leq k$, with the norm

$$\|f\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)} = \sum_{|\alpha| \leq k} \left(\int_\Omega |D^\alpha f(x)|^p \right)^{\frac{1}{p}} dV(x), \quad 1 \leq p < \infty.$$

Let λ be a positive continuous function on Ω. We define the weighted Sobolev space $W^{k,p}_\lambda(\Omega)$ as the space of f in $W^{k,p}_{\text{loc}}(\Omega)$ such that the following norm is finite:

$$\|f\|_{W^{k,p}(\Omega;\lambda)} := \sum_{|\alpha| \leq k} \|\lambda D^\alpha f\|_{L^p(\Omega)} = \sum_{|\alpha| \leq k} \left(\int_\Omega |D^\alpha f(x)|^p \lambda(x)^p \, dV(x) \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

If Ω is a domain in $\mathbb{C}^n$ with complex variable z, we write instead $\lambda(z)^p dV(z)$.

In our application we will take $\lambda(x) = \operatorname{dist}(x, b\Omega)^s$ for some $s \in \mathbb{R}$.

We shall use $\mathscr{S}(\mathbb{R}^N)$ to denote the space of Schwartz functions, and $\mathscr{S}'(\mathbb{R}^N)$ for the space of tempered distributions.

Definition 2.2. A *special Lipschitz domain* is an open set $\omega \subset \mathbb{R}^N$ of the form $\omega = \{(x', x_N) : x_N > \rho(x')\}$ with $\|\nabla \rho\|_{L^\infty} < 1$. A *bounded Lipschitz* domain is a bounded open set Ω whose boundary is locally the graph of some Lipschitz function. In other words, $b\Omega = \bigcup_{\nu=1}^M U_\nu$, where for each $1 \leq \nu \leq M$, there exists an invertible linear transformation $\Phi_\nu : \mathbb{R}^d \to \mathbb{R}^d$ and a special Lipschitz domain ω_ν such that

$$U_\nu \cap \Omega = U_\nu \cap \Phi_\nu(\omega_\nu).$$

Definition 2.3 (Hölder-Zygmund)**.** Let $U \subseteq \mathbb{R}^N$ be an open subset. We define the Hölder-Zygmund space $\Lambda^s(U)$ for $s > 0$ by the following:

- For $0 < s < 1$, $\Lambda^s(U)$ consists of all $f \in C^0(U)$ such that $\|f\|_{\Lambda^s(U)} := \sup_U |f| + \sup_{x,y \in U} \frac{|f(x)-f(y)|}{|x-y|^s} < \infty$.
- $\Lambda^1(U)$ consists of all $f \in C^0(U)$ such that $\|f\|_{\Lambda^1(U)} := \sup_U |f| + \sup_{x,y \in U; \frac{x+y}{2} \in U} \frac{|f(x)+f(y)-2f(\frac{x+y}{2})|}{|x-y|} < \infty$.
- For $s > 1$, $\Lambda^s(U)$ consists of all $f \in \Lambda^{s-1}(U)$ such that $\nabla f \in \Lambda^{s-1}(U)$. We define $\|f\|_{\Lambda^s(U)} := \|f\|_{\Lambda^{s-1}(U)} + \sum_{j=1}^N \|D_j f\|_{\Lambda^{s-1}(U)}$.

Definition 2.4. Let $s \in \mathbb{R}$, $1 < p < \infty$. We define $H^{s,p}(\mathbb{R}^N)$ to be the fractional Sobolev space consisting of all (complex-valued) tempered distributions $f \in \mathscr{S}'(\mathbb{R}^N)$ such that $(I - \Delta)^{\frac{s}{2}} f \in L^p(\mathbb{R}^N)$, with norm

$$\|f\|_{H^{s,p}(\mathbb{R}^N)} := \|(I - \Delta)^{\frac{s}{2}} f\|_{L^p(\mathbb{R}^N)},$$

where $(I - \Delta)^{\frac{s}{2}} f$ is given by

$$(I - \Delta)^{\frac{s}{2}} f = ((1 + 4\pi|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi))^\vee.$$

Here for a Schwartz function g we set the Fourier transform $\widehat{g}(\xi) = \int_{\mathbb{R}^N} g(x) e^{-2\pi i x \cdot \xi} dx$, and the definition extends naturally to tempered distributions.

Remark 2.5. The Sobolev space $H^{s,p}(\mathbb{R}^N)$ defined above is sometimes called the Bessel potential space. There is another type of commonly-used fractional Sobolev spaces called *Sobolev-Slobodeckij spaces*, which is also known as the Besov spaces $\mathscr{B}^s_{pp}(\mathbb{R}^N)$, see [DNPV12] for example. We will not use this type of space in our paper with the exception of $\mathscr{B}^s_{\infty,\infty}$, which agrees with the Hölder-Zygmund space Λ^s.

We also have the following definition for functions and distributions defined on open sets of $\mathbb{R}^N$:

Definition 2.6. Let $\Omega \subseteq \mathbb{R}^N$ be an open set.

(i) Define $\mathscr{S}'(\Omega) = \{\tilde{f}|_\Omega : \tilde{f} \in \mathscr{S}'(\mathbb{R}^N)\}$.

(ii) For $s \in \mathbb{R}$ and $1 < p < \infty$, define $H^{s,p}(\Omega) = \{\tilde{f}|_\Omega : \tilde{f} \in H^{s,p}(\mathbb{R}^N)\}$ with norm

$$\|f\|_{H^{s,p}(\Omega)} = \inf_{\tilde{f}|_\Omega = f} \|\tilde{f}\|_{H^{s,p}(\mathbb{R}^N)}.$$

(iii) For $s \in \mathbb{R}$ and $1 < p < \infty$, define $H^{s,p}_0(\Omega)$ to be the subspace of $H^{s,p}(\mathbb{R}^N)$ which is the completion of $C_c^\infty(\Omega)$ under the norm $\|\cdot\|_{H^{s,p}(\mathbb{R}^N)}$. We will write $\|g\|_{H^{s,p}_0(\Omega)} = \|g\|_{H^{s,p}(\mathbb{R}^N)}$ if $g \in H^{s,p}_0(\Omega)$.

Remark 2.7. In our paper, $H^{s,p}_0(\Omega)$ is defined to be the closed subspace of $H^{s,p}(\mathbb{R}^N)$, which is different from some other literature. For example in [Tri06, Definition 1.95(ii)] Triebel defines the space $\mathring{H}^{s,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^{s,p}(\Omega)}}$, which is a subspace of $H^{s,p}(\Omega)$.

Nevertheless, when $s > \frac{1}{p} - 1$ and Ω is a bounded Lipschitz domain, we have $\mathring{H}^{s,p}(\Omega) = H^{s,p}_0(\Omega)$, in the sense that the natural map $H^{s,p}_0(\Omega) \to \mathring{H}^{s,p}(\Omega)$ induced by the restriction map $[\tilde{f} \mapsto \tilde{f}|_\Omega] : H^{s,p}(\mathbb{R}^N) \to H^{s,p}(\Omega)$ is a bijection, see equation (2.3) below.

For $f \in H^{s,p}_0(\Omega)$, we write $\|f\|_{H^{s,p}_0(\Omega)} = \|f\|_{H^{s,p}(\Omega)}$.

Remark 2.8. For $1 < p < \infty$ and $s \in \mathbb{R}$, the Bessel-Sobolev space $H^{s,p}(\mathbb{R}^N)$ is in fact a special case of the Triebel-Lizorkin space $\mathscr{F}^s_{p2}(\mathbb{R}^N)$ with equivalent norm. See [Tri83, Definition 2.3.1/2 and Theorem 2.5.6(i)]. More precisely we have the following:

Proposition 2.9 (Littlewood-Paley Theorem)**.** *Let $\phi_0 \in \mathscr{S}(\mathbb{R}^N)$ be a Schwartz function whose Fourier transform satisfies*

$$\operatorname{supp}\widehat{\phi}_0 \subseteq \{|\xi| < 2\}, \qquad \widehat{\phi}_0|_{\{|\xi|\le 1\}} \equiv 1, \qquad 0 \le \widehat{\phi}_0 \le 1.$$

For $j \ge 1$, let ϕ_j be the Schwartz function whose Fourier transform is $\widehat{\phi}_0(2^{-j}\xi) - \widehat{\phi}_0(2^{-(j-1)}\xi)$. Then for $s \in \mathbb{R}$ and $1 < p < \infty$, there exists a $C = C_{\phi_0,p,s} > 0$ such that

$$C^{-1}\|f\|_{H^{s,p}(\mathbb{R}^N)} \le \left(\int_{\mathbb{R}^N} \Big(\sum_{j=0}^\infty 2^{2js}|\phi_j * f(x)|^2\Big)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \le C\|f\|_{H^{s,p}(\mathbb{R}^N)}, \quad \forall f \in \mathscr{S}'(\mathbb{R}^N), \tag{2.1}$$

provided that either term in the inequality is finite.

Following the notation from [Tri83, Section 2.3.1], we denote the middle term in (2.1) by $\|f\|_{\mathscr{F}^s_{p2}(\mathbb{R}^N;\phi)}$, which is a Triebel-Lizorkin norm on $\mathbb{R}^N$.

By way of Definition 2.6, one can also define for an arbitrary open set $\Omega \subseteq \mathbb{R}^N$ the space $\mathscr{F}^s_{p2}(\Omega) = \{\tilde{f}|_\Omega : \tilde{f} \in \mathscr{F}^s_{p2}(\mathbb{R}^N)\}$ equipped with the norm $\|f\|_{\mathscr{F}^s_{p2}(\Omega)} = \inf_{\tilde{f}|_\Omega = f} \|\tilde{f}\|_{\mathscr{F}^s_{p2}(\mathbb{R}^N)}$ (see [Tri06, Definition 1.95(i)]. It follows that $H^{s,p}(\Omega) = \mathscr{F}^s_{p2}(\Omega)$.

In the special case that s is a non-negative integer and $1 < p < \infty$, $H^{s,p}$ becomes the familiar Sobolev space $W^{k,p}$.

Lemma 2.10. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then*

(i) $H^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$ with equivalent norm.

(ii) *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then $W^{k,p}(\Omega) = H^{k,p}(\Omega)$ where the norms are equivalent.*

Proof. The proof of (i) can be found in [Tri83, Theorem 2.5.6(ii)].

For (ii), see [Tri06, Theorem 1.222(i)]. Notice that we have $H^{k,p}(\Omega) = \mathscr{F}^k_{p2}(\Omega)$ as discussed above. □

Remark 2.11. As explained in the proof of [Tri06, Theorem 1.222(i)], the key to the proof of (ii) is the use of an extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)$. In our paper we need to use a different extension operator that have some nicer properties.

Proposition 2.12. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain. Suppose $1 < p < \infty$ and $s \in \mathbb{R}$. Then we have the following equalities of spaces, where the norms are equivalent.*

(i) $H^{s,p}(\mathbb{R}^N) = H^{-s,p'}(\mathbb{R}^N)'$, *where* $p' = \frac{p}{p-1}$.

(ii) *For* $s > \frac{1}{p} - 1$, $H^{s,p}_0(\Omega) = \{f \in H^{s,p}(\mathbb{R}^N) : f|_{\overline{\Omega}^c} = 0\}$.

(iii) $H^{s,p}_0(\Omega) = H^{-s,p'}(\Omega)'$ *and* $H^{-s,p'}(\Omega) = H^{s,p}_0(\Omega)'$, *provided that* $s > \frac{1}{p} - 1$.

Proof. For proof of (i) see [Tri95, Theorem 2.6.1(a)].

The proof of (ii) and (iii) are the combination of several results in [Tri02]. We now offer some explanations. Recall in Remark 2.8 we can use $H^{s,p} = \mathscr{F}^s_{p2}$ for $s \in \mathbb{R}$ and $1 < p < \infty$.

In [Tri02, Section 3.2], Triebel defines $\widetilde{H}^{s,p}(\overline{\Omega}) \subseteq H^{s,p}(\mathbb{R}^N)$ and $\widetilde{H}^{s,p}(\Omega) \subseteq H^{s,p}(\Omega)$ as

$$\widetilde{H}^{s,p}(\overline{\Omega}) := \{f \in H^{s,p}(\mathbb{R}^N) : \operatorname{supp} f \subseteq \overline{\Omega}\}, \quad \widetilde{H}^{s,p}(\Omega) := \{f|_\Omega : f \in \widetilde{H}^{s,p}(\overline{\Omega})\}. \tag{2.2}$$

So $\{f \in H^{s,p}(\mathbb{R}^N) : f|_{\overline{\Omega}^c} = 0\} = \widetilde{H}^{s,p}(\overline{\Omega})$.

Clearly $\widetilde{H}^{s,p}(\overline{\Omega})$ (resp. $\widetilde{H}^{s,p}(\Omega)$) is a closed subspace of $H^{s,p}(\mathbb{R}^N)$ (resp. $H^{s,p}(\Omega)$), and we have a surjective restriction map $[f \mapsto f|_\Omega] : \widetilde{H}^{s,p}(\overline{\Omega}) \to \widetilde{H}^{s,p}(\Omega)$.

When $s > \frac{1}{p} - 1$, by [Tri02, Proposition 3.1] we have $\widetilde{H}^{s,p}(\overline{\Omega}) = \widetilde{H}^{s,p}(\Omega)$ in the sense that the restriction map $f \mapsto f|_\Omega$ is bijective.

Recall that by definition $H^{s,p}_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{H^{s,p}(\mathbb{R}^N)}}$ is a closed subspace of $H^{s,p}(\mathbb{R}^N)$. Also observe that $H^{s,p}_0(\Omega) \subseteq \widetilde{H}^{s,p}(\overline{\Omega})$, since if $f = \lim_{j\to\infty} f_j$ and $\operatorname{supp} f_j \subseteq \Omega$, then $\operatorname{supp} f \subseteq \overline{\Omega}$. Thus $H^{s,p}_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{\widetilde{H}^{s,p}(\overline{\Omega})}}$.

By [Tri02, Theorem 3.5(i)], for $s > \frac{1}{p} - 1$, $C^\infty_c(\Omega)$ is dense in $\widetilde{H}^{s,p}(\Omega)$. Hence by using the identification $\widetilde{H}^{s,p}(\overline{\Omega}) = \widetilde{H}^{s,p}(\Omega)$, we get for $s > \frac{1}{p} - 1$,

$$H^{s,p}_0(\Omega) = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{\widetilde{H}^{s,p}(\overline{\Omega})}} = \overline{C^\infty_c(\Omega)}^{\|\cdot\|_{\widetilde{H}^{s,p}(\Omega)}} = \widetilde{H}^{s,p}(\Omega). \tag{2.3}$$

Using $\widetilde{H}^{s,p}(\overline{\Omega}) = \widetilde{H}^{s,p}(\Omega)$ we obtain $H^{s,p}_0(\Omega) = \{f \in H^{s,p}(\mathbb{R}^N) : f|_{\overline{\Omega}^c} = 0\}$, which proves (ii).

By [Tri02, Definition 3.3 and (43)], we have duality $H^{-s,p'}(\Omega) = \widetilde{H}^{s,p}(\Omega)'$ and $\widetilde{H}^{s,p}(\Omega) = H^{-s,p'}(\Omega)'$ when $s > \frac{1}{p} - 1$, where the norms are equivalent. Using the identification $\widetilde{H}^{s,p}(\Omega) = H^{s,p}_0(\Omega)$, we get $H^{s,p}_0(\Omega) = H^{-s,p'}(\Omega)'$ and $H^{-s,p'}(\Omega) = H^{s,p}_0(\Omega)'$ for the given range of s, proving (iii). □

We also need some interpolations results.

Definition 2.13. Let X_0, X_1 be two Banach spaces that belong to a larger ambient space. For $0 < \theta < 1$. The *complex interpolation space* $[X_0, X_1]_\theta$ is defined to be the space consisting of all $f(\theta) \in X_0 + X_1$, where $f : \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\} \to X_0 + X_1$ is a continuous map that is analytic in the interior, such that $f(it) \in X_0$ and $f(1 + it) \in X_1$ for all $t \in \mathbb{R}$. The norm is given by

$$\|u\|_{[X_0,X_1]_\theta} = \inf_f \{\sup_{t\in\mathbb{R}} (\|f(it)\|_{X_0} + \|f(1 + it)\|_{X_1}) : u = f(\theta)\}.$$

Proposition 2.14 (Complex interpolation theorem). *Let X_0, X_1, Y_0, Y_1 be Banach spaces that belong to some larger ambient spaces. Suppose $T : X_0 + X_1 \to Y_0 + Y_1$ is a linear operator such that for each $i = 0, 1$, $\|Tu\|_{Y_i} \le C_0 \|u\|_{X_i}$ for all $u \in X_i$. Then $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is bounded linear with $\|Tu\|_{[Y_0,Y_1]_\theta} \le C_0^{1-\theta} C_1^\theta \|u\|_{[X_0,X_1]_\theta}$ for all $u \in [X_0, X_1]_\theta$.*

See [Tri95, Theorem 1.9.3(a) and Definition 1.2.2/2].

We also have the following facts:

Proposition 2.15. *Let Ω be an open set of $\mathbb{R}^N$. Let $1 < p < \infty$ and $s_0, s_1 \in \mathbb{R}$. Denote $\delta(x) := \operatorname{dist}(x, b\Omega)$ and set $s_\theta := \theta s_1 + (1-\theta)s_0$ for $0 < \theta < 1$. Then the following hold:*

(i) $[L^p(\Omega, \delta^{s_0}), L^p(\Omega, \delta^{s_1})]_\theta = L^p(\Omega, \delta^{s_\theta})$.

(ii) $[H^{s_0,p}(\Omega), H^{s_1,p}(\Omega)]_\theta = H^{s_\theta,p}(\Omega)$, *provided that Ω is a bounded Lipschitz domain.*

The proof of (i) can be found in [Tri95, Theorem 1.18.5]. The proof of (ii) can be found in [Tri06, Corollary 1.111 (1.372)].

3. The Universal Extension operator

In this section we recall the construction of the universal extension operator by Rychkov [Ryc99]. None of the results here is new, although we shall present the proof in a slightly different way from that of Rychkov.

In the rest of the paper we shall denote by $\mathbb{K}$ the positive cone in $\mathbb{R}^N$:

$$\mathbb{K} = \{(x', x_N) : x_N > |x'|\}.$$

Remark 3.1. In many literature, for example [Tri06, Section 1.11.4 (1.322) p. 63], the definition for a special Lipschitz domain only requires ρ to be a Lipschitz function. In other words, $\|\nabla\rho\|_{L^\infty(\mathbb{R}^{n-1};\mathbb{R}^{n-1})}$ is finite but can be arbitrary large. By taking invertible linear transformation we can make $\nabla\rho$ small in new coordinates.

Definition 3.2. A *regular dyadic resolution* is a sequence $\phi = (\phi_j)_{j=0}^\infty$ of Schwartz functions, denoted by $\phi \in \mathfrak{D}$, such that

- $\int \phi_0 = 1$, $\int x^\alpha \phi_0(x)dx = 0$ for all $\alpha \in \mathbb{N}^N \backslash \{0\}$.
- $\phi_j(x) = 2^{nj}\phi_0(2^j x) - 2^{n(j-1)}\phi_0(2^{j-1}x)$, for $j \ge 1$.

A *generalized dyadic resolution* is a sequence $\psi = (\psi_j)_{j=0}^\infty$ of Schwartz functions, denoted by $\psi \in \mathfrak{G}$, such that

- $\int x^\alpha \psi_1(x)dx = 0$ for all $\alpha \in \mathbb{N}^N$.
- $\psi_j(x) = 2^{n(j-1)}\psi_1(2^{j-1}x)$, for $j \ge 1$.

Here ψ_0 can be an arbitrary Schwartz function.

Lemma 3.3 ([Ryc99, Theorem 4.1(a)]). *There exists a function $g \in \mathscr{S}(\mathbb{R})$ such that $\operatorname{supp} g \subseteq [1, \infty)$, $\int_\mathbb{R} g = 1$ and $\int_\mathbb{R} t^k g(t)dt = 0$ for all $k \in \mathbb{Z}^+$.*

Proof. Define

$$G(z) := \exp(-(z-1)^{\frac{1}{8}} - (z-1)^{-\frac{1}{8}}), \quad z \in \mathbb{C}\backslash[1, \infty).$$

Here we use $(z-1)^{\frac{1}{8}} = |z-1|^{\frac{1}{8}} e^{i\frac{1}{8}\arg(z-1)}$ with $0 < \arg(z-1) < 2\pi$. It is easy to check that the two branches $G(t + i0)$ and $G(t - i0)$ are both smooth functions which are flat at $t = 1$.

For $0 < \varepsilon < \frac{1}{2}$, take an oriented loop $\Gamma_\varepsilon \subseteq \mathbb{C}$ with

$$\begin{aligned}\Gamma_\varepsilon = &\{t + i0 : 1 + \varepsilon \le t \le \varepsilon^{-1}\} \cup \{\varepsilon^{-1}e^{i\theta} : 0 \le \theta \le 2\pi\} \\ &\cup \{-t - i0 : -\varepsilon^{-1} \le t \le -1 - \varepsilon\} \cup \{1 + \varepsilon e^{-i\theta} : -2\pi \le \theta \le 0\}.\end{aligned}$$

By Cauchy's theorem,

$$\frac{1}{2\pi i}\int_0^\infty t^k(G(t+i0)-G(t-i0))dt = \lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} z^k G(z)dz = \begin{cases} G(0)\neq 0, & k=-1 \\ 0, & k\geq 0. \end{cases} \tag{3.1}$$

Define

$$g(t) := \frac{1}{(2\pi i)G(0)}\frac{G(t+i0)-G(t-i0)}{t}, \quad t\in\mathbb{R}.$$

Then $g\equiv 0$ on $(-\infty,1)$. Also g vanishes to infinite order at both $t=\infty$ and $t=1$. In view of (3.1), we have

$$\int_0^\infty g(t)\,dt = 1, \quad \int_0^\infty t^k g(t)dt = 0, \quad \forall\, k\in\mathbb{Z}^+. \qquad \square$$

Lemma 3.4 ([Ryc99, Proposition 2.1])**.** *Recall* $-\mathbb{K}=\{x_N<-|x'|\}$ *and let* $\mathfrak{D},\mathfrak{G}$ *be given in Definition 3.2.*

(i) There is a $\phi=(\phi_j)_{j=0}^\infty\in\mathfrak{D}$ *on* $\mathbb{R}^N$ *such that* $\operatorname{supp}\phi_j\subseteq -\mathbb{K}\cap\{x_N<-2^{-j}\}$ *for all* $j\in\mathbb{N}$.

(ii) For any $\phi=(\phi_j)$ *satisfying (i), there is a* $\psi=(\psi_j)_{j=0}^\infty\in\mathfrak{G}$ *such that* $\operatorname{supp}\psi_j\subseteq -\mathbb{K}\cap\{x_N<-2^{-j}\}$ *for all* $j\in\mathbb{N}$ *and* $f=\sum_{j=0}^\infty \psi_j*\phi_j*f$ *for all* $f\in\mathscr{S}'(\mathbb{R}^N)$.

Definition 3.5. We call $(\phi,\psi)=(\phi_j,\psi_j)_{j=0}^\infty$ with above-mentioned properties a $\mathbb{K}$-*dyadic pair.*

Proof of Lemma 3.4. Let $g\in\mathscr{S}(\mathbb{R})$ be as in Lemma 3.3 which is supported in $[1,\infty)$. Take an invertible linear transformation $\Theta=(\theta_1,\dots,\theta_N):\mathbb{R}^N\to\mathbb{R}^N$ such that $\Theta^{-1}([1,\infty)^N)\subseteq -\mathbb{K}\cap\{x_N<-1\}$. Define

$$\phi_0(x_1,\dots,x_N) = C_0\, g(\theta_1(x))\cdots g(\theta_N(x)),$$

where $C_0\neq 0$ is the constant chosen so that $\int_{\mathbb{R}^N}\phi_0=1$, or $\widehat{\phi}_0(0)=1$. Then $\phi_0\in\mathscr{S}(\mathbb{R}^N)$ satisfies $\operatorname{supp}\phi_0\subseteq\Theta^{-1}([1,\infty)^N)$. Moreover, ϕ_0 satisfies $\int_{\mathbb{R}^N}x^\alpha\phi_0(x)\,dx=0$ for all $|\alpha|>0$ since $\int t^k g(t)dt=0$ for all $k\in\mathbb{Z}^+$.

Define $\phi_j(x)=2^{Nj}\phi_0(2^jx)-2^{N(j-1)}\phi_0(2^{j-1}x)$ for $j\geq 1$, so then $\operatorname{supp}\phi_j\subseteq\{x_N<-2^{-j}\}\cap -\mathbb{K}$. This proves (i).

To prove (ii), let

$$\rho_0 := \phi_0*\phi_0\in\mathscr{S}(\mathbb{R}^N), \quad \rho_j(x) := 2^{Nj}\rho_0(2^jx)-2^{N(j-1)}\rho_0(2^{j-1}x), \quad j\geq 1.$$

Then $\operatorname{supp}\rho_0\subseteq\operatorname{supp}\phi_0+\operatorname{supp}\phi_0\subseteq -\mathbb{K}\cap\{x_N<-2\}$ and therefore

$$\operatorname{supp}\rho_j\subseteq\{x_N\leq -2\cdot 2^{-(j-1)}\}\cap -\mathbb{K} = \{x_N<-2^{-j}\}\cap -\mathbb{K}.$$

So $\rho \in \mathfrak{D}$ satisfies $\operatorname{supp}\rho_j \subseteq -\mathbb{K} \cap \{x_N < -2^{-j}\}$ for all $j \geq 0$ and $\widehat{\rho}_j(\xi) = \widehat{\phi}_j(\xi)(\widehat{\phi}_0(2^{-j}\xi) + \widehat{\phi}_0(2^{-(j-1)}\xi))$ for $j \geq 1$. Therefore

$$
\begin{aligned}
1 &= \sum_{j,k=0}^{\infty} \widehat{\rho}_j(\xi)\widehat{\rho}_k(\xi) \\
&= \sum_{j=0}^{\infty} \widehat{\rho}_j(\xi)\Big(\widehat{\rho}_j(\xi) + 2\sum_{k=j+1}^{\infty} \widehat{\rho}_k(\xi)\Big) \\
&= \sum_{j=0}^{\infty} \widehat{\rho}_j(\xi)\big(\widehat{\rho}_j(\xi) + 2 - 2\widehat{\rho}_0(2^{-j}\xi)\big) \\
&= \widehat{\rho}_0(\xi)(2 - \widehat{\rho}_0(\xi)) + \sum_{j=1}^{\infty} \widehat{\rho}_j(\xi)(2 - \widehat{\rho}_0(2^{-j}\xi) - \widehat{\rho}_0(2^{-(j-1)}\xi)) \\
&= [\widehat{\phi}_0(\xi)]^2(2 - \widehat{\rho}_0(\xi)) + \sum_{j=1}^{\infty} \widehat{\phi}_j(\xi)(\widehat{\phi}_0(2^{-j}\xi) + \widehat{\phi}_0(2^{-(j-1)}\xi))(2 - \widehat{\rho}_0(2^{-j}\xi) - \widehat{\rho}_0(2^{-(j-1)}\xi)).
\end{aligned}
$$

We can now define ψ via its Fourier transform as

$$
\begin{aligned}
&\widehat{\psi}_0(\xi) := 2\widehat{\phi}_0(\xi) - \widehat{\phi}_0(\xi)^3; \\
&\widehat{\psi}_j(\xi) := (\widehat{\phi}_0(2^{-j}\xi) + \widehat{\phi}_0(2^{-(j-1)}\xi))(2 - \widehat{\rho}_0(2^{-j}\xi) - \widehat{\rho}_0(2^{-(j-1)}\xi)), \quad j \geq 1.
\end{aligned}
$$

Then $\sum_{j=0}^{\infty} \widehat{\phi}_j \widehat{\psi}_j = 1$. Note that $\widehat{\psi}_j(\xi) = \widehat{\psi}_1(2^{-(j-1)}\xi)$ for $j \geq 1$, and therefore

$$\psi_j(x) = 2^{N(j-1)}\psi_1(2^{j-1}x), \quad j \geq 1.$$

Also we have

$$\psi_j(x) = \Big(2^{Nj}\phi_0(2^j x) + 2^{N(j-1)}\phi_0(2^{j-1}x)\Big) * \Big(2\delta_0 - 2^{Nj}\rho_0(2^j x) - 2^{N(j-1)}\rho_0(2^{j-1}x)\Big), \quad j \geq 1.$$

Since $\operatorname{supp}\phi_0$ and $\operatorname{supp}\rho_0$ are contained in $-\mathbb{K} \cap \{x_N < -1\}$, we have $\operatorname{supp}\psi_j \subseteq -\mathbb{K} \cap \{x_N < -2^{-j}\}$. Also we get $\widehat{\psi}_1(\xi) = O(|\xi|^\infty)$ from $\widehat{\phi}_0(\xi) = 1 + O(|\xi|^\infty)$, which implies $\int x^\alpha \psi_1(x)dx = 0$ for all α with $|\alpha| \geq 0$. □

We can now define the universal extension operator, first for special Lipschitz domains, and then for bounded Lipschitz domains.

Definition 3.6. Let (ϕ, ψ) be a $\mathbb{K}$-dyadic pair, and let ω be a special Lipschitz domain. The universal extension operator E_ω associated with (ϕ, ψ) is defined by

$$E_\omega f := \sum_{j=0}^{\infty} \psi_j * (\mathbf{1}_\omega \cdot (\phi_j * f)), \tag{3.2}$$

where $\mathbf{1}_\omega$ is the characteristic function on ω.

Here by extension, we mean for any tempered distribution $f \in \mathscr{S}'(\omega)$, $(Ef)|_\omega = f$ as distributions on ω. Indeed since $\omega + \mathbb{K} = \omega$, we have $(\psi_j * (\mathbf{1}_\omega h))|_\omega = h|_\omega$ for $h \in L^1_{\text{loc}}(\mathbb{R}^N)$. Thus

$$(E_\omega f)|_\omega = \sum_{j=0}^{\infty} (\psi_j * (\mathbf{1}_\omega(\phi_j * f)))|_\omega = \sum_{j=0}^{\infty} (\psi_j * \phi_j * f)|_\omega = f.$$

More generally for a bounded Lipschitz domain Ω, and $\mathcal{U}$ an open set containing $\overline{\Omega}$, we can use partition of unity to define extension operator $\mathcal{E} = \mathcal{E}_\Omega$ for Ω such that $\operatorname{supp}\mathcal{E}\varphi \subset \mathcal{U}$ for all φ: Let $\{U_\nu\}_{\nu=0}^M$ be a finite open cover of $\overline{\Omega}$, such that $U_0 \subset\subset \Omega$, $b\Omega \subseteq \bigcup_{\nu=1}^M U_\nu$ and $\bigcup_{\nu=0}^M U_\nu \subseteq \mathcal{U}$. Furthermore we may assume that for each ν, there exists a special Lipschitz domain ω_ν and an invertible affine linear transformation $\Phi_\nu : \mathbb{R}^N \to \mathbb{R}^N$, such that $U_\nu = \Phi_\nu(\mathbb{B}^N)$ and $U_\nu \cap \Phi_\nu(\omega_\nu) = U_\nu \cap \Omega$.

Choose $\chi_\nu \in C_c^\infty(U_\nu)$ such that $\chi_0 + \sum_{\nu=1}^M \chi_\nu^2 \equiv 1$ in some neighborhood of Ω. For a function g defined on $\Omega \cap U_\nu$, let $E_\nu g := E_{\omega_\nu}(g \circ \Phi_\nu) \circ \Phi_\nu^{-1}$.

$$\mathcal{E} f := \chi_0 f + \sum_{\nu=1}^M \chi_\nu E_\nu(\chi_\nu f). \tag{3.3}$$

Proposition 3.7. *Let ω be a special Lipschitz domain, and E_ω be given by (3.2). Then*

(i) $E_\omega : H^{s,p}(\omega) \to H^{s,p}(\mathbb{R}^N)$ is a bounded operator for all $s \in \mathbb{R}$ and $1 < p < \infty$.
(ii) $E_\omega : \Lambda^s(\omega) \to \Lambda^s(\mathbb{R}^N)$ is a bounded operator for all $s > 0$.

In particular $\mathcal{E}$ is a continuous map from $C^\infty(\overline{\omega})$ to $C^\infty(\mathbb{R}^N)$.

The reader can find the proof in [Ryc99, Theorem 4.1(b)]. By partition of unity we see that $\mathcal{E} : H^{s,p}(\Omega) \to H^{s,p}(\mathbb{R}^N)$ and $\mathcal{E} : \Lambda^s(\Omega) \to \Lambda^s(\mathbb{R}^N)$ are also bounded operators. In particular $\mathcal{E}$ is a continuous map from $C^\infty(\overline{\omega})$ to $C^\infty(\mathbb{R}^N)$.

There is also a useful "Littlewood-Paley type" characterization of $H^{s,p}(\omega)$.

Proposition 3.8. *Let ω be a special Lipschitz domain and $\phi = (\phi_j)_{j=0}^\infty$ be constructed as in Lemma 3.4 (i).*

(i) For $s \in \mathbb{R}$ and $1 < p < \infty$, $H^{s,p}(\omega)$ has equivalent norm

$$\|f\|_{\mathscr{F}^s_{p2}(\omega;\phi)} := \Big\| \Big(\sum_{j=0}^\infty 2^{2js} |\phi_j * f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\omega)}.$$

(ii) For $s > 0$, the Hölder-Zygmund space $\Lambda^s(\omega)$ has an equivalent norm

$$\|f\|_{\mathscr{B}^s_{\infty\infty}(\omega;\phi)} := \sup_{j \in \mathbb{N}} 2^{js} \|\phi_j * f\|_{L^\infty(\omega)}.$$

The proof is in [Ryc99, Theorem 3.2], where the assumption is $\phi_0 \in C_c^\infty(-\mathbb{K})$, but based on [Ryc99, Theorem 4.1(b)] same proof works for $\phi_0 \in \mathscr{S}(-\mathbb{K})$.

4. Commutator Estimate

The main result of this section is the following commutator estimate on special Lipschitz domains. We will write D for the gradient operator, and $D^k = (D^\alpha)_{|\alpha|=k}$.

Theorem 4.1. *Let $1 < p < \infty$ and $s \in \mathbb{R}$, and let ω be a special Lipschitz domain. Suppose (ϕ, ψ) is a $\mathbb{K}$-dyadic pair and let E_ω be defined as in Definition 3.6. Then there exists a constant $C = C_{s,p} > 0$ such that for $\delta(x) = \operatorname{dist}(x, b\omega)$,*

$$\|\delta^{1-s}[D, E_\omega] f\|_{L^p(\overline{\omega}^c)} \le C \|f\|_{H^{s,p}(\omega)}, \quad \forall f \in \mathscr{S}'(\mathbb{R}^N). \tag{4.1}$$

Remark 4.2.

(i) By Proposition 3.8, the $H^{s,p}(\omega)$ norm is equivalent to the $\mathscr{F}^s_{p,2}(\omega;\phi)$ norm. In fact we will establish the following stronger estimate: for $s \in \mathbb{R}$, $1 \le p \le \infty$,

$$\|\delta^{1-s}[D, E_\omega] f\|_{L^p(\overline{\omega}^c)} \le C_{s,p,\phi} \|f\|_{\mathscr{F}^s_{p,\infty}(\omega;\phi)}, \quad \forall f \in \mathscr{S}'(\mathbb{R}^N), \tag{4.2}$$

where

$$\|f\|_{\mathscr{F}^s_{p,\infty}(\omega;\phi)} := \Big\| \sup_{j \in \mathbb{N}} 2^{js} |\phi_j * f| \Big\|_{L^p(\omega)} \le \Big\| \Big(\sum_{j=0}^\infty 2^{2js} |\phi_j * f|^2 \Big)^{\frac{1}{2}} \Big\|_{L^p(\omega)} = \|f\|_{\mathscr{F}^s_{p,2}(\omega;\phi)}.$$

(ii) When $f \in H^{s,p}(\omega)$ for $s > 1$, Theorem 4.1 follows from Proposition 5.6, which gives that

$$\|\delta^{1-s} g\|_{L^p(\overline{\omega}^c)} \le C\|g\|_{H^{s-1,p}(\mathbb{R}^N)}, \quad \text{for any } g \in H_0^{s-1,p}(\overline{\omega}^c). \tag{4.3}$$

This is because $[D, E_\omega] \equiv 0$ in ω, and therefore $[D, E_\omega]f \in H_0^{s-1,p}(\overline{\omega}^c)$ by Proposition 2.12 (ii). Letting $g = [D, E_\omega]f$ in (4.3) we obtain (4.1).

(iii) When $s > \frac{1}{p}$ and $f \in H^{s,p}(\omega)$, Theorem 4.1 implies that $[D, E_\omega]f \in L^1_{\text{loc}}(\mathbb{R}^N)$. Indeed, since $[D, E_\omega]f$ is supported in ω^c,

$$\begin{aligned} \|[D, E_\omega]f\|_{L^1(B^N(0,R))} \le& \|\delta^{s-1}\|_{L^{p'}(B^N(0,R))} \|\delta^{1-s}[D, E_\omega]f\|_{L^p(B^N(0,R))} \\ \le& \|\delta^{s-1}\|_{L^p(B^N(0,R))} \|\delta^{1-s}[D, E_\omega]f\|_{L^p(\overline{\omega}^c)} \\ \lesssim& \|\delta^{(s-1)p'}\|^{1/p'}_{L^1(B^N(0,R))} \|f\|_{H^{s,p}(\omega)} \end{aligned}$$

for every R.

Note that $(s-1)p' > (\frac{1}{p} - 1)\frac{p}{p-1} = -1$, so $\delta^{(s-1)p'}$ is a locally integrable in $\mathbb{R}^N$, which implies that the right-hand side is finite. When $s \le \frac{1}{p}$, $\delta(x)^{(s-1)p'}$ is no longer integrable near the boundary of ω, and we can only interpret the commutator as a distribution.

Note that when $p = \infty$ and $s > 0$, we have $\sup_{j\in\mathbb{N}} 2^{js}\|\phi_j * f\|_{L^\infty(\omega)} = \big\|\sup_{j\in\mathbb{N}} 2^{js}|\phi_j * f|\big\|_{L^\infty(\omega)}$, or $\mathscr{B}^s_{\infty,\infty}(\omega;\phi) = \mathscr{F}^s_{\infty,\infty}(\omega;\phi)$ (also see [Tri83, Remark 2.3.4/3]). Thus by (4.2) and Proposition 3.8 (ii), we have the following estimate for the Hölder-Zygmund space:

Corollary 4.3. *Let $s > 0$ and let E_ω, ω be as in Theorem 4.1. There is a $C > 0$ such that*

$$\|\delta^{1-s}[D, E_\omega]f\|_{L^\infty(\overline{\omega}^c)} \le C\|f\|_{\Lambda^s(\omega)}, \quad f \in \Lambda^s(\omega). \tag{4.4}$$

To prove Theorem 4.1 we need a sequence of lemmas.

Lemma 4.4. *Let ϕ, ψ be two generalized dyadic resolutions. Then for any $M > 0$ and $\gamma \in \mathbb{N}^N$, there is a $C = C_{M,N,\gamma} > 0$ such that*

$$\int_{|x|\ge 2^{-l}} |D^\gamma \phi_j * \psi_k(x)|\, dx \le C 2^{\min(j,k)|\gamma| - M\{|j-k| + \max(j-l,k-l,0)\}}, \quad j,k \ge 0, \quad l \in \mathbb{Z}. \tag{4.5}$$

Proof. By symmetry we can assume $j \le k$. We first use the scaling properties of ϕ and ψ to show that the estimate can be reduced to the cases $j = 0$ and $j = 1$. When $1 < j \le k$, recall that $\phi_j(x) = 2^{N(j-1)}\phi_1(2^{j-1}x)$ and $\psi_k(x) = 2^{N(k-1)}\psi_1(2^{k-1}x)$, so

$$\begin{aligned} \phi_j * \psi_k(x) &= 2^{N(j+k-2)} \int \phi_1(2^{j-1}x - 2^{j-1}y)\psi_1(2^{k-1}y)dy \\ &= 2^{N(k-1)} \int \phi_1(2^{j-1}x - \tilde{y})\psi_1(2^{k-j}\tilde{y})d\tilde{y} \\ &= 2^{N(j-1)}\phi_1 * \psi_{k-j+1}(2^{j-1}x). \end{aligned}$$

Therefore taking substitution $\tilde{x} = 2^{j-1}x$ we have

$$\int_{|x|\ge 2^{-l}} |D^\gamma \phi_j * \psi_k(x)|\, dx = 2^{(j-1)|\gamma|} \int_{|\tilde{x}|\ge 2^{j-1-l}} |D^\gamma \phi_1 * \psi_{k-j+1}(\tilde{x})| d\tilde{x}, \quad 1 \le j \le k. \tag{4.6}$$

Suppose (4.5) is true for $j = 1 \le k$. Since $k \ge j$, the right hand side of (4.6) is bounded by

$$\begin{aligned} C2^{(j-1)|\gamma|}2^{|\gamma| - M(|1-(k-j+1)| + \max(1+(j-1-l), k-j+1+(j-1-l), 0))} &= C2^{j|\gamma| - M(|j-k| + \max(k-l,0))} \\ &= C2^{\min(j,k)|\gamma| - M(|j-k| + \max(j-l,k-l,0))}. \end{aligned}$$

This proves the reduction.

Next we consider the case for $j \in \{0,1\}$ and $k \geq 1$. Write $k = 1+m$, for $m \geq 0$. Since $\int_{\mathbb{R}^N} x^\alpha \psi_1(x) = 0$ for any $\alpha \in \mathbb{N}^N$, we have

$$\begin{aligned}
D^\gamma \phi_j * \psi_{m+1}(x) &= \int_{\mathbb{R}^N} D^\gamma \phi_j(x-y)\psi_{m+1}(y)\,dy \\
&= \int_{\mathbb{R}^N} D^\gamma \phi_j(x-y) 2^{Nm}\psi_1(2^m y)\,dy \\
&= \int_{\mathbb{R}^N} D^\gamma \phi_j(x-2^{-m}y)\psi_1(y)\,dy \\
&= \int_{\mathbb{R}^N} \left(D^\gamma \phi_j(x-2^{-m}y) - \sum_{|\alpha|\leq M'-1} (-2^{-m}y)^\alpha \frac{D^{(\gamma+\alpha)}\phi_j(x)}{\alpha!} \right) \psi_1(y)\,dy,
\end{aligned}$$

where M' is some large number to be chosen. By Taylor's theorem, the expression in parenthesis is bounded in absolute value by

$$\frac{1}{M'!}|2^{-m}y|^{M'} \sup_{B(x,2^{-m}|y|)} \left| D^{|\gamma|+M'}\phi_j \right|.$$

Since ϕ_0 and ϕ_1 are Schwartz, we have for $j=0,1$

$$\sup_{B(x,2^{-m}|y|)} \left| D^{|\gamma|+M'}\phi_j \right| \lesssim_{\gamma,M'} \begin{cases} (1+|x|)^{-M'}, & |x| \geq 2^{1-m}|y| \\ 1, & |x| < 2^{1-m}|y| \end{cases} \quad \text{for } M'>0, \gamma \in \mathbb{N}^N. \tag{4.7}$$

Therefore for $j=0$ or 1 we have

$$\begin{aligned}
&\int_{|x|\geq 2^{-l}} |D^\gamma \phi_j * \psi_{m+1}(x)|\,dx \\
&\quad\lesssim_{\gamma,M'} \int_{|x|\geq 2^{-l}} \left[\left(\int_{|y|\leq 2^{m-1}|x|} + \int_{|y|\geq 2^{m-1}|x|} \right) \frac{|2^{-m}y|^{M'}}{M'!} \sup_{B(x,2^{-m}|y|)} \left| D^{|\gamma|+M'}\phi_j \right| |\psi_1(y)|\,dy \right] dx \\
&\quad\lesssim_{\gamma,M'} \int_{|x|\geq 2^{-l}} \left[\int_{|y|\leq 2^{m-1}|x|} |2^{-m}y|^{M'}(1+|x|)^{-M'}|\psi_1(y)|\,dy + \int_{|y|\geq 2^{m-1}|x|} |2^{-m}y|^{M'}|\psi_1(y)|\,dy \right] dx.
\end{aligned}$$

Using polar coordinates and (4.7) we can bound the above expression by

$$\begin{aligned}
&2^{-mM'} \int_{2^{-l}}^\infty \left[(1+\rho)^{-M'} \int_0^\infty r^{M'} r^{N-1}(1+r)^{-2M'-N}\,dr \right. \\
&\qquad \left. + \int_{2^{m-1}\rho}^\infty r^{M'}(1+r)^{-2M'-N} r^{N-1}\,dr \right] \rho^{N-1}\,d\rho \\
&\lesssim_{\gamma,M'} 2^{-mM'} \int_{2^{-l}}^\infty \rho^{N-1} \left[(1+\rho)^{-M'} + \int_{2^{m-1}\rho}^\infty (1+r)^{-M'-1}\,dr \right] d\rho \\
&\lesssim_{\gamma,M'} 2^{-mM'} \int_{2^{-l}}^\infty \rho^{N-1} \left((1+\rho)^{-M'} + (1+2^{m-1}\rho)^{-M'} \right) d\rho \\
&\lesssim_{\gamma,M'} 2^{-mM'} \int_{2^{-l}}^\infty \rho^{N-1}(1+\rho)^{-M'}\,d\rho.
\end{aligned}$$

Taking $M' \geq 2M+N$, then the above is bounded by $C_{\gamma,M'}2^{-mM'}\min\{2^{l(M'-N)},1\} \leq C_{\gamma,M'}2^{-2Mm} \times \min\{2^{Ml},1\} \leq C_{\gamma,M'}2^{-M(m+\max\{m-l,0\})}$, which is what we need for the estimate.

Finally, if $j = k = 0$, we use the fact that $\phi_0 * \psi_0$ is Schwartz, and therefore

$$\int_{|x|\geq 2^{-l}} |D^\gamma[\phi_0 * \psi_0](x)\,dx| \leq C2^{-l}, \quad l < 0;$$
$$\int_{|x|\geq 2^{-l}} |D^\gamma[\phi_0 * \psi_0](x)\,dx| \leq C, \quad l \geq 0,$$

which implies (4.5).

□

Corollary 4.5. *Let $(\psi_j)_{j=0}^\infty \subseteq \mathscr{S}(\mathbb{R}^N)$ be a generalized dyadic decomposition. Then for any $M > 0$ and $\gamma \in \mathbb{N}^N$ there is a $C = C_{\psi,M,\gamma} > 0$ such that*

$$\int_{|x|\geq 2^{-l}} |D^\alpha \psi_k(x)|dx \leq C2^{k|\gamma|-M\max(0,k-l)}, \quad \forall k \in \mathbb{N}, \quad l \in \mathbb{Z}. \tag{4.8}$$

Proof. Let $\phi = (\phi_j)_{j=0}^\infty$ be any regular dyadic resolution, so $\psi_k = \sum_{j=0}^\infty \phi_j * \psi_k$ and we have $\int_{|x|\geq 2^{-l}} |D^\alpha \psi_k| \leq \sum_{j=0}^\infty \int_{|x|\geq 2^{-l}} |D^\alpha(\phi_j * \psi_k)|$. Taking sum over $j \in \mathbb{N}$ on the right hand side of (4.5) we get (4.8). □

We remark that Corollary 4.5 can also be proved independently without the use of Lemma 4.4.

In our proof we use the following dyadic decomposition:

$$P_k := \{(x', x_N) : 2^{-\frac{1}{2}-k} < x_N - \rho(x') < 2^{\frac{1}{2}-k}\} \subseteq \omega, \quad k \in \mathbb{Z}, \tag{4.9}$$

$$S_k := \{(x', x_N) : -2^{\frac{1}{2}-k} < x_N - \rho(x') < -2^{-\frac{1}{2}-k}\} \subseteq \overline{\omega}^c, \quad k \in \mathbb{Z}. \tag{4.10}$$

Up to sets of measure zero, we have disjoint unions $\omega = \bigsqcup_{k\in\mathbb{Z}} P_k$ and $\overline{\omega}^c = \bigsqcup_{k\in\mathbb{Z}} S_k$.

Lemma 4.6. *Let $1 \leq p \leq \infty$. For any M there is a $C_M > 0$ such that for every $j, j' \in \mathbb{N}$ and $k, k' \in \mathbb{Z}$,*

$$\|\psi_j * (D\mathbf{1}_\omega \cdot (\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f)))\|_{L^p(S_k)} \leq C2^{j-M(|j-k|+|j-j'|+|j'-k'|)}\|\phi_{j'} * f\|_{L^p(P_{k'})}. \tag{4.11}$$

Proof. Since $D\mathbf{1}_\omega = -D\mathbf{1}_{\overline{\omega}^c}$ as distributions on $\mathbb{R}^N$, the term $\psi_j * (D\mathbf{1}_\omega \cdot (\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f)))$ in (4.11) has two identical expressions:

$$A_{jj'k'} := D\psi_j * (\mathbf{1}_\omega(\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))) - \psi_j * (\mathbf{1}_\omega(D\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))), \tag{4.12}$$

$$B_{jj'k'} := -D\psi_j * (\mathbf{1}_{\omega^c}(\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))) + \psi_j * (\mathbf{1}_{\omega^c}(D\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))). \tag{4.13}$$

First we show that the left-hand side of (4.11) is 0 if $j \leq k - 2$ or $j' \leq k' - 2$. By (4.12) and the fact that $\operatorname{supp}\phi_j, \operatorname{supp}\psi_j \subseteq \{x_N < -2^{-j}\}$, we see if $j' \leq k' - 2$,

$$\begin{aligned}\operatorname{supp}(\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f))) &\subseteq \operatorname{supp}\phi_j + \operatorname{supp}\psi_{j'} + \operatorname{supp}\left(\mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f)\right) \\ &\subseteq \{x_N - \rho(x') < -2^{1-j} - 2^{1-j'} + 2^{-k'}\} \\ &\subseteq \{x_N - \rho(x') < 0\},\end{aligned}$$

and similarly,

$$\operatorname{supp}(D\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f))) \subseteq \{x_N - \rho(x') < 0\}.$$

Hence the left-hand side of (4.11) vanishes if $j' \leq k' - 2$. On the other hand, the expression in (4.13) is supported in $\operatorname{supp}\psi_j + \overline{\omega}^c \subseteq \{x_N - \rho(x') < -2^{-j}\}$, which is disjoint from $S_k = \{-2^{-k+\frac{1}{2}} < x_N - \rho(x') < -2^{-k-\frac{1}{2}}\}$ when $j < k - 1$. Hence the left-hand side of (4.11) again vanishes. We shall now assume that $j \geq k - 2$ and $j' \geq k' - 2$.

We first estimate the left-hand side of (4.11) using (4.12). Write (4.12) as $A_{jj'k'} =: A^1_{jj'k'} - A^2_{jj'k'}$ where

$$A^1_{jj'k'} := D\psi_j * (\mathbf{1}_\omega(\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))),$$
$$A^2_{jj'k'} := \psi_j * (\mathbf{1}_\omega(D\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))).$$

Denoting $h_{jj'k'} := \phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f))$, we have

$$\|A^1_{jj'k'}\|_{L^p(S_k)} = \|D\psi_j * (\mathbf{1}_\omega \cdot h_{jj'k'})\|_{L^p(S_k)}.$$

For $x \in S_k$ and $y \in \omega$, we have $|x-y| \geq \operatorname{dist}(S_k, \omega) \geq 2^{-1-k}$, and

$$D\psi_j * (\mathbf{1}_\omega h_{jj'k'})(x) = \int_{\omega \cap |x-y| \geq 2^{-1-k}} D\psi_j(x-y) h_{jj'k'}(y)\, dy.$$

Since $\psi_j = 2^{N(j-1)}\psi_1(2^{j-1}x)$ and ψ_1 and $D\psi_1$ are Schwartz functions, we have

$$\int_{|x|>2^{-1-k}} |D\psi_j(x)|\, dx = \int_{|x|>2^{j-2-k}} 2^{j-1}|D\psi_1(x)|\, dx \lesssim_M 2^{j-M(j-k)} \tag{4.14}$$

$$\int_{|x|>2^{-1-k}} |\psi_j(x)|\, dx = \int_{|x|>2^{j-2-k}} |\psi_1(x)| \lesssim_M 2^{-M(j-k)}. \tag{4.15}$$

Now $\operatorname{supp}(\phi_j * \psi_{j'}) \subset \{x_N < -2^{-j}\}$ by Definition 3.5. Hence by Lemma 4.4 applied with $l = j$, we have

$$\|D^\gamma \phi_j * \psi_{j'}\|_{L^1(\mathbb{R}^N)} = \|D^\gamma \phi_j * \psi_{j'}\|_{L^1(|x| \geq 2^{-j})} \lesssim 2^{j|\gamma| - M|j-j'|}. \tag{4.16}$$

Applying Young's inequality and estimates (4.14), (4.16), we have

$$\begin{aligned}
\|A^1_{jj'k'}\|_{L^p(S_k)} &\leq \|D\psi_j\|_{L^1(\{|x| \geq 2^{-1-k}\})} \|h_{jj'k'}\|_{L^p(\omega)} \\
&\lesssim 2^{j-M(j-k)} \|\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f))\|_{L^p(\omega)} \\
&\lesssim 2^{j-M(j-k)} \|\phi_j * \psi_{j'}\|_{L^1(\mathbb{R}^N)} \|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\lesssim 2^j 2^{-M|j-j'|} 2^{-M(j-k)} \|\phi_{j'} * f\|_{L^p(P_{k'})}.
\end{aligned}$$

Similarly by (4.15) and we can show that

$$\begin{aligned}
\|A^2_{jj'k'}\|_{L^p(S_k)} &\lesssim \|\psi_j\|_{L^1(|x|>2^{-1-k})} \|D\phi_j * \psi_{j'}\|_{L^1(\mathbb{R}^N)} \|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\lesssim 2^j 2^{-M|j-j'|} 2^{-M(j-k)} \|\phi_{j'} * f\|_{L^p(P_{k'})}.
\end{aligned}$$

Next, we estimate the left-hand side of (4.11) using (4.13). Write (4.13) as $B_{jj'k'} =: -B^1_{jj'k'} + B^2_{jj'k'}$ where

$$B^1_{jj'k'} := D\psi_j * (\mathbf{1}_{\omega^c}(\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))),$$
$$B^2_{jj'k'} := \psi_j * (\mathbf{1}_{\omega^c}(D\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}(\phi_{j'} * f)))).$$

Denoting $h'_{j'k'} = \mathbf{1}_{P_{k'}} \cdot (\phi_{j'} * f)$ and applying Young's inequality we get

$$\|B^1_{jj'k'}\|_{L^p(S_k)} \lesssim \|D\psi_j\|_{L^1(\mathbb{R}^N)} \|\phi_j * \psi_{j'} * h'_{j'k'}\|_{L^p(\overline{\omega}^c)}. \tag{4.17}$$

For $x \in \overline{\omega}^c$ and $y \in P_{k'}$, we have $|x-y| \geq \operatorname{dist}(P_{k'}, \overline{\omega}^c) \geq 2^{-1-k'}$. Hence for $x \in \overline{\omega}^c$,

$$D^\gamma \phi_j * \psi_{j'} * h'_{j'k'}(x) = \int_{P_{k'} \cap \{y: |x-y| \geq 2^{-1-k'}\}} (D^\gamma \phi_j * \psi_{j'})(x-y) h'_{j'k'}(y)\, dy.$$

By Young's inequality and Lemma 4.4,

$$\|D^\gamma \phi_j * \psi_{j'} * h'_{j'k'}\|_{L^p(\overline{\omega}^c)} \le \left(\int_{|x|\ge 2^{-1-k'}} |D^\gamma \phi_j * \psi_{j'}|\right) \|h'_{j'k'}\|_{L^p(\mathbb{R}^N)}$$
$$\lesssim 2^{j|\gamma|-M(|j-j'|+j'-k')}\|\phi_{j'} * f\|_{L^p(P_{k'})}.$$

Since $\|D\psi_j\|_{L^1(\mathbb{R}^N)} = 2^{j-1}\|D\psi_1\|_{L^1(\mathbb{R}^N)} \le C2^{j-1}$, we get (4.17) that

$$\|B^1_{jj'k'}\|_{L^p(S_k)} \lesssim 2^{j-1-M|j-j'|-M(j'-k')}\|\phi_{j'} * f\|_{L^p(P_{k'})}.$$

In the same way we can show that B_2 satisfies the same estimate

$$\|B^2_{jj'k'}\|_{L^p(S_k)} \lesssim 2^{j-M|j-j'|-M(j'-k')}\|\phi_{j'} * f\|_{L^p(P_{k'})}.$$

Finally combining the estimates for $A^1_{jj'k'}$, $A^2_{jj'k'}$, $B^1_{jj'k'}$ and $B^2_{jj'k'}$ we get

$$\begin{aligned}
&\|\psi_j * ((D\mathbf{1}_\omega)\cdot(\phi_j * \psi_{j'} * (\mathbf{1}_{P_{k'}}\cdot(\phi_{j'} * f))))\|_{L^p(S_k)} \\
&\le 2^{j-M|j-j'|}\min\left\{2^{-M(j-k)}, 2^{-M(j'-k')}\right\}\|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\le 2^{j-M|j-j'|}2^{-M\max\{j-k,j'-k'\}}\|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\le 2^{j-M|j-j'|}2^{-\frac{M}{2}[(j-k)+(j'-k')]}\|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\lesssim 2^{j-\frac{M}{2}[|j-j'|+|j-k|+|j'-k'|]}\|\phi_{j'} * f\|_{L^p(P_{k'})},
\end{aligned}$$

where in the last step we use $j \ge k-2$ and $j' \ge k'-2$. Replacing $M/2$ by M we get the result. □

Lemma 4.7. *Let $M > 1$. Then there is a $C_M > 0$ such that*

$$\sum_{b\in\mathbb{Z}} 2^{-M(|a-b|+|b-c|)} \le C_M 2^{-(M-1)|a-c|}, \quad \text{for all } a, c \in \mathbb{Z}.$$

Proof. By a substitution $\widetilde{b} = a-b$ we see that $\sum_{b\in\mathbb{Z}} 2^{-M(|a-b|+|b-c|)} = \sum_{\widetilde{b}\in\mathbb{Z}} 2^{-M(|\widetilde{b}|+|a-c-\widetilde{b}|}$. So it suffices to show that

$$\sum_{b\in\mathbb{Z}} 2^{-M(|b|+|a-b|)} \le C_M 2^{-(M-1)|a|}.$$

By symmetry we can assume $a > 0$ in the above inequality. It follows that

$$\begin{aligned}
\sum_{b\in\mathbb{Z}} 2^{-M(|b|+|a-b|)} &\le \sum_{b\,\le\,0 \text{ or } b\,\ge\,2a} 2^{-M(|b|+|a-b|)} + \sum_{b=1}^{a} 2^{-M(b+(a-b))} + \sum_{b=a+1}^{2a-1} 2^{-M(b+(b-a))} \\
&\le C_M \sum_{b\in\mathbb{Z}} 2^{-M(|b|+a)} + \sum_{b=1}^{a} 2^{-Ma} + \sum_{b=a+1}^{2a-1} 2^{-Ma} \\
&= C_M 2^{-Ma}\left(\sum_{b\in\mathbb{Z}} 2^{-M|b|} + \sum_{b=1}^{2a-1} 1\right) \\
&\lesssim C_M(a+1)2^{-Ma}.
\end{aligned}$$

Clearly $a+1 \le 2^a$, for $a \in \mathbb{Z}^+$. Hence $\sum_{b\in\mathbb{Z}} 2^{-M(|b|+|a-b|)} \le 2^{-(M-1)|a|}$. □

We are now ready to prove the main result of the section.

Proof of Theorem 4.1. As mentioned in Remark 4.2 (i), we will prove the following stronger estimate

$$\|\delta^{1-s}[D, E_\omega]f\|_{L^p(\overline{\omega}^c)} \le C_{p,s}\Big\|\sup_{j\in\mathbb{N}} 2^{js}|\phi_j * f|\Big\|_{L^p(\omega)}, \quad \forall f \in \mathscr{S}'(\omega), \tag{4.2}$$

for $1 \le p \le \infty$, provided that the norm on the right hand side is finite.

Let P_k and S_k be the dyadic strips defined in (4.9) and (4.10). Since $\delta \approx 2^{-k}$ on S_k, we can replace the function δ by $\sum_{k\in\mathbb{Z}} 2^{-k}\mathbf{1}_{S_k}$. Also

$$\begin{aligned}
[D, E_\omega]f &= DE_\omega f - E_\omega D f \\
&= \sum_{j=0}^{\infty} \psi_j * [(D\mathbf{1}_\omega)\cdot(\phi_j * f) + \mathbf{1}_\omega \cdot (\phi_j * Df)] - \sum_{j=0}^{\infty} \psi_j * (\mathbf{1}_\omega \cdot (\phi_j * Df)) \\
&= \sum_{j=0}^{\infty} \psi_j * ((D\mathbf{1}_\omega)\cdot(\phi_j * f)) \\
&= \sum_{j,j'=0}^{\infty} \psi_j * \big((D\mathbf{1}_\omega)\cdot \big(\phi_j * \psi_{j'} * (1_\omega \cdot (\phi_{j'} * f))\big)\big) \\
&= \sum_{j,j'\in\mathbb{N},k'\in\mathbb{Z}} \psi_j * \big((D\mathbf{1}_\omega)\cdot (\phi_j * \psi_{j'} * (1_{P_{k'}} \cdot (\phi_{j'} * f)))\big).
\end{aligned}$$

Denoting the summand on the right-hand side by $A_{jj'k'}$, we have by (4.11),

$$\|A_{jj'k'}\|_{L^p(S_k)} \lesssim 2^{j-M(|j-j'|+|j-k|+|j'-k'|)}\|\phi_{j'} * f\|_{L^p(P_{k'})}.$$

Therefore

$$\begin{aligned}
\|\delta^{1-s}[D,E_\omega]f\|_{L^p(S_k)} &\lesssim 2^{-k(1-s)}\, \|[D,E_\omega]f\|_{L^p(S_k)} \\
&\lesssim 2^{(s-1)k} \sum_{j,j'\in\mathbb{N},k'\in\mathbb{Z}} 2^{j-M(|j-j'|+|j-k|+|j'-k'|)}\|\phi_{j'} * f\|_{L^p(P_{k'})}.
\end{aligned}$$

Write $k = j' + (k-j')$ and $j = j' + (j-j')$. Then the above is bounded by

$$\begin{aligned}
\|\delta^{1-s}[D,E_\omega]f\|_{L^p(S_k)} &\lesssim \sum_{j,j'\in\mathbb{N},k'\in\mathbb{Z}} 2^{(s-1)j'+j'}2^{|s-1||k-j'|+|j-j'|}2^{-M(|j-k|+|j-j'|+|j'-k'|)}\|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\lesssim \sum_{j,j'\in\mathbb{N},k'\in\mathbb{Z}} 2^{(|s-1|+1)(|j-j'|+|k-j|)}2^{-M(|j-k|+|j-j'|+|j'-k'|)}2^{sj'}\|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\lesssim \sum_{j,j'\in\mathbb{N},k'\in\mathbb{Z}} 2^{-(M-|s-1|-1)(|j-k|+|j-j'|+|j'-k'|)}2^{sj'}\|\phi_{j'} * f\|_{L^p(P_{k'})}.
\end{aligned}$$

Applying Lemma 4.7 to the sum over j and then again to the sum over j', we get

$$\begin{aligned}
\|\delta^{1-s}[D,E_\omega]f\|_{L^p(S_k)} &\lesssim \sum_{j'\in\mathbb{N},k'\in\mathbb{Z}} 2^{-(M-|s-1|-2)(|k-j'|+|j'-k'|)}2^{sj'}\|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\lesssim \sum_{k'\in\mathbb{Z}} 2^{-(M-|s-1|-3)|k-k'|} \sup_{j'\in\mathbb{N}} 2^{sj'}\|\phi_{j'} * f\|_{L^p(P_{k'})} \\
&\lesssim \sum_{k'\in\mathbb{Z}} 2^{-(M-|s-1|-3)|k-k'|} \|\sup_{j'\in\mathbb{N}} 2^{sj'}|\phi_{j'} * f|\|_{L^p(P_{k'})}.
\end{aligned}$$

Define sequences u, v, w by

$$u[j] := \|\delta^{1-s}[D,E_\omega]f\|_{L^p(S_j)}, \quad v[j] := 2^{-(M-|s-1|-3)|j|}, \quad w[j] := \|\sup_{l\in\mathbb{N}} 2^{sl}|\phi_l * f|\|_{L^p(P_j)}.$$

Then we have shown that $u \lesssim v * w$. By Young's inequality we get $\|u\|_{\ell^p} \le \|v\|_{\ell^1}\|w\|_{\ell^p}$. Clearly $\|u\|_{\ell^p} \approx \|\delta^{1-s}[D,E_\omega]f\|_{L^p(\overline{\omega}^c)}$ and $\|w\|_{\ell^p} = \|f\|_{\mathscr{F}^s_{p,\infty}(\omega;\phi)}$ (see Remark 4.2 (i)). By choosing M sufficiently large so that $\|v\|_{\ell^1} < \infty$ we obtain the desired estimate (4.2). □

5. Hardy-Littlewood Lemma of Sobolev Type

In the last section we estimated $L^p(\Omega, \lambda)$ norm using $H^{s,p}(\Omega)$ norm, where the weight λ is some power of the boundary distance function. To show that our solution for $\overline{\partial}$ is in $H^{s+\frac{1}{2},p}(\Omega)$, we also need to bound the $H^{s,p}(\Omega)$ norms by weighted Sobolev norms $W^{k,p}(\Omega, \lambda)$. We will call this kind of estimates Hardy-Littlewood lemma of Sobolev type, after the classical version for Hölder spaces.

Lemma 5.1. *Let $s > -\frac{1}{p}$ and $1 < p < \infty$. Then*

(i) There is a $C_{s,p} > 0$ such that for all $v \in W^{1,p}_{\mathrm{loc}}(0, 2)$ such that $v \equiv 0$ near 2,

$$\int_0^2 t^{sp}|v(t)|^p\,dt \le C_{s,p}\int_0^2 t^{(s+1)p}|v'(t)|^p\,dt. \tag{5.1}$$

(ii) Let $\omega = \{x_N > \rho(x')\} \subseteq \mathbb{R}^N$ be a bounded special Lipschitz domain. Suppose $u \in W^{1,p}_{\mathrm{loc}}(\omega)$ and $\operatorname{supp} u \subseteq \overline{\omega} \cap \mathbb{B}^N$. Then

$$\|\delta^s u\|_{L^p(\omega)} \le C'_{s,p}\|\delta^{s+1}Du\|_{L^p(\omega)},$$

where $\delta(x) = \operatorname{dist}(x, b\omega)$ and $C'_{s,p} > 0$ is the constant that does not depend on u.

Proof. (i) By assumption $v \in W^{1,1}_{\mathrm{loc}}(0, 2)$ is locally absolutely continuous, hence $v(t)$ can be defined point-wise.

Let ε be a small positive number. Using integration by parts we have

$$\begin{aligned}\int_\delta^2 t^{sp}|v(t)|^p\,dt &= -\frac{\delta^{sp+1}}{sp+1} + \int_\delta^2 \frac{t^{sp+1}}{sp+1}p|v(t)|^{p-1}v'(t)\operatorname{sign}(v(t))\,dt\\ &\le \int_\delta^2 \frac{t^{sp+1}}{sp+1}p|v(t)|^{p-1}|v'(t)|\,dt\\ &\le C_{s,p}\|t^{s(p-1)}|v|^{p-1}\|_{L^{\frac{p}{p-1}}([\delta,2])}\|t^{s+1}v'\|_{L^p([\delta,2])}\\ &= C_{s,p}\|t^s v\|^{p-1}_{L^p([\delta,2])}\|t^{s+1}v'\|_{L^p([\delta,2])}.\end{aligned}$$

Here $\operatorname{sign} x = \frac{x}{|x|}$ when $x \neq 0$ and $\operatorname{sign} x = 0$.

Note that the left-hand side of the above inequality is $\|t^s v\|^p_{L^p([\delta,2])}$. Dividing by $\|t^s v\|^{p-1}_{L^p([\delta,2])}$ (which is finite) from both sides and taking the limit as $\delta \to 0$ we get (5.1).

(ii) By assumption u vanishes outside $\mathbb{B}^N$, so

$$\begin{aligned}\|\delta^s u\|^p_{L^p(\omega)} &\lesssim \int_{|y'|<1}\int_{y_N=\rho(y')}^1 (y_N - \rho(y'))^{sp}|u(y', y_N)|^p\,dy_N\,dy'\\ &= \int_{|y'|<1}\int_{t=0}^{1-\rho(y')} t^{sp}|u(y', t+\rho(y'))|^p\,dt\,dy'.\end{aligned}$$

Set $\widetilde{u}(y', t) := u(y', t + \rho(y'))$. Then $\widetilde{u}(y', t)$ vanishes near $t = 1 - \rho(y')$. Since $\sup|\rho| < 1$, for every $y' \in \mathbb{R}^{N-1}$, we see that the map $t \mapsto u(y', t + \rho(y'))$ is supported in $[0, 1 - \rho(y'))$ and vanishes near $1 - \rho(y')$. Since $1 - \rho(y') < 2$, by part (i) we have

$$\begin{aligned}\|\delta^s u\|_{L^p(\omega)} &\lesssim \int_{|y'|<1}\int_0^{1-\rho(y')} t^{(s+1)p}|D_t u(y', t+\rho(y'))|^p\,dt\,dy'\\ &= \int_{|y'|<1}\int_{\rho(y')}^1 (y_N - \rho(y'))^{(s+1)p}|D_{y_N}u(y', y_N)|^p\,dy_N\,dy'\\ &\lesssim \|\delta^{s+1}Du\|_{L^p(\omega)}.\end{aligned}$$

This completes the proof. ∎

The following result can be viewed as a weighted version of the Poincaré inequality.

Proposition 5.2. *Let $1 < p < \infty$ and k, l be non-negative integers with $l < k$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and define $\delta(x)$ to be the distance function to the boundary $b\Omega$. If $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ and*

$$\sum_{|\gamma|\le k} \|\delta^{k-l} D^\gamma u\|_{L^p(\Omega)} < \infty,$$

then $u \in W^{l,p}(\Omega)$. Furthermore, there exists a constant C that does not depend on u such that

$$\|u\|_{W^{l,p}(\Omega)} \le C \sum_{|\gamma|\le k} \|\delta^{k-l} D^\gamma u\|_{L^p(\Omega)}.$$

Proof. For each $0 \le i \le l < k$ and each $|\alpha| = i$, we show that

$$\int_\Omega |D^\alpha u|^p \, dV(x) \lesssim \sum_{|\gamma|\le k} \int_\Omega \delta(x)^{p(k-l)} |D^\gamma u|^p \, dV(x). \tag{5.2}$$

It suffices to show that for every non-negative integer j and $1 < p < \infty$, one has

$$\|\delta^j v\|_{L^p(\Omega)} \lesssim_{j,p,\Omega} \|\delta^{j+1} v\|_{L^p(\Omega)} + \|\delta^{j+1} Dv\|_{L^p(\Omega)}. \tag{5.3}$$

Indeed, setting $v = D^\alpha u$ and using (5.3) $(k-i)$ times we get

$$\begin{aligned}\|D^\alpha u\|_{L^p(\Omega)} &\lesssim \|\delta D^\alpha u\|_{L^p(\Omega)} + \|\delta D D^\alpha u\|_{L^p(\Omega)} \\ &\lesssim \cdots \lesssim \sum_{|\gamma|\le k-i} \|\delta^{k-i} D^{\alpha+\gamma} u\|_{L^p(\Omega)} \\ &\lesssim \sum_{|\gamma|\le k} \|\delta^{k-l} D^\gamma u\|_{L^p(\Omega)}.\end{aligned}$$

It remains to prove (5.3). Take a finite open cover $\{U_\nu\}_{\nu=0}^M$ of Ω such that $U_0 \subset\subset \Omega$ and $\bigcup_{\nu=1}^M U_\nu \supset b\Omega$. Let $\{\chi_\nu\}$ be a partition of unity such that $\chi_\nu \in C_c^\infty(U_\nu)$, $0 \le \chi_\nu \le 1$, and $\sum_{\nu=0}^M \chi_\nu \equiv 1$ in some neighborhood of Ω. We can assume that for each $1 \le \nu \le M$ there exists an invertible affine linear transformation $\psi_\nu : \mathbb{B}^N \to U_\nu$, where $\mathbb{B}^N$ is the unit ball in $\mathbb{R}^N$, such that

$$\psi_\nu(\mathbb{B}^N \cap \omega_\nu) = \mathcal{U}_\nu \cap \Omega, \quad 1 \le \nu \le M.$$

Here $\omega_\nu = \{y_N > \rho_\nu(y')\}$ are special Lipschitz domains. For $y \in \mathbb{B}^N \cap \omega_\nu$, $\delta \circ \psi_\nu(y) \approx \delta_\nu(y) := y_N - \rho_\nu(y')$, thus

$$\begin{aligned}\|\delta^j v\|_{L^p(\Omega)} &\lesssim \|\chi_0 \delta^j v\|_{L^p(\Omega)} + \sum_{\nu=1}^M \|\chi_\nu \delta^j v\|_{L^p(\Omega\cap\mathcal{U}_\nu)} \\ &\lesssim \|\chi_0 v\|_{L^p(\Omega)} + \sum_{\nu=1}^M \|\delta_\nu^j [(\chi_\nu v)\circ\psi_\nu]\|_{L^p(\omega_\nu\cap\mathbb{B}^N)}.\end{aligned}$$

Clearly $\|\chi_0 v\|_{L^p(\Omega)} \lesssim \|\delta^{j+1}\chi_0 v\|_{L^p(\Omega)} \le \|\delta^{j+1} v\|_{L^p(\Omega)}$. By Lemma 5.1 (ii), we have for $1 \le \nu \le M$,

$$\begin{aligned}\|\delta_\nu^j [(\chi_\nu v)\circ\psi_\nu]\|_{L^p(\mathbb{B}^N\cap\omega_\nu)} &\lesssim \|\delta_\nu^{j+1} D[(\chi_\nu v)\circ\psi_\nu]\|_{L^p(\mathbb{B}^N\cap\omega_\nu)} \\ &\lesssim \|\delta_\nu^{j+1} D(\chi_\nu v)\circ\psi_\nu\|_{L^p(\mathbb{B}^N\cap\omega_\nu)} \\ &\lesssim \|\delta^{j+1} D(\chi_\nu v)\|_{L^p(U_\nu\cap\Omega)} \\ &\lesssim \|\delta^{j+1} v\|_{L^p(U_\nu\cap\Omega)} + \|\delta^{j+1} Dv\|_{L^p(U_\nu\cap\Omega)} \\ &\lesssim \|\delta^{j+1} v\|_{L^p(\Omega)} + \|\delta^{j+1} Dv\|_{L^p(\Omega)}.\end{aligned}$$

Taking sum over $0 \le \nu \le M$, this proves (5.3) and thus the proposition. $\square$

Lemma 5.3. *Let Ω be a bounded Lipschitz domain. Denote $\delta(x) = \operatorname{dist}(x, b\Omega)$. Then for any $k \in \mathbb{N}$ and $1 < p < \infty$ there is a $C = C_{k,p,\Omega} > 0$ such that*

$$\|\delta^{-k} f\|_{L^p(\Omega)} \le C\|f\|_{H_0^{k,p}(\Omega)}.$$

Proof. We only need to prove the statement for special Lipschitz domain $\omega = \{(x', x_N) \in \mathbb{R}^N : x_N > \rho(x')\}$ and for $f \in H_0^{k,p}(\omega)$ which is supported in $\mathbb{B}^N$, namely,

$$\|\delta^{-k} f\|_{L^p(\omega)} \le C\|f\|_{H_0^{k,p}(\omega)}, \quad \forall f \in H_0^{k,p}(\omega), \quad \operatorname{supp} f \subseteq \mathbb{B}^N. \tag{5.4}$$

For a bounded Lipschitz domain one can use partition of unity and the result for special Lipschitz domains. We leave the reader to check the details.

The case $k = 0$ is trivial, so we assume $k > 0$. Since $H_0^{k,p}(\omega)$ is the completion of $C_c^\infty(\omega)$ under the norm $H^{k,p}(\mathbb{R}^N)$ (see Definition 2.6 (iii)), it suffices to prove (5.4) for $f \in C_c^\infty(\omega)$ with uniform bounds. Indeed, for a general $f \in H^{k,p}(\omega)$, take $(f_j)_{j=1}^\infty \subset C_c^\infty(\omega)$ such that $\|f_j - f\|_{H_0^{k,p}(\omega)} \to 0$, so then $(\delta^{-k} f_j)_{j=1}^\infty \subset L^p(\omega)$ is a Cauchy sequence, and $\|\delta^{-k} f\|_{L^p(\omega)} \le C\|f\|_{H_0^{k,p}(\omega)}$ with the same constant.

Since $\|\rho\|_{C^{0,1}} < 1$, we know that $\frac{1}{2}\delta(x) \le |x_N - \rho(x')| \le 2\delta(x)$ for all $x \in \mathbb{R}^N$, so we can replace $\delta(x)$ by $|x_N - \rho(x')|$.

Let $g(t) \in C_c^\infty(0, 2)$. By Taylor's theorem

$$g(t) = \frac{1}{(k-1)!}\int_0^t g^{(k)}(s)(t-s)^{k-1} ds, \quad t > 0.$$

Therefore

$$\begin{aligned}
\|t^{-k} g(t)\|_{L_t^p(\mathbb{R}_+)} &\le \frac{1}{(k-1)!}\Big\|\frac{1}{t}\int_0^t |g^{(k)}(s)| ds\Big\|_{L_t^p(\mathbb{R}_+)} \\
&= \frac{1}{(k-1)!}\Big\|\int_0^2 |g^{(k)}(\lambda t)| d\lambda\Big\|_{L^p(\mathbb{R}_+)} \\
&\le \frac{1}{(k-1)!}\int_0^2 \|g^{(k)}(\lambda\cdot)\|_{L^p(\mathbb{R}_+)} d\lambda \\
&= \frac{1}{(k-1)!}\int_0^2 \|g^{(k)}\|_{L^p(\mathbb{R}_+)} \lambda^{-\frac{1}{p}} d\lambda \\
&= \frac{p/(p-1)}{(k-1)!}\|g^{(k)}\|_{L^p(\mathbb{R}_+)}.
\end{aligned}$$

Now for each $x' \in \mathbb{R}^{N-1}$, set $g_{x'}(t) := f(x', t + \rho(x'))$ so $g_{x'}^{(k)}(t) = \partial_t^k f(x', t + \rho(x')) = (\partial_{x_N}^k f)(x', t + \rho(x'))$, we see that $\operatorname{supp} g \subseteq [0, 2)$ since $\operatorname{supp} f \subseteq \mathbb{B}^N$. By Fubini theorem we have

$$\begin{aligned}
\int_\omega \Big| |x_N - \rho(x')|^{-k} f(x', x_N)\Big|^p dV(x) &= \int_{\mathbb{R}^{N-1}} dx' \int_0^\infty |t^{-k} f(x', t + \rho(x'))|^p dt \\
&= \int_{\mathbb{R}^{N-1}} \|t^{-k} g_{x'}(t)\|_{L_t^p(\mathbb{R}_+)}^p dx' \\
&\le C_{k,p}\int_{\mathbb{R}^{N-1}} \|g_{x'}^{(k)}\|_{L^p(\mathbb{R}_+)}^p dx' \\
&= C_{k,p}\int_{\mathbb{R}^{N-1}} \left(\int_{\mathbb{R}_+} |(\partial_{x_N}^k f)(x', t + \rho(x'))|^p dt\right) dx' \\
&= C_{k,p}\int_\omega |\partial_{x_N}^k f(x)|^p dx \le C_{k,p}\int_\omega |D^k f|^p \, dV(x).
\end{aligned}$$

Thus we have $\|\delta^{-k} f\|_{L^p(\omega)} \lesssim \|D^k f\|_{L^p(\omega)} \le \|f\|_{W^{k,p}(\omega)} = \|f\|_{W^{k,p}(\mathbb{R}^N)}$ uniformly for all $f \in C_c^\infty(\omega)$.

Note that by Lemma 2.10, $H^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$ with equivalent norm, and by density of $C_c^\infty(\omega)$ in $H_0^{k,p}(\omega)$ (see Definition 2.6 (iii)) we conclude that $\|\delta^{-k}f\|_{L^p(\omega)} \lesssim \|f\|_{H^{k,p}(\mathbb{R}^N)}$ for all $f \in H_0^{k,p}(\omega)$. □

Proposition 5.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain. Denote $\delta(x) = \operatorname{dist}(x, b\Omega)$. Then for $s \geq 0$ and $1 < p < \infty$, there is a $C = C(s, p, \Omega) > 0$ such that*

$$\|u\|_{H^{-s,p}(\Omega)} \leq C\|\delta^s u\|_{L^p(\Omega)}, \quad \forall u \in L^p_{\mathrm{loc}}(\Omega). \tag{5.5}$$

Remark 5.5. For the special case when $p = 2$, $s \geq 0$ is not a half integer, and Ω has smooth boundary, the above result is proved in [CS01, Theorem C.4].

Proof. Note that the estimate is equivalent to showing the boundedness of the inclusion operator

$$\iota : L^p(\Omega, \delta^s) \hookrightarrow H^{-s,p}(\Omega)$$

for $s \geq 0$ and $1 < p < \infty$. We will argue by duality and interpolation.

Let p' be the conjugate of p. By Proposition 5.3, for $k \in \mathbb{N}$,

$$\|g\|_{L^{p'}(\Omega,\delta^{-k})} \leq C_{k,p'}\|g\|_{H_0^{k,p'}(\Omega)}, \quad \forall g \in H_0^{k,p'}(\Omega),$$

is a bounded operator. By Proposition 2.12 (iii), we have $H^{-k,p}(\Omega) = H_0^{k,p'}(\Omega)'$. Using Hölder's inequality, we have for every $f \in L^p(\Omega, \delta^k)$,

$$\begin{aligned}
\|f\|_{H^{-k,p}(\Omega)} &= \sup_{g \in H_0^{k,p'}(\Omega); \|g\|_{H_0^{k,p'}(\Omega)} \leq 1} \langle f, g\rangle \\
&\leq \sup_{g \in L^{p'}(\Omega,\delta^{-k}); \|g\|_{L^{p'}(\Omega,\delta^{-k})} \leq C_{k,p'}} \int_\Omega |fg| \\
&= \sup_{\|\delta^{-k}g\|_{L^{p'}(\Omega)} \leq C_{k,p'}} \int_\Omega |\delta^k f||\delta^{-k}g| \\
&\leq \sup_{\|\delta^{-k}g\|_{L^{p'}(\Omega)} \leq C_{k,p'}} \|\delta^k f\|_{L^p(\Omega)}\|\delta^{-k}g\|_{L^{p'}(\Omega)} \\
&\leq C_{k,p'}\|\delta^k f\|_{L^p(\Omega)} = C_{k,p'}\|f\|_{L^p(\Omega,\delta^k)}.
\end{aligned}$$

Hence the inclusion $\iota : L^p(\Omega, \delta^k) \hookrightarrow H^{-k,p}(\Omega)$ is bounded for $k \in \mathbb{N}$.

For general $s > 0$, take any integer $k > s$ and denote $\theta = s/k$. By Proposition 2.15 we have

$$[L^p(\Omega), L^p(\Omega, \delta^k)]_\theta = L^p(\Omega, \delta^s),$$

and

$$[L^p(\Omega), H^{-k,p}(\Omega)]_\theta = H^{-s,p}(\Omega).$$

Using interpolation we obtain the boundedness of inclusion $\iota : L^p(\Omega, \delta^s) \to H^{-s,p}(\Omega)$. □

We now use Proposition 5.4 to extend Lemma 5.3 to all $s \geq 0$.

Proposition 5.6. *Let Ω be a bounded Lipschitz domain. Denote $\delta(x) = \operatorname{dist}(x, b\Omega)$. Then for any $s \geq 0$ and $1 < p < \infty$ there is a $C = C_{s,p,\Omega} > 0$ such that $\|\delta^{-s}f\|_{L^p(\Omega)} \leq C\|f\|_{H_0^{s,p}(\Omega)}$.*

Proof. Since the dual space of $L^p(\Omega, \delta^{-s})$ is $L^{p'}(\Omega, \delta^s)$, we have

$$\|\delta^{-s}f\|_{L^p(\Omega)} = \sup_{\substack{g \in L^{p'}(\Omega,\delta^s), \\ \|g\|_{L^{p'}(\Omega,\delta^s)} \leq 1}} |\langle f, g\rangle|.$$

Since $L^{p'}(\Omega,\delta^s)\subset H^{-s,p'}(\Omega)$ by Proposition 5.4 and since $H_0^{s,p}(\Omega)=H^{-s,p'}(\Omega)'$ from Proposition 2.12 (iii), we have

$$\|\delta^{-s}f\|_{L^p(\Omega)}\leq\sup_{\substack{g\in H^{-s,p'}(\Omega),\\ \|g\|_{H^{-s,p'}(\Omega)}\leq C}}|\langle f,g\rangle|\approx\|f\|_{H_0^{s,p}(\Omega)},$$

which completes the proof. □

Remark 5.7. Proposition 5.6 holds for Hölder-Zygmund space as well. Namely, for any $s\geq0$, there exists a $C=C_{s,\Omega}>0$ such that $\|\delta^{-s}f\|_{L^\infty(\Omega)}\leq C\|f\|_{\Lambda_0^s(\Omega)}$. Here $\Lambda_0^s(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ in $\Lambda^s(\overline{\Omega})$.

As a consequence to Proposition 5.6, we can now prove a weighted estimate for the commutator $[D,\mathcal{E}]$. Recall that for a bounded Lipschitz domain Ω, we define the extension operator $\mathcal{E}_\Omega$ by formula (3.3),

$$\mathcal{E}_\Omega f=\chi_0 f+\sum_{\nu=1}^M\chi_\nu E_\nu(\chi_\nu f),\tag{5.6}$$

where $E_\nu g:=E_{\omega_\nu}(g\circ\Phi_\nu)\circ\Phi_\nu^{-1}$, and $\omega_\nu,U_\nu,\Phi_\nu$ are given in the remark before (3.3).

Lemma 5.8. *Let $\Omega\subseteq\mathbb{R}^N$ be a bounded Lipschitz domain, and let $\mathcal{E}$ be defined as above. Then for $1<p<\infty$, we have*

$$\|\delta^{1-s}[D,\mathcal{E}]f\|_{L^p(\overline{\Omega}^c)}\leq C_{s,p}\|f\|_{H^{s,p}(\Omega)},\quad\forall f\in H^{s,p}(\Omega),\quad s>0;$$
$$\|\delta^{1-s}[D,\mathcal{E}]f\|_{L^\infty(\overline{\Omega}^c)}\leq C_{s,p}\|f\|_{\Lambda^s(\Omega)},\quad\forall f\in\Lambda^s(\Omega),\quad s>0$$

Proof. We have

$$\begin{aligned}[D,\mathcal{E}]f&=D\left[(\chi_0 f)+\sum_{\nu=1}^M\chi_\nu E_\nu(\chi_\nu f)\right]-\left[\chi_0(Df)+\sum_{\nu=1}^M\chi_\nu E_\nu(\chi_\nu(Df))\right]\\&=(D\chi_0)f+\sum_{\nu=1}^M D(\chi_\nu)E_\nu(\chi_\nu f)+\chi_\nu DE_\nu(\chi_\nu f)-\chi_\nu E_\nu(\chi_\nu(Df))\\&=(D\chi_0)f+\sum_{\nu=1}^M D(\chi_\nu)E_\nu(\chi_\nu f)+\chi_\nu[D,E_\nu](\chi_\nu f)+\chi_\nu E_\nu D(\chi_\nu f)-\chi_\nu E_\nu(\chi_\nu(Df))\\&=(D\chi_0)f+\sum_{\nu=1}^M\chi_\nu[D,E_\nu](\chi_\nu f)+\sum_{\nu=1}^M D(\chi_\nu)E_\nu(\chi_\nu f)-\chi_\nu E_\nu(D(\chi_\nu)f)\end{aligned}$$

The first term above is identically 0 in $\overline{\Omega}^c$. For the last two terms, we note that $g:=D(\chi_\nu)E_\nu(\chi_\nu f)-\chi_\nu E_\nu(D(\chi_\nu)f)\equiv0$ in Ω. Moreover, by the estimate for E_ν we have $\|g\|_{H^{s,p}(\mathbb{R}^N)}\leq C_{s,p}\|f\|_{H^{s,p}(\Omega)}$. Hence $g\in H_0^{s,p}(\overline{\Omega}^c)$ by Proposition 2.12 (ii). It now follows from Proposition 5.6 and Remark 5.7 that

$$\|\delta^{-s}g\|_{L^p(\overline{\Omega}^c)}\leq C_{s,p}\|g\|_{H_0^{s,p}(\overline{\Omega}^c)}\leq C_{s,p}\|f\|_{H^{s,p}(\Omega)},\quad s\geq0;$$
$$\|\delta^{-s}g\|_{L^\infty(\overline{\Omega}^c)}\leq C_s\|g\|_{\Lambda_0^s(\overline{\Omega}^c)}\leq C_s\|f\|_{\Lambda^s(\overline{\Omega})},\quad s\geq0.$$

To finish the proof we will show that

$$\|\delta^{1-s}[D,E_\nu](\chi_\nu f)\|_{L^p(U_\nu\cap\overline{\Omega}^c)}\leq C_{s,p}\|f\|_{H^{s,p}(\Omega)},\quad s\in\mathbb{R};\tag{5.7}$$

$$\|\delta^{1-s}[D,E_\nu](\chi_\nu f)\|_{L^\infty(U_\nu\cap\overline{\Omega}^c)}\leq C_{s,p}\|f\|_{\Lambda^s(\Omega)},\quad s\in\mathbb{R}.\tag{5.8}$$

Indeed we have

$$
\begin{aligned}
[D,E_\nu](\chi_\nu f) &= D(E_\nu(\chi_\nu f)) - E_\nu(D(\chi_\nu f)) \\
&= D(E_{\omega_\nu}[(\chi_\nu f)\circ\Phi_\nu]\circ\Phi_\nu^{-1}) - E_{\omega_\nu}[D(\chi_\nu f)\circ\Phi_\nu]\circ\Phi_\nu^{-1} \\
&= (\nabla E_{\omega_\nu}[(\chi_\nu f)\circ\Phi_\nu])\circ\Phi_\nu^{-1}\cdot D\Phi_\nu^{-1} - E_{\omega_\nu}[\nabla((\chi_\nu f)\circ\Phi_\nu)]\circ\Phi_\nu^{-1}\cdot D\Phi_\nu^{-1} \\
&\quad + E_{\omega_\nu}[\nabla((\chi_\nu f)\circ\Phi_\nu)]\circ\Phi_\nu^{-1}\cdot D\Phi_\nu^{-1} - E_{\omega_\nu}[D(\chi_\nu f)\circ\Phi_\nu]\circ\Phi_\nu^{-1} \\
&= [\nabla, E_{\omega_\nu}]((\chi_\nu f)\circ\Phi_\nu)\circ\Phi_\nu^{-1}\cdot D\Phi_\nu^{-1}.
\end{aligned}
$$

Note that in the last step above we used the fact that Φ_ν is a linear transformation so that

$$
\begin{aligned}
E_{\omega_\nu}[\nabla((\chi_\nu f)\circ\Phi_\nu)]\circ\Phi_\nu^{-1}\cdot D\Phi_\nu^{-1} &= E_{\omega_\nu}[\nabla(\chi_\nu f)\circ\Phi_\nu]\circ\Phi_\nu^{-1}\cdot D(\Phi_\nu\cdot\Phi_\nu^{-1}) \\
&= E_{\omega_\nu}[D(\chi_\nu f)\circ\Phi_\nu]\circ\Phi_\nu^{-1}.
\end{aligned}
$$

Applying Theorem 4.1 to the domain ω_ν, we have for any $s\in\mathbb{R}$:

$$
\begin{aligned}
\|\delta^{1-s}[D,E_\nu](\chi_\nu f)\|_{L^p(U_\nu\cap\overline{\Omega}^c)} &\lesssim \|\delta^{1-s}[D,E_{\omega_\nu}]((\chi_\nu f)\circ\Phi_\nu)\|_{L^p(\Phi_\nu^{-1}(U_\nu\cap\overline{\Omega}^c))} \\
&= \|\delta^{1-s}[D,E_{\omega_\nu}]((\chi_\nu f)\circ\Phi_\nu)\|_{L^p(\mathbb{B}^N\cap\overline{\omega}_\nu^c)} \\
&\lesssim \|(\chi_\nu f)\circ\Phi_\nu\|_{H^{s,p}(\omega_\nu)} \\
&\lesssim \|f\|_{H^{s,p}(\Omega)},
\end{aligned}
$$

where we used that $U_\nu = \Phi_\nu(\mathbb{B}^N)$ and $\Phi_\nu(\mathbb{B}^N\cap\omega_\nu) = U_\nu\cap\Omega$. This proves (5.7).

In a similar way, we obtain (5.8) from Corollary 4.3. The proof is now complete. □

Proposition 5.9. *Let $\Omega\subseteq\mathbb{R}^N$ be a bounded domain with C^2 boundary. Then for $1<p<\infty$ and $0\le r\le 2$, there is a $C=C_{\Omega,r,p}>0$ such that*

(i) $\|f\|_{H^{r,p}(\Omega)} \le C(\|\delta^{1-r}f\|_{L^p(\Omega)} + \|\delta^{1-r}Df\|_{L^p(\Omega)})$ *for $0\le r\le 1$ and $f\in W^{1,p}(\Omega,\delta^{1-r})$.*

(ii) $\|f\|_{H^{r,p}(\Omega)} \le C(\|\delta^{2-r}f\|_{L^p(\Omega)} + \|\delta^{2-r}Df\|_{L^p(\Omega)} + \|\delta^{2-r}D^2f\|_{L^p(\Omega)})$ *for $1\le r\le 2$ and $f\in W^{2,p}(\Omega,\delta^{2-r})$.*

Proof. (i) Since $b\Omega$ is C^2 and $0\le s\le 1$, we can apply Proposition A.1 (i) to get

$$\|f\|_{H^{s,p}(\Omega)} \lesssim \|f\|_{H^{s-1,p}(\Omega)} + \|Df\|_{H^{s-1,p}(\Omega)}.$$

Now $s-1\le 0$ so Proposition 5.4 applies, and we have

$$
\begin{aligned}
\|f\|_{H^{s-1,p}(\Omega)} &\lesssim \|\delta^{1-s}f\|_{L^p(\Omega)}; \\
\|Df\|_{H^{s-1,p}(\Omega)} &\lesssim \|\delta^{1-s}Df\|_{L^p(\Omega)}.
\end{aligned}
$$

Combining them we get $\|f\|_{H^{s,p}(\Omega)} \lesssim \|\delta^{1-s}f\|_{L^p(\Omega)} + \|\delta^{1-s}Df\|_{L^p(\Omega)}$, which proves (i).

(ii) Since $1\le s\le 2$ we have $s, s-1\in[0,2]$. So by Proposition A.1 (ii),

$$\|f\|_{H^{s,p}(\Omega)} \lesssim \|f\|_{H^{s-2,p}(\Omega)} + \|Df\|_{H^{s-2,p}(\Omega)} + \|D^2f\|_{H^{s-2,p}(\Omega)}.$$

Since $s-2\le 0$, we again apply (5.4) to get

$$\|D^jf\|_{H^{s-2,p}(\Omega)} \lesssim \|\delta^{2-s}D^jf\|_{L^p(\Omega)}, \quad j=0,1,2.$$

Thus

$$\|f\|_{H^{s,p}(\Omega)} \lesssim \|\delta^{2-s}f\|_{L^p(\Omega)} + \|\delta^{2-s}Df\|_{L^p(\Omega)} + \|\delta^{2-s}D^2f\|_{L^p(\Omega)},$$

which proves (ii). □

6. Sobolev Estimates of Homotopy Operators

In this section we derive the weighted estimates for the homotopy operator. Together with the commutator estimate and Hardy-Littlewood lemma, this leads to the proof of Theorem 1.1. Unlike in [Gon19] and [Shi21], no integration by parts is used in our proof.

In what follows we let ρ be a C^2 defining function of Ω which is strictly plurisubharmonic in a neighborhood of $b\Omega$. We will adopt the following notation:

$$\Omega_\varepsilon = \{z \in \mathbb{C}^n : \operatorname{dist}(z,\Omega) < \varepsilon\}, \quad \Omega_{-\varepsilon} = \{z \in \Omega : \operatorname{dist}(z, b\Omega) > \varepsilon\}.$$

Proposition 6.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with C^2 boundary. Suppose $W(z,\zeta) \in C^1(\Omega_\varepsilon \times (\Omega_\varepsilon \setminus \overline{\Omega_{-\varepsilon}}))$ is a Leray mapping, that is, W is holomorphic in $z \in \Omega_\varepsilon$ and satisfies*

$$\Phi(z,\zeta) := W(z,\zeta)\cdot(\zeta - z) \neq 0, \quad z \in \Omega, \quad \zeta \in \Omega_\varepsilon \setminus \overline{\Omega}.$$

Let $\mathcal{U}$ be a bounded neighborhood of $\overline{\Omega}$ such that $U \subset \Omega_\varepsilon$. Suppose φ is a $(0,q)$-form with $1 \le q \le n$ such that φ and $\overline{\partial}\varphi$ are in $C^\infty(\overline{\Omega})$. Then

$$\varphi = \overline{\partial}\mathcal{H}_q\varphi + \mathcal{H}_{q+1}\overline{\partial}\varphi.$$

Here $\mathcal{H}_q$ is the operator defined by

$$\mathcal{H}_q\varphi = \int_{\mathcal{U}} K^0_{0,q-1} \wedge E\varphi + \int_{\mathcal{U}\setminus\overline{\Omega}} K^{01}_{0,q-1} \wedge [\overline{\partial}, E]\varphi, \tag{6.1}$$

where E is any extension operator that maps $C^\infty(\overline{\Omega})$ into $C^\infty(\mathbb{C}^n)$ with $\operatorname{supp} E\varphi \subseteq \mathcal{U}$ for all φ, and

$$K^0(z,\zeta) = \frac{1}{(2\pi i)^n}\frac{\langle \overline{\zeta} - \overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2} \wedge \left(\overline{\partial}_{\zeta,z}\frac{\langle \overline{\zeta} - \overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2}\right)^{n-1}, \quad \overline{\partial}_{\zeta,z} = \overline{\partial}_\zeta + \overline{\partial}_z;$$

$$\begin{aligned} K^{0,1}(z,\zeta) &= \frac{1}{(2\pi i)^n}\frac{\langle \overline{\zeta} - \overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2} \wedge \frac{\langle W, d\zeta\rangle}{\langle W\,,\, \zeta - z\rangle} \\ &\quad \wedge \sum_{i+j=n-2}\left[\frac{\langle d\overline{\zeta} - d\overline{z}\,,\, d\zeta\rangle}{|\zeta - z|^2}\right]^i \wedge \left[\overline{\partial}_{\zeta,z}\frac{\langle W, d\zeta\rangle}{\langle W, \zeta - z\rangle}\right]^j. \end{aligned} \tag{6.2}$$

We set $K^1_{0,-1} = 0$ and $K^{0,1}_{0,-1} = 0$.

The reader can find the proof of Proposition 6.1 in [Gon19], where E is taken to be Stein extension operator. Here we note that on any bounded strictly pseudoconvex domain Ω with C^2 boundary, there exists an $\varepsilon > 0$ such that W satisfies the assumptions in Proposition 6.1 on $\Omega_\varepsilon \times (\Omega_\varepsilon \setminus \overline{\Omega_{-\varepsilon}})$. Furthermore, near every $\zeta^* \in b\Omega$, one can find a small neighborhood $\mathcal{V}$ of ζ^* such that for all $z \in \mathcal{V}$, there exists a coordinate map $\phi_z : \mathcal{V} \to \mathbb{R}^{2n}$ given by $\phi_z : \zeta \mapsto (s,t) = (s_1, s_2, t_3, \dots, t_{2n})$, where $s_1 = \rho(\zeta)$ (we have $s_1 \approx \delta(\zeta)$ for $\zeta \in \mathcal{V} \setminus \Omega$). Moreover for $z \in \mathcal{V} \cap \Omega$, $\zeta \in \mathcal{V} \setminus \overline{\Omega}$, the function $\Phi(z,\zeta) = W(z,\zeta)\cdot(\zeta - z)$ satisfies

$$|\Phi(z,\zeta)| \ge c\left(\delta(z) + s_1 + |s_2| + |t|^2\right), \quad \delta(z) = \operatorname{dist}(z, b\Omega), \tag{6.3}$$

$$|\Phi(z,\zeta)| \ge c|z - \zeta|^2, \qquad |\zeta - z| \ge c(s_1 + |s_2| + |t|), \tag{6.4}$$

for some constant c depending on the domain. We call such Φ a holomorphic support function. The reader can refer to [Gon19] for details.

From now on we shall fix an open set $\mathcal{U}$ such that $\Omega \subset\subset \mathcal{U} \subset\subset \Omega_\varepsilon$, and we will use the extension operator $\mathcal{E}$ defined by formula (3.3) with $\operatorname{supp}\mathcal{E}\varphi \subset \mathcal{U}$ for all φ.

Theorem 6.2. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2 boundary. Given $1 \le q \le n$, let $\mathcal{H}_q$ be defined as in (6.1), where the extension operator $\mathcal{E}$ is given by formula (3.3). Then for any $1 < p < \infty$ and $s > \frac{1}{p}$, $\mathcal{H}_q$ is a bounded linear operator $\mathcal{H}_q : H^{s,p}_{(0,q)}(\Omega) \to H^{s+\frac{1}{2},p}_{(0,q-1)}(\Omega)$.*

Remark 6.3. In fact, when $q = n$, then any extension of φ is automatically $\overline{\partial}$ closed, so $[\overline{\partial}, E]\varphi \equiv 0$ and

$$\mathcal{H}_n \varphi = \int_{\mathcal{U}} K^0_{0,q-1} \wedge E\varphi.$$

In this case for all $s \ge 0$ and $1 < p < \infty$, $\mathcal{H}_n : H^{s,p}_{(0,n)}(\Omega) \to H^{s+1,p}_{(0,n-1)}(\Omega)$ where Ω is any bounded Lipschitz domain. See Proposition 6.6.

Theorem 6.2 allows us to prove a homotopy formula under much weaker regularity assumption.

Theorem 6.4 (Homotopy formula)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2 boundary. Given $1 < p < \infty$ and $1 \le q \le n$, suppose $\varphi \in H^{s,p}_{(0,q)}(\Omega)$ satisfies $\overline{\partial}\varphi \in H^{s,p}_{(0,q+1)}(\Omega)$ where $s > \frac{1}{p}$. Let $\mathcal{H}_q$ be defined by (6.1), where the extension operator $E = \mathcal{E}_\Omega$ is given by formula (3.3). Then the following homotopy formula holds in the sense of distributions:*

$$\varphi = \overline{\partial}\mathcal{H}_q\varphi + \mathcal{H}_{q+1}\overline{\partial}\varphi. \tag{6.5}$$

In particular for a $\overline{\partial}$-closed φ which is in $H^{s,p}_{(0,q)}(\Omega)$ for $s > \frac{1}{p}$, the equation $\overline{\partial}\mathcal{H}_q\varphi = \varphi$ holds and $\mathcal{H}_q\varphi \in H^{s+\frac{1}{2},p}_{(0,q-1)}(\Omega)$.

Proof. Formula (6.5) is proved in [Gon19] for Stein's extension operator for $\varphi, \overline{\partial}\varphi \in C^1(\overline{\Omega})$. The statement holds for smooth forms by Proposition 6.1, and we shall use approximation for the general case. First we show that there exists a sequence $\varphi_\varepsilon \in C^\infty(\overline{\Omega})$ such that

$$\varphi_\varepsilon \xrightarrow{\varepsilon\to 0} \varphi \quad \text{in } H^{s,p}(\Omega),$$
$$\overline{\partial}\varphi_\varepsilon \xrightarrow{\varepsilon\to 0} \overline{\partial}\varphi \quad \text{in } H^{s,p}(\Omega).$$

The smoothing is done componentwise, and for simplicity we will continue to denote the coefficient functions of φ by φ. Take an open covering $\{U_\nu\}_{\nu=0}^M$ of Ω such that

$$U_0 \subset\subset \Omega, \quad b\Omega \subseteq \bigcup_{\nu=1}^M U_\nu, \quad U_\nu \cap \Omega = U_\nu \cap \Phi_\nu(\{x_N > \rho_\nu(x')\}), \quad \nu = 1, \dots, M.$$

Here Φ_ν, $1 \le \nu \le M$ are some invertible affine linear transformations. Let χ_ν be a partition of unity associated with $\{U_\nu\}_{\nu=0}^\infty$, i.e. $\chi_\nu \in C^\infty_c(U_\nu)$ and $\sum_{\nu=0}^M \chi_\nu = 1$.

Let $\mathbb{B}^{2n}$ be the unit ball in $\mathbb{C}^n$ and $B(0,r)$ be the ball centered at 0 with radius r. For each $1 \le \nu \le M$, we can find an open cone K_ν and some ε_ν such that $(U_\nu \cap \Omega) + (K_\nu \cap B(0,\varepsilon_\nu)) \subseteq \Omega$.

Take $\psi_0 \in C^\infty_c(\mathbb{B}^{2n})$ with $\psi_0 \ge 0$ and $\int_{\mathbb{C}^n} \psi_0 = 1$. For $1 \le \nu \le N$, take $\psi_\nu \in C^\infty_c(-K_\nu)$ with $\psi_\nu \ge 0$ and $\int_{\mathbb{C}^n} \psi_\nu = 1$. Write $\psi_{\nu,\varepsilon}(x) = \varepsilon^{-2n}\psi_\nu(\frac{x}{\varepsilon})$. For $\varepsilon > 0$ sufficiently small, we can define

$$(\chi_0\varphi) * \psi_{0,\varepsilon}(z) = \int_{\mathbb{B}^N} (\chi_0\varphi)(z - \varepsilon\zeta)\psi_0(\zeta)\, dV(\zeta), \quad z \in U_0,$$
$$(\chi_\nu\varphi) * \psi_{\nu,\varepsilon}(z) = \int_{-K} (\chi_\nu\varphi)(z - \varepsilon\zeta)\psi_\nu(\zeta)\, dV(\zeta), \quad z \in U_\nu \cap \Omega, \quad \nu = 1, \dots, M.$$

We now set $\varphi_\varepsilon := \sum_{\nu=0}^M (\chi_\nu\varphi) * \psi_\nu^\varepsilon \in C^\infty(\overline{\Omega})$. Clearly $\|\varphi_\varepsilon - \varphi\|_{H^{s,p}(\Omega)} \to 0$ since $\|(\chi_\nu\varphi) * \psi_{\nu,\varepsilon} - \chi_\nu\varphi\|_{H^{s,p}(\Omega)} \xrightarrow{\varepsilon\to 0} 0$ for each $0 \le \nu \le M$. Meanwhile,

$$\overline{\partial}\varphi_\varepsilon = \sum_{\nu=0}^M \psi_\nu^\varepsilon * \left(\varphi\overline{\partial}\chi_\nu + \chi_\nu\overline{\partial}\varphi\right).$$

By assumption, both $\varphi\overline{\partial}\chi_\nu$ and $\chi_\nu\overline{\partial}\varphi$ are in $H^{s,p}(\Omega)$, so $\psi_\nu^\varepsilon * \left(\overline{\partial}\chi_\nu\varphi + \chi_\nu\overline{\partial}\varphi\right)$ converges to $\chi_\nu\overline{\partial}\varphi + \overline{\partial}\chi_\nu\varphi$ in $H^{s,p}(\Omega)$ as $\varepsilon \to 0$. Taking the sum over ν we see that $\|\overline{\partial}\varphi_\varepsilon - \overline{\partial}\varphi\|_{H^{s,p}(\Omega)} \to 0$.

Now (6.5) holds for φ replaced with φ_ε. By Theorem 6.2,

$$\begin{aligned}\|\overline{\partial}\mathcal{H}_q(\varphi_\varepsilon - \varphi)\|_{H^{s-\frac{1}{2},p}(\Omega)} &\leq \|\mathcal{H}_q(\varphi_\varepsilon - \varphi)\|_{H^{s+\frac{1}{2},p}(\Omega)} \\ &\leq \|\varphi_\varepsilon - \varphi\|_{H^{s,p}(\Omega)},\end{aligned}$$

and also

$$\|\mathcal{H}_{q+1}\overline{\partial}(\varphi_\varepsilon - \varphi)\|_{H^{s+\frac{1}{2},p}(\Omega)} \leq \|\overline{\partial}(\varphi_\varepsilon - \varphi)\|_{H^{s,p}(\Omega)}.$$

Then (6.5) follows by taking $\varepsilon \to 0$.

Note that if $\varphi \in H^{s,p}(\Omega)$ is $\overline{\partial}$-closed, then the distribution $\overline{\partial}\varphi (\equiv 0)$ is in $H^{s,p}(\Omega)$. Therefore (6.5) holds for this φ. In particular $u = \mathcal{H}_q\varphi \in H^{s+\frac{1}{2},p}_{(0,q-1)}$ and $\overline{\partial}u = \varphi$. □

First we prove a lemma which will be useful later.

Lemma 6.5. *Let $n \geq 2$, $\beta \geq 0$, $\alpha > -1$, and let $0 < \delta < \frac{1}{2}$. If $\alpha < \beta - \frac{1}{2}$, then*

$$\int_0^1\int_0^1\int_0^1 \frac{s_1^\alpha t^{2n-3}\, ds_1\, ds_2\, dt}{(\delta + s_1 + s_2 + t^2)^{2+\beta}(\delta + s_1 + s_2 + t)^{2n-3}} \leq C\delta^{\alpha-\beta+\frac{1}{2}}.$$

Proof. Partition the domain of integration into seven regions:

$R_1 : t > t^2 > \delta, s_1, s_2$. We have

$$I \leq \int_{\sqrt{\delta}}^1 \frac{t^{2n-3}}{t^{4+2\beta}t^{2n-3}}\left(\int_0^{t^2} s_1^\alpha\, ds_1\right)\left(\int_0^{t^2} ds_2\right) dt \leq C\int_{\sqrt{\delta}}^1 t^{2\alpha-2\beta}\, dt \leq C\delta^{\alpha-\beta+\frac{1}{2}}.$$

$R_2 : t > \delta > t^2, s_1, s_2$. We have

$$I \leq \delta^{-2-\beta}\left(\int_\delta^{\sqrt{\delta}} \frac{t^{2n-3}}{t^{2n-3}}\, dt\right)\left(\int_0^\delta s_1^\alpha\, ds_1\right)\left(\int_0^\delta ds_2\right) \leq C\delta^{\alpha-\beta+\frac{1}{2}}.$$

$R_3 : t > s_1 > \delta, t^2, s_2$. We have

$$I \leq \int_\delta^1 \frac{s_1^\alpha}{s_1^{2+\beta}}\left(\int_0^{\sqrt{s_1}} \frac{t^{2n-3}}{t^{2n-3}}\, dt\right)\left(\int_0^{s_1} ds_2\right) ds_1 \leq C\int_\delta^1 s_1^{\alpha-\beta+\frac{1}{2}-1}\, ds_1 \leq C\delta^{\alpha-\beta+\frac{1}{2}}.$$

$R_4 : t > s_2 > \delta, t^2, s_1$. We have

$$I \leq \int_\delta^1 \frac{1}{s_2^{2+\beta}}\left(\int_0^{\sqrt{s_2}} \frac{t^{2n-3}}{t^{2n-3}}\, dt\right)\left(\int_0^{s_2} s_1^\alpha\, ds_1\right) ds_2 \leq C\int_\delta^1 s_2^{\alpha-\beta+\frac{1}{2}-1}\, ds_2 \leq C\delta^{\alpha-\beta+\frac{1}{2}}.$$

$R_5 : \delta > t, t^2, s_1, s_2$. We have

$$I \leq \delta^{-2-\beta}\delta^{-(2n-3)}\left(\int_0^\delta t^{2n-3}\, dt\right)\left(\int_0^\delta s_1^\alpha\, ds_1\right)\left(\int_0^\delta ds_2\right) \leq C\delta^{\alpha-\beta+1}.$$

$R_6 : s_1 > \delta, t, t^2, s_2$. We have

$$I \leq \int_\delta^1 \frac{s_1^\alpha}{s_1^{2+\beta}s_1^{2n-3}}\left(\int_0^{s_1} t^{2n-3}\, dt\right)\left(\int_0^{s_1} ds_2\right) ds_1 \leq C\int_\delta^1 s_1^{\alpha-\beta}\, ds_1.$$

$R_7 : s_2 > \delta, t, t^2, s_1$. We have

$$I \leq \int_\delta^1 \frac{1}{s_2^{2+\beta}s_2^{2n-3}}\left(\int_0^{s_2} t^{2n-3}\, dt\right)\left(\int_0^{s_2} s_1^\alpha\, ds_1\right) ds_2 \leq C\int_\delta^1 s_2^{\alpha-\beta}\, ds_2.$$

Here the constants depend only on n, α and β. For R_6 and R_7, we have

$$\int_\delta^1 r^{\alpha-\beta} dr \le \begin{cases} C, & \alpha-\beta > -1, \\ C(1+|\log\delta|), & \alpha-\beta=-1, \\ C\delta^{\alpha-\beta+1}, & \alpha-\beta<-1, \end{cases}$$

which is bounded by $C\delta^{\alpha-\beta+\frac{1}{2}}$ in all cases. □

We now write the homotopy operator $\mathcal{H}_q\varphi$ as

$$\mathcal{H}_q\varphi = \mathcal{H}_q^0\varphi + \mathcal{H}_q^1\varphi, \tag{6.6}$$

where

$$\mathcal{H}_q^0\varphi := \int_{\mathcal{U}} K_{0,q-1}^0 \wedge \mathcal{E}\varphi, \quad \mathcal{H}_q^1\varphi := \int_{\mathcal{U}\setminus\Omega} K_{0,q-1}^{01} \wedge [\overline{\partial},\mathcal{E}]\varphi.$$

For the operator $\mathcal{H}_q^0$, we can gain one derivative for any $\varphi \in H^{s,p}(\Omega)$, $s \ge 0$.

Proposition 6.6. *Let $1<p<\infty$ and $s\ge 0$. Suppose $\varphi \in H^{s,p}_{(0,q)}(\Omega)$ with $q\ge 1$. Then $\mathcal{H}_q^0\varphi$ is in $H^{s+1,p}_{(0,q-1)}(\mathcal{U})$, and*

$$\|\mathcal{H}_q^0\varphi\|_{H^{s+1,p}(\mathcal{U})} \lesssim \|\mathcal{E}\varphi\|_{H^{s,p}(\mathcal{U})} \lesssim \|\varphi\|_{H^{s,p}(\Omega)}.$$

Proof. The proof for integers s can be found in [Shi21, Proposition 3.2]. The general case follows from interpolation (see Propositions 2.14 and 2.15). □

Proposition 6.7. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2 boundary, and $1<p<\infty$. For $q\ge 1$, let $\mathcal{H}_q^1\varphi$ be given by (6.6), where the extension operator E is defined by formula (3.3). Suppose $\varphi \in H^{s,p}_{(0,q)}(\Omega)$ for $s > \frac{1}{p}$ and m is a positive integer such that $m-s-\frac{1}{2}>0$. Then there exists some constant $C=C(\Omega,p)$ such that*

$$\|\delta^{m-s-\frac{1}{2}} D^m \mathcal{H}_q^1\varphi\|_{L^p(\Omega)} < C\|\varphi\|_{H^{s,p}(\Omega)}.$$

Proof. Note that in view of Remark 4.2 (iii), $\mathcal{H}_q^1\varphi$ is C^∞ in the interior of Ω. To prove the statement we will show that

$$\|\delta^{m-s-\frac{1}{2}} D^m \mathcal{H}_q^1\varphi\|_{L^p(\Omega)} \le C\|\delta^{1-s}[\overline{\partial},\mathcal{E}]\varphi\|_{L^p(\mathcal{U}\setminus\overline{\Omega})}. \tag{6.7}$$

Then by Theorem 4.1 the right-hand side above is bounded by $\|\varphi\|_{H^{s,p}(\Omega)}$. We now proceed with the proof of (6.7), for which we will estimate

$$\int_\Omega \delta(z)^{p(m-s-\frac{1}{2})} \left| D_z^m \int_{\mathcal{U}\setminus\overline{\Omega}} K_{0,q}^{01}(z,\zeta)\wedge[\overline{\partial},\mathcal{E}]\varphi(\zeta)\,dV(\zeta)\right|^p dV(z), \tag{6.8}$$

where in the definition of $K^{01}_{0,q}$ (see (6.2)) we set W to be a C^1 Leray map. Writing $\Phi(z,\zeta) = W(z,\zeta)\cdot(\zeta-z)$, the inner integral can be expanded to a linear combination of

$$\mathcal{K}f(z) = \int_{\mathcal{U}\setminus\overline{\Omega}} f(\zeta) P(W_1(z,\zeta),z,\zeta)\frac{N_1(\zeta-z)}{\Phi^{n-l}(z,\zeta)|\zeta-z|^{2l}}\,dV(\zeta), \quad 1\le l\le n-1. \tag{6.9}$$

$$W_1 = (W, \partial_\zeta W, \partial_z^{k_0} W),\ k_0 \le m.$$

Here f is a coefficient function of $[\overline{\partial},\mathcal{E}]\varphi$. $P(w)$ denotes a polynomial in w and $\overline{w}$, and N_1 denotes a monomial of degree 1 in $\zeta - z$ and $\overline{\zeta-z}$. P may differ when recurs.

By the remark after Proposition 6.1, we can take a small neighborhood $\mathcal{V}$ of a fixed boundary point $\zeta^* \in b\Omega$. For $z\in\mathcal{V}$, let $\phi_z : \mathcal{V} \to \mathbb{C}^n$ be the coordinate transformation satisfying (6.3) and (6.4). Using a partition of unity in ζ space and replacing f by χf for a C^∞ cut-off function χ, we may assume

$$\operatorname{supp}_\zeta f \subseteq \mathcal{V}\setminus\overline{\Omega}.$$

Similarly by a partition of unity in z space and replacing $K^{01}_{0,q}$ by $\chi K^{01}_{0,q}$ we may assume

$$\text{supp}_z\ K^{01}_{0,q}(z,\zeta) \subseteq \mathcal{V}.$$

Write $\widetilde{N}_{1-2l}(z,\zeta) = \frac{N_1(\zeta-z)}{|\zeta-z|^{2l}}$. For $z \in \mathcal{V}\cap\Omega$ and $\zeta \in \mathcal{V}\setminus\overline{\Omega}$, we have

$$|\partial^j_z \widetilde{N}_{1-2l}(\zeta - z)| \lesssim |\zeta - z|^{1-2l-j}, \tag{6.10}$$

$$|\partial^j_z \Phi^{-(n-l)}(z,\zeta)| \lesssim |\Phi^{-(n-l)-j}(z,\zeta)|, \tag{6.11}$$

where we use the fact that W is holomorphic in $z \in \mathcal{U}$. Write

$$D^m_z \mathcal{K} f(z) = \int_{\mathcal{U}\setminus\overline{\Omega}} A_m(z,\zeta) f(\zeta)\, dV(\zeta),$$

where $A_m(z,\zeta)$ is a sum of terms of the form

$$A_m(z,\zeta) = \frac{P_1(z,\zeta)}{\Phi^{n-l+\mu_1}(z,\zeta)} \partial^{\mu_2}_z \widetilde{N}_{1-2l}(z,\zeta), \quad 1 \le l \le n-1, \quad \mu_1 + \mu_2 \le m. \tag{6.12}$$

Setting $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\begin{aligned} |D^m_z \mathcal{K} f(z)| &\le \int_{\mathcal{U}\setminus\overline{\Omega}} |A_m(z,\zeta)|^{\frac{1}{p}} |A_m(z,\zeta)|^{\frac{1}{p'}} |f(\zeta)|\, dV(\zeta) \\ &= \int_{\mathcal{U}\setminus\overline{\Omega}} \left[\delta(\zeta)^{-\eta} |\delta(\zeta)^{\eta} A_m(z,\zeta)|^{\frac{1}{p}}\right] \left[|f(\zeta)| |\delta(\zeta)^{\eta}| A_m(z,\zeta)|^{\frac{1}{p'}}\right] dV(\zeta), \end{aligned}$$

where η is a number to be specified. By Hölder's inequality, we get

$$|D^m_z \mathcal{K} f(z)|^p \le \left[\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{-\eta p + \eta} |A_m(z,\zeta)| |f(\zeta)|^p\, dV(\zeta)\right] \left[\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{\eta} |A_m(z,\zeta)|\, dV(\zeta)\right]^{\frac{p}{p'}}. \tag{6.13}$$

By (6.4), we have $C'|\zeta - z| \ge |\Phi(z,\zeta)| \ge C|\zeta - z|^2$. In view of (6.10) and (6.11), it suffices to estimate $A_m(z,\zeta)$ for $l = n-1$, $\mu_1 = m$ and $\mu_2 = 0$. Thus from now on we can just assume

$$A_m(z,\zeta) = \frac{P(W_1, z, \zeta)}{\Phi^{m+1}(z,\zeta)} \widetilde{N}_{-(2n-3)}.$$

By estimate (6.3), we have for $z \in \mathcal{V}\cap\Omega$ and $\zeta \in \mathcal{V}\setminus\overline{\Omega}$,

$$|\Phi(z,\zeta)| \ge c(\delta(z) + |s_1| + |s_2| + |t|^2), \quad |\zeta - z| \ge c(\delta(z) + |s_1| + |s_2| + |t|), \tag{6.14}$$

where $(s_1, s_2, t) = \left(\phi^1_z(\zeta), \phi^2_z(\zeta), \phi'_z(\zeta)\right)$, $\phi^1_z(\zeta) = \rho(\zeta)$.

By (6.14) and integrating in polar coordinates $t = (t_1, \dots, t_{2n-2}) \in \mathbb{R}^{2n-2}$, we obtain

$$\begin{aligned} &\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{\eta} |A_m(z,\zeta)|\, dV(\zeta) \\ &\le C_0 \int_{s_1=0}^1 \int_{s_2=0}^1 \int_{t=0}^1 \frac{s_1^{\eta} t^{2n-3}\, ds_1\, ds_2\, dt}{(\delta(z) + s_1 + s_2 + t^2)^{2+(m-1)} (\delta(z) + s_1 + s_2 + t)^{2n-3}} \\ &\le C_0 \delta(z)^{\eta + \frac{1}{2} - (m-1)} = C_0 \delta(z)^{\eta - m + \frac{3}{2}}, \end{aligned} \tag{6.15}$$

where we apply Lemma 6.5 using $\alpha = \eta$, $\beta = m - 1 \ge 0$ and by choosing

$$-1 < \eta < \beta - \frac{1}{2} = m - \frac{3}{2}. \tag{6.16}$$

Note that the constant C_0 depends only on the domain Ω and the defining function ρ. Hence by (6.13), (6.15) and the Fubini theorem we get

$$\begin{aligned}
&\int_\Omega \delta(z)^{p(m-s-\frac{1}{2})}|D_z^m \mathcal{K}f(z)|^p\, dV(z)\\
\lesssim &\int_\Omega \delta(z)^\gamma \left(\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{(1-p)\eta}|A_m(z,\zeta)||f(\zeta)|^p\, dV(\zeta)\right) dV(z)\\
\lesssim &\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{(1-p)\eta}\left[\int_\Omega \delta(z)^\gamma |A_m(z,\zeta)|\, dV(z)\right]|f(\zeta)|^p\, dV(\zeta),
\end{aligned} \tag{6.17}$$

where

$$\gamma = p\left(m-s-\frac{1}{2}\right)+\left(\eta-m+\frac{3}{2}\right)\frac{p}{p'} = m-\frac{3}{2}+(1-s)p+\eta(p-1). \tag{6.18}$$

We now estimate the inner integral in the last line of (6.17). Recall that for each $z \in \mathcal{V}$, we define C^1 coordinate transformation ϕ_z for $\zeta \in \mathcal{V}$:

$$\phi_z^1(\zeta) = \rho(\zeta), \quad \phi_z^2(\zeta) = \operatorname{Im}(\rho_\zeta\cdot(\zeta-z)), \quad \phi_z'(\zeta) = \left(\operatorname{Re}(\zeta'-z'), \operatorname{Im}(\zeta'-z')\right).$$

Now for $\zeta \in \mathcal{V}$, we can similarly define a new coordinate $\widetilde{\phi}_\zeta : \mathcal{V} \to \mathbb{C}^n$ for $z \in \mathcal{V}$:

$$\widetilde{\phi}_\zeta^1(z) = \rho(z), \quad \widetilde{\phi}_\zeta^2(z) = \operatorname{Im}(\rho_\zeta\cdot(\zeta-z)), \quad \widetilde{\phi}_\zeta'(z) = \left(\operatorname{Re}(\zeta'-z'), \operatorname{Im}(\zeta'-z')\right).$$

Write $(\tilde{s}_1, \tilde{s}_2, \tilde{t}) = (\widetilde{\phi}_\zeta^1(z), \widetilde{\phi}_\zeta^2(z), \widetilde{\phi}_\zeta'(z))$ where $|\widetilde{\phi}_\zeta^1(z)| = |\rho(z)| \approx \delta(z)$. By (6.14) we have for $z \in \mathcal{V}\cap\Omega$ and $\zeta \in \mathcal{V}\setminus\overline{\Omega}$,

$$\begin{aligned}
|\Phi(z,\zeta)| &\geq c(\delta(z)+\phi_z^1(\zeta)+|\phi_z^2(\zeta)|+|\phi_z'(\zeta)|^2)\\
&\geq c(\delta(\zeta)+|\widetilde{\phi}_\zeta^1(z)|+|\widetilde{\phi}_\zeta^2(z)|+|\widetilde{\phi}_\zeta'(z)|^2)\\
&= c(\delta(\zeta)+|\tilde{s}_1|+|\tilde{s}_2|+|\tilde{t}|^2),
\end{aligned} \tag{6.19}$$

and

$$|\zeta - z| \geq c(|\delta(\zeta)+|\tilde{s}_1|+|\tilde{s}_2|+|\tilde{t}|) \tag{6.20}$$

Writing in polar coordinates $\tilde{t} = (t_1, \dots, t_{2n-2}) \in \mathbb{R}^{2n-2}$, we have

$$\int_\Omega \delta(z)^\gamma |A_m(z,\zeta)|\, dV(z) \leq C\int_{\tilde{s}_1=0}^1\int_{\tilde{s}_2=0}^1\int_{\tilde{t}=0}^1 \frac{\tilde{s}_1^\gamma\, \tilde{t}^{2n-3}\, d\tilde{s}_1\, d\tilde{s}_2\, d\tilde{t}}{(\delta(\zeta)+\tilde{s}_1+\tilde{s}_2+\tilde{t}^2)^{m+1}\,(\delta(\zeta)+\tilde{s}_1+\tilde{s}_2+\tilde{t})^{2n-3}}. \tag{6.21}$$

To apply Lemma 6.5 we take $\alpha = \gamma$ and $\beta = m-1$, and we need $-1 < \gamma < \beta - \frac{1}{2} = m - \frac{3}{2}$. In view of (6.18), this is the same as

$$-1 < m - \frac{3}{2} + (1-s)p + \eta(p-1) < m - \frac{3}{2},$$

which translates to

$$\frac{(s-1)p}{p-1} - \frac{m-\frac{1}{2}}{p-1} < \eta < \frac{(s-1)p}{p-1} = (s-1)p'. \tag{6.22}$$

Note that this is always possible by our assumption on m. Indeed, by (6.16) and (6.22) we need to choose

$$\max\left\{-1, \frac{(s-1)p}{p-1} - \frac{m-\frac{1}{2}}{p-1}\right\} < \eta < \min\left\{m-\frac{3}{2}, \frac{(s-1)p}{p-1}\right\}, \quad \frac{1}{p}+\frac{1}{p'} = 1. \tag{6.23}$$

By assumption $s > \frac{1}{p}$ and $m > s + \frac{1}{2}$, so the range of admissible η is non-empty.

Now applying Lemma 6.5 to (6.21) we obtain

$$\int_\Omega \delta(z)^\gamma |A_m(z,\zeta)|\, dV(z) \le C\delta(\zeta)^{\gamma+\frac{1}{2}-(m-1)} = C\delta(\zeta)^{\gamma-m+\frac{3}{2}}, \tag{6.24}$$

where the constant depends on Ω only.

Putting (6.24) in (6.17) we get

$$\begin{aligned}\left[\int_\Omega \delta(z)^{(m-s-\frac{1}{2})p}|D_z^m Kf(z)|^p\, dV(z)\right]^{\frac{1}{p}} &\lesssim \left[\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{(1-p)\eta}\delta(\zeta)^{\gamma-m+\frac{3}{2}}|f(\zeta)|^p\, dV(\zeta)\right]^{\frac{1}{p}}\\ &\lesssim \left[\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{(1-s)p}|[\overline{\partial},\mathcal{E}]\varphi(\zeta)|^p\, dV(\zeta)\right]^{\frac{1}{p}}.\end{aligned}$$

This proves (6.7) and thus the proposition. □

Next we extend the result of Proposition 6.7 to all lower order derivatives of u.

Proposition 6.8. *Keeping the assumptions of Proposition 6.7, the following holds*

$$\|D^k\mathcal{H}_q^1\varphi\|_{L^p(\Omega,\,\delta(z)^{m-s-\frac{1}{2}})} \le C(\Omega,p)\|\varphi\|_{H^{s,p}(\Omega)}, \quad 0\le k\le m.$$

Proof. We need to estimate

$$\int_\Omega \delta(z)^{p(m-s-\frac{1}{2})}\left|D_z^k\mathcal{K}f\right|^p\, dV(z),$$

where

$$\mathcal{K}f(z) = \int_{\mathcal{U}\setminus\overline{\Omega}} K_{0,q}^{01}(z,\zeta)\wedge[\overline{\partial},\mathcal{E}]\varphi(\zeta)\, dV(\zeta), \quad f = [\overline{\partial},\mathcal{E}]\varphi.$$

As before we write

$$D_z^k\mathcal{K}f(z) = \int_{\mathcal{U}\setminus\overline{\Omega}} A_k(z,\zeta)f(\zeta)\, dV(\zeta),$$

and

$$|D_z^k\mathcal{K}f(z)|^p \le \left[\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{(1-p)\alpha}|A_k(z,\zeta)||f(\zeta)|^p\, dV(\zeta)\right]\left[\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{\alpha}|A_k(z,\zeta)|\, dV(\zeta)\right]^{\frac{p}{p'}}. \tag{6.25}$$

for some α to be chosen. Now A_k is a sum of the form (see (6.12))

$$A_k(z,\zeta) = \frac{P_1(z,\zeta)}{\Phi^{n-l+\mu_1}(z,\zeta)}\partial_z^{\mu_2}\left\{\tilde{N}_{1-2l}(\zeta-z)\right\}, \quad 1\le l\le n-1, \quad \mu_1+\mu_2\le k.$$

By the same reasoning as before, it suffices to estimate the term for $l=n-1$, $\mu_1=k$, namely we have

$$|A_k(z,\zeta)| \lesssim \frac{1}{|\Phi^{k+1}(z,\zeta)||\zeta-z|^{2n-3}}.$$

By a partition of unity in both z and ζ space, we can assume that $\operatorname{supp}_z A_k(z,\zeta)\subseteq V\cap\Omega$ and $\operatorname{supp}_\zeta f\subseteq V\setminus\overline{\Omega}$, where V is some small neighborhood of a fixed point $\zeta_0\in b\Omega$.

Now as $|\Phi(z,\zeta)|<C|z-\zeta|$, we can assume that $|\Phi(z,\zeta)|<1$ for any $z,\zeta\in V$. Hence for $k\le m$,

$$|A_k(z,\zeta)| \le \frac{C}{|\Phi^{m+1}(z,\zeta)||\zeta-z|^{2n-3}}, \quad z,\zeta\in V.$$

In view of (6.25), the rest of proof is identical to that of previous theorem. □

Together with the Hardy-Littlewood lemmas from Section 4, we can prove the gain of 1/2 derivative in Sobolev space for the operator $\mathcal{H}_q^1$.

Proposition 6.9. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2 boundary. For $q \geq 1$, let $\mathcal{H}_q\varphi$ be given by (6.1), where the extension operator $\mathcal{E}$ is given by (3.3). Suppose $\varphi \in H^{s,p}_{(0,q)}(\Omega)$ with $s > \frac{1}{p}$, then $\mathcal{H}^1_q\varphi \in H^{s+\frac{1}{2},p}_{(0,q-1)}(\Omega)$.*

Proof. We divide into cases.
Case 1: $s = \frac{2k+1}{2}$, $k \in \mathbb{Z}^+$.

We have $s+\frac{1}{2} \in \mathbb{Z}^+$ and also $s > \frac{1}{p}$. Take m to be any integer greater than $s+\frac{1}{2}$. By Propositions 6.8 and 5.2 we obtain

$$\|\mathcal{H}^1_q\varphi\|_{W^{s+\frac{1}{2},p}(\Omega)} \lesssim \sum_{|\gamma|\leq k} \|\delta^{m-s-\frac{1}{2}} D^\gamma \mathcal{H}^1_q\varphi\|_{L^p(\Omega)} \lesssim \|\varphi\|_{H^{s,p}(\Omega)}.$$

Since $\|\mathcal{H}^1_q\varphi\|_{W^{s+\frac{1}{2},p}(\Omega)} = \|\mathcal{H}^1_q\varphi\|_{H^{s+\frac{1}{2},p}(\Omega)}$ for $s + \frac{1}{2}$ a positive integer, the result follows.
Case 2: $s \in \left(\frac{1}{p}, \frac{3}{2}\right)$ and $s + \frac{1}{2} \in [0, 1]$.

We apply Proposition 5.9 (i) and Proposition 6.8 for $m = 1$ to get

$$\|\mathcal{H}^1_q\varphi\|_{H^{s+\frac{1}{2},p}(\Omega)} \leq \|\delta^{1-s-\frac{1}{2}} \mathcal{H}^1_q\varphi\|_{L^p(\Omega)} + \|\delta^{1-s-\frac{1}{2}} D\mathcal{H}^1_q\varphi\|_{L^p(\Omega)} \leq C\|\varphi\|_{H^{s,p}(\Omega)}.$$

Case 3: $s \in \left(\frac{1}{p}, \frac{3}{2}\right)$ and $s + \frac{1}{2} \in [1, 2]$.

We apply Proposition 5.9 (ii) and Proposition 6.8 for $m = 2$ to get

$$\begin{aligned}\|\mathcal{H}^1_q\varphi\|_{H^{s+\frac{1}{2},p}(\Omega)} &\leq C\|\mathcal{H}^1_q\varphi\|_{L^p(\Omega)} + \|\delta^{2-s-\frac{1}{2}} D\mathcal{H}^1_q\varphi\|_{L^p(\Omega)} + \|\delta^{2-s-\frac{1}{2}} D^2\mathcal{H}^1_q\varphi\|_{L^p(\Omega)} \\ &\leq C\|\varphi\|_{H^{s,p}(\Omega)}.\end{aligned}$$

Finally the remaining cases can be done by interpolation. □

By combining Proposition 6.6 and Proposition 6.9 we obtain Theorem 6.2.

7. Λ^r estimate for $r > 0$

Let $r > 0$ and let $\Omega \subseteq \mathbb{C}^n$ be a bounded Lipschitz domain. We recall from Definition 2.3 that $\Lambda^r(\Omega)$ is the space of Hölder-Zygmund functions of order r up to the boundary.

We first recall the interpolation result of Hölder-Zygmund spaces.

Proposition 7.1 (Complex interpolation of Λ^r-spaces)**.** *Let $r_0, r_1 > 0$ and let $\Omega \subset \mathbb{C}^n$ be a bounded Lipschitz domain. For $0 < \theta < 1$, let $r_\theta = (1-\theta)r_0 + \theta r_1$. Then $[\Lambda^{r_0}(\Omega), \Lambda^{r_1}(\Omega)]_\theta = \Lambda^{r_\theta}(\Omega)$.*

The proof is a combination of [BL76, Theorem 6.4.5(6)] and [Tri06, Theorems 1.110 and 1.122].

In [Gon19], Gong constructed a solution operator $\mathcal{S}_q\varphi$ to $\overline{\partial}u = \varphi$ which maps any $(0, q)$ form $\varphi \in \Lambda^r(\Omega)$ to a $(0, q-1)$ form in $\Lambda^{r+\frac{1}{2}}(\Omega)$ for all $r > 1$. We now extend this result to all $r > 0$.

First let us recall the classical Hardy-Littlewood lemma for Hölder continuous functions.

Lemma 7.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let $\delta(x)$ denote the distance function from x to the boundary of Ω. If u is a C^1 function in Ω and there exists an $0 < \alpha < 1$ and $C > 0$ such that*

$$|Du(x)| \leq C\delta(x)^{-1+\alpha} \quad \text{for every } x \in \Omega,$$

then $u \in \Lambda^\alpha(\Omega)$.

The reader can refer to [CS01, p. 345] for a proof.

Theorem 7.3. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with C^2 boundary. Let $1 \leq q \leq n$ and let $\mathcal{H}_q$ be given by formula (6.1), where the extension operator E is defined by formula (3.3). Then for any $r > 0$, $\mathcal{H}_q$ is a bounded linear operator $\mathcal{H}_q : \Lambda^r_{(0,q)}(\Omega) \to \Lambda^{r+\frac{1}{2}}_{(0,q-1)}(\Omega)$.*

Proof. For $\varphi \in \Lambda^r(\Omega)$ with $r > 1$, see [Gon19], where the homotopy operator is defined by formula (6.1) in the classical sense. We note that Gong used a different extension operator than ours, but in the case $r > 1$ the proofs work the same since the only property of the extension operator used in his proof is the fact $[D, \mathcal{E}] : \Lambda^r(\Omega) \to \Lambda^{r-1}(\Omega)$ for $r > 1$, which obviously holds for our extension operator as well by Proposition 3.7 (ii).

By interpolation we only need to prove for $0 < r < \frac{1}{2}$. In view of Lemma 7.2, it suffices to show that

$$\sup_{z\in\Omega} \delta(z)^{1-(r+\frac{1}{2})} \left| D\mathcal{H}_q^1 \varphi(z) \right| \le C\|\varphi\|_{\Lambda^r(\Omega)}.$$

We have

$$\delta(z)^{1-(r+\frac{1}{2})} |D\mathcal{H}_q^1\varphi(z)| = \delta(z)^{\frac{1}{2}-r} \left| \int_{\mathcal{U}\setminus\overline{\Omega}} D_z K^{01}_{0,q}(z,\zeta) \wedge [\overline{\partial}, \mathcal{E}]\varphi(\zeta)\, dV(\zeta) \right|.$$

Write $A_1(z,\zeta) = D_z K^{01}_{0,q}(z,\zeta)$ and $f = [\overline{\partial}, \mathcal{E}]\varphi$. By Lemma 5.8, we get $\|\delta^{1-r} f\|_{L^\infty(U\setminus\overline{\Omega})} \le \|\varphi\|_{\Lambda^r(\Omega)}$, so

$$\begin{aligned}
\delta(z)^{1-(r+\frac{1}{2})} |D\mathcal{H}_q^1\varphi(z)| &= \delta(z)^{\frac{1}{2}-r} \int_{\mathcal{U}\setminus\overline{\Omega}} |A_1(z,\zeta)||f(\zeta)|\, dV(\zeta) \\
&= \delta(z)^{\frac{1}{2}-r} \int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{r-1}|A_1(z,\zeta)|\delta(\zeta)^{1-r}|f(\zeta)|\, dV(\zeta) \\
&\le \delta(z)^{\frac{1}{2}-r} \left(\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{r-1}|A_1(z,\zeta)|\, dV(\zeta) \right) \|\varphi\|_{\Lambda^r(\Omega)} \\
&\le C\|\varphi\|_{\Lambda^r(\Omega)},
\end{aligned}$$

where in the last inequality we apply Lemma 6.5 for $\alpha = r - 1$ and $\beta = 0$ (which is possible since $0 < r < \frac{1}{2}$ and $-1 < \alpha < \beta + \frac{1}{2}$) to get

$$\begin{aligned}
\int_{\mathcal{U}\setminus\overline{\Omega}} \delta(\zeta)^{r-1}|A_1(z,\zeta)|\, dV(\zeta) &\le C \int_{s_1=0}^1 \int_{s_2=0}^1 \int_{t=0}^1 \frac{s_1^{r-1} t^{2n-3}\, ds_1\, ds_2\, dt}{(\delta(z)+s_1+s_2+t^2)^2(\delta(z)+s_1+s_2+t)^{2n-3}} \\
&\le C\delta(z)^{r-\frac{1}{2}}.
\end{aligned}$$

This completes the proof. ∎

Appendix A. An equivalent norm property

Proposition A.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded C^2-domain and let $1 < p < \infty$.*

(i) For $0 < s < 2$, $H^{s,p}(\Omega)$ has equivalent norm

$$\|f\|_{H^{s,p}(\Omega)} \approx \|f\|_{H^{s-1,p}(\Omega)} + \|Df\|_{H^{s-1,p}(\Omega)}.$$

(ii) For $1 < s < 2$, $H^{s,p}(\Omega)$ has equivalent norm

$$\|f\|_{H^{s,p}(\Omega)} \approx \|f\|_{H^{s-2,p}(\Omega)} + \|Df\|_{H^{s-2,p}(\Omega)} + \|D^2 f\|_{H^{s-2,p}(\Omega)}.$$

Note that the above results are known for C^∞-domain (see [Tri83, Theorem 3.3.5(ii)]), and we shall adopt similar method here.

To prove Proposition A.1 we first need a lemma.

Lemma A.2. *Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a C^2-diffeomorphism such that $D\Phi$ and $D\Phi^{-1}$ both have bounded C^1 norms. Then*

(i) $\tilde{f} \mapsto \tilde{f} D\Phi$ defines a bounded linear map $H^{s,p}(\mathbb{R}^N) \to H^{s,p}(\mathbb{R}^N)$ for all $1 < p < \infty$ and $-1 \le s \le 1$.

(ii) $\tilde{f} \mapsto \tilde{f} \circ \Phi$ defines a bounded linear map $H^{s,p}(\mathbb{R}^N) \to H^{s,p}(\mathbb{R}^N)$ for all $1 < p < \infty$ and $-1 \le s \le 2$.

See also [Tri92, Theorem 4.3.2] for (ii). Note that in the reference the result is only proved for the range $N(\frac{1}{p}-1)<s<2$, which is not enough for us if $\frac{1}{p}-1<-\frac{1}{N}$.

Proof. Clearly the map $[\tilde f\mapsto \tilde f D\Phi]:W^{1,p}(\mathbb{R}^N)\to W^{1,p}(\mathbb{R}^N)$ is bounded linear, and $[\tilde f\mapsto \tilde f D\Phi]:W^{1,p'}(\mathbb{R}^N)\to W^{1,p'}(\mathbb{R}^N)$ is also bounded linear.

Since the product operator is self-adjoint and $H^{-1,p}(\mathbb{R}^N)=W^{1,p'}(\mathbb{R}^N)'$(Proposition 2.12), the map $[\tilde f\mapsto \tilde f D\Phi]:H^{-1,p}(\mathbb{R}^N)\to H^{-1,p}(\mathbb{R}^N)$ is also bounded linear.

By using interpolation (Proposition 2.14 and Proposition 2.15) , we get $[\tilde f\mapsto \tilde f D\Phi]:H^{s,p}(\mathbb{R}^N)\to H^{s,p}(\mathbb{R}^N)$ for all $-1\le s\le 1$. This finishes the proof of (i).

For (ii), clearly $[\tilde f\mapsto \tilde f\circ\Phi]:W^{k,p}(\mathbb{R}^N)\to W^{k,p}(\mathbb{R}^N)$ are bounded linear for $k=0,1,2$ since $D\Phi, D^2\Phi$ are bounded. Since $W^{k,p}=H^{k,p}$ we have the boundedness for $s=0,1,2$.

Using change of variables, the adjoint map of $\tilde f\mapsto \tilde f\circ\Phi$ is $\tilde g\mapsto|\det D\Phi^{-1}|\cdot(\tilde g\circ\Phi^{-1})$. Clearly $[\tilde g\mapsto\tilde g\circ\Phi^{-1}]:W^{1,p'}(\mathbb{R}^N)\to W^{1,p'}(\mathbb{R}^N)$ is bounded linear. Since $\det D\Phi^{-1}$ is bounded C^1 and non-vanishing everywhere, we have $|\det D\Phi^{-1}|\in C^1(\mathbb{R}^N)$. Therefore the map $[\tilde g\mapsto|\det D\Phi^{-1}|\cdot(\tilde g\circ\Phi^{-1})]:W^{1,p'}(\mathbb{R}^N)\to W^{1,p'}(\mathbb{R}^N)$ is bounded linear.

Since $H^{-1,p}(\mathbb{R}^N)=W^{1,p'}(\mathbb{R}^N)'$, taking the adjoint back we get the boundedness $[\tilde f\mapsto\tilde f\circ\Phi]:H^{-1,p}(\mathbb{R}^N)\to H^{-1,p}(\mathbb{R}^N)$, which proves the case $s=-1$. Finally by interpolation, we get $[\tilde f\mapsto\tilde f\circ\Phi]:H^{s,p}(\mathbb{R}^N)\to H^{s,p}(\mathbb{R}^N)$ for all $-1\le s\le 2$. ☐

Proof of Proposition A.1. Recall the definition $\|f\|_{H^{s,p}(\Omega)}=\min_{\tilde f|_\Omega=f}\|\tilde f\|_{H^{s,p}(\mathbb{R}^N)}$. The "$\gtrsim$"-part follows from the equivalent norm in $\mathbb{R}^N$ (see [Tri83, Theorem 2.3.8(ii)]) and the fact that if $\tilde f\in H^{s,p}(\mathbb{R}^N)$ extends $f\in H^{s,p}(\Omega)$ then $D^\alpha\tilde f\in H^{s-1,p}(\mathbb{R}^N)$ extends $D^\alpha f$. Also see the proof of [Tri83, Theorem 3.3.5(ii)].

We now prove the "$\lesssim$"-part. It suffices to prove (i), since (ii) follows by applying (i) twice.

By [Tri83, Theorem 3.3.5(ii)], the result holds on the half space $\mathbb{R}^N_+=\{x_N>0\}$, namely, for every $r\in\mathbb{R}$ and $1<p<\infty$, the relation

$$\|g\|_{H^{r,p}(\mathbb{R}^N_+)}\approx_{r,p}\|g\|_{H^{r-1,p}(\mathbb{R}^N_+)}+\|Dg\|_{H^{r-1,p}(\mathbb{R}^N_+)}, \tag{A.1}$$

holds for all $g\in H^{r,p}(\mathbb{R}^N_+)$ supported in $\mathbb{B}^N\cap\overline{\mathbb{R}^N_+}$. Here $\mathbb{B}^N$ is the unit ball in $\mathbb{R}^N$.

By partition of unity, we can find the following:

- Open sets $(U_\nu)_{\nu=1}^M$ such that $b\Omega\subseteq\bigcup_{\nu=1}^M U_\nu$.
- Functions $\chi_0\in C_c^\infty(\Omega)$, $\chi_\nu\in C_c^\infty(U_\nu)$ for $1\le\nu\le M$ such that $\sum_{\nu=0}^M\chi_\nu|_\Omega\equiv 1$.
- C^2-maps $\Phi_\nu:\mathbb{R}^N\to\mathbb{R}^N$ for $1\le\nu\le M$ such that $\Phi_\nu(\mathbb{B}^N)=U_\nu$, $\Phi_\nu(\mathbb{B}^N\cap\mathbb{R}^N_+)=U_\nu\cap\Omega$ and $D\Phi_\nu, D\Phi_\nu^{-1}$ have bounded C^1 norm.

Therefore by Lemma A.2 (ii),

$$\|f\|_{H^{s,p}(\Omega)}\le\sum_{\nu=0}^M\|\chi_\nu f\|_{H^{s,p}(\Omega)}\lesssim\|\chi_0 f\|_{H^{s,p}(\mathbb{R}^N)}+\sum_{\nu=1}^M\|(\chi_\nu f)\circ\Phi_\nu\|_{H^{s,p}(\mathbb{R}^N_+)}.$$

By [Tri83, Theorem 2.3.8(ii)] we have

$$\begin{aligned}\|\chi_0 f\|_{H^{s,p}(\mathbb{R}^N)}&\approx\|\chi_0 f\|_{H^{s-1,p}(\mathbb{R}^N)}+\|D(\chi_0 f)\|_{H^{s-1,p}(\mathbb{R}^N)}\\&\le\|\chi_0 f\|_{H^{s-1,p}(\mathbb{R}^N)}+\|fD\chi_0\|_{H^{s-1,p}(\mathbb{R}^N)}+\|\chi_0 Df\|_{H^{s-1,p}(\mathbb{R}^N)}\\&\lesssim\|f\|_{H^{s-1,p}(\Omega)}+\|Df\|_{H^{s-1,p}(\Omega)}.\end{aligned}$$

For $1 \le \nu \le M$, we apply Lemma A.2 with $-1 \le s-1 \le 1$,

$$\begin{aligned}\|(\chi_\nu f)\circ\Phi_\nu\|_{H^{s,p}(\mathbb{R}^N_+)} &\lesssim \|(\chi_\nu f)\circ\Phi_\nu\|_{H^{s-1,p}(\mathbb{R}^N_+)} + \|D((\chi_\nu f)\circ\Phi_\nu)\|_{H^{s-1,p}(\mathbb{R}^N_+)} \\ &\lesssim \|(\chi_\nu f)\circ\Phi_\nu\|_{H^{s-1,p}(\mathbb{B}^N\cap\mathbb{R}^N_+)} + \|((D(\chi_\nu f))\circ\Phi_\nu)\cdot D\Phi_\nu\|_{H^{s-1,p}(\mathbb{R}^N_+)} \\ &\lesssim \|(\chi_\nu f)\circ\Phi_\nu\|_{H^{s-1,p}(\mathbb{B}^N\cap\mathbb{R}^N_+)} + \|(D(\chi_\nu f)\circ\Phi_\nu\|_{H^{s-1,p}(\mathbb{R}^N_+)} \\ &\lesssim \|\chi_\nu f\|_{H^{s-1,p}(\Omega)} + \|D(\chi_\nu f)\|_{H^{s-1,p}(\Omega)} \lesssim \|f\|_{H^{s-1,p}(\Omega)} + \|Df\|_{H^{s-1,p}(\Omega)}.\end{aligned}$$

By taking sum over $0 \le \nu \le M$ we complete the proof. □

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