# EVERY COUNTABLE GROUP ADMITS AMENABLE ACTIONS ON STABLY FINITE SIMPLE C*-ALGEBRAS

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*Dedicated to Professor Yasuyuki Kawahigashi on the occasion of his 60th birthday.*

Abstract. We give the first examples of (non-amenable group) amenable actions on stably finite simple C*-algebras. More precisely, we give such actions for any countable group in an explicit way. The main ingredients of our construction are the full Fock space of the regular representation and a trace-scaling automorphism.

## 1. Introduction

Recently amenability of C*-dynamical systems attracted much attention. This is a consequence of the discovery of amenable C*-dynamical systems on *simple* C*-algebras [21]. The formulation and characterizations of amenability of (non-commutative) C*-dynamical systems had been unclear for a long time, but this problem was recently settled in [17] (see also [3], [6], [22]). Here we only recall the definition in the discrete group case. An action $\alpha\colon \Gamma \curvearrowright A$ of a discrete group $\Gamma$ on a C*-algebra $A$ is *amenable* [2] if and only if there exists a $\Gamma$-equivariant conditional expectation $\ell^\infty(\Gamma) \mathbin{\bar{\otimes}} A^{**} \to 1 \otimes A^{**}$ when $\ell^\infty(\Gamma) \mathbin{\bar{\otimes}} A^{**}$ is equipped with the diagonal $\Gamma$-action of the left translation action and $\alpha^{**}$. For useful characterizations and basic properties of amenability of C*-dynamical systems, we refer the reader to [2], [6], [17]. A recent highlight around this subject is the successful classification results [22], [7], [8] of certain amenable actions on *purely infinite simple* C*-algebras.

As an application of new characterizations of amenability, Ozawa and the author have established a powerful method to produce amenable actions on simple C*-algebras (see [17], Theorem 6.1). However, in all previously known constructions [21], [17] of amenable actions (of non-amenable groups) on simple C*-algebras, the resulting underlying C*-algebras end up being *purely infinite.* Here we recall that the class of *classifiable* simple C*-algebras (see e.g., the recent survey [23]) splits into two classes: *stably finite* C*-algebras and purely infinite C*-algebras. As K-theoretic invariants of a C*-algebra have a much richer structure in the former class, it is a natural and important question to ask if there also exists an amenable action on a stably finite simple C*-algebra. In this article we settle this question in the non-unital case. More precisely, we show the following theorem.

**Theorem A.** *Every countable group $\Gamma$ admits an amenable action on a stably finite simple separable nuclear C*-algebra.*

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To show the stable finiteness of the resulting (simple, non-unital) C*-algebra, we construct a *proper tracial weight* on it. Here we recall that a tracial weight on a C*-algebra is said to be *proper* if it is densely defined, lower semi-continuous, and nonzero. We refer the reader to Chapter 5 of [18] for basic facts on tracial weights on C*-algebras.

On this opportunity, we also record the following generalization of Corollary 6.4 in [17], originally shown for the free groups, to exact locally compact groups satisfying the strong Baum–Connes conjecture ([14], page 304). We recall that groups with the Haagerup property satisfy the strong Baum–Connes conjecture [10].

**Theorem B.** *Let $G$ be an exact locally compact second countable group. Then $G$ admits an amenable action $\alpha$ on the Cuntz algebra $\mathcal{O}_\infty$. Moreover, when $G$ satisfies the strong Baum–Connes conjecture, one can arrange $\alpha$ to be $KK^G$-equivalent to $\mathbb{C}$.*

**Remark.** (1): The author is informed from Gábor Szabó that he, together with James Gabe, has independently proved the statement for exact groups with the Haagerup property by a different method: Their proof is based on their classification result (the existence theorem) [8] and Corollary 6.3 of [17].

(2): The exactness assumption in Theorem B is necessary by Corollary 3.6 of [17].

Throughout the article, all C*-algebras are assumed to be nonzero.

For basic facts on C*-algebras and discrete groups, we refer the reader to Chapters 2 to 4 of the book [5]. For basic facts on Pedersen ideals, we refer the reader to Chapter 5.6 of the book [18].

**Notations.** For a Hilbert space $\mathfrak{H}$, denote by $\mathbb{K}(\mathfrak{H})$ the C*-algebra of compact operators on $\mathfrak{H}$. When $\mathfrak{H} = \ell^2(\mathbb{N})$, we simply denote by $\mathbb{K}$.

For a C*-algebra $A$, denote by $\mathcal{M}(A)$ its multiplier algebra. For an isometric element $v$ in $\mathcal{M}(A)$, denote by $\mathrm{Ad}_v \colon A \to A$ the endomorphism given by

$$\mathrm{Ad}_v(a) := vav^*, \quad a \in A.$$

For a C*-algebra $A$, denote by $\mathrm{Ped}(A)$ the Pedersen ideal of $A$ (see [18], Theorem 5.6.1).

For C*-algebras $A_i$ and a proper tracial weight $\tau_i$ on $A_i$, $i = 1, 2$, denote by $\tau_1 \otimes \tau_2$ the unique proper tracial weight on the minimal tensor product $A_1 \otimes A_2$ which coincides with $\tau_1|_{\mathrm{Ped}(A_1)} \odot \tau_2|_{\mathrm{Ped}(A_2)}$ on the algebraic tensor product $\mathrm{Ped}(A_1) \odot \mathrm{Ped}(A_2) \subset \mathrm{Ped}(A_1 \otimes A_2)$. (Cf. [18], Proposition 5.6.7.)

## 2. Proofs

*Proof of Theorem A.* The Higman–Neumann–Neumann embedding theorem [9] states that any countable group embeds into a finitely generated group. It is clear that the restriction of a discrete group amenable action to a subgroup is again amenable. Therefore it suffices to show the statement for finitely generated groups.

Let $\Gamma$ be a non-trivial finitely generated group. Take a finite subset $S \subset \Gamma$ with $\bigcup_{n=1}^{\infty} S^n = \Gamma$, $|S| > 1$. Take a simple separable nuclear C*-algebra $A$, an automorphism $\rho\colon A \to A$, and a proper tracial weight $\tau_A$ on $A$ satisfying

$$\tau_A \circ \rho = \frac{1}{|S|}\tau_A.$$

(For instance, define $A := \mathbb{K} \otimes (\bigotimes_{\mathbb{N}} \mathbb{M}_{|S|}(\mathbb{C}))$. Take any isomorphisms

$$\rho_1\colon \mathbb{K} \to \mathbb{K} \otimes \mathbb{M}_{|S|}(\mathbb{C}), \quad \rho_2\colon \bigotimes_{\mathbb{N}} \mathbb{M}_{|S|}(\mathbb{C}) \to \bigotimes_{n=2}^{\infty} \mathbb{M}_{|S|}(\mathbb{C}),$$

and define an automorphism $\rho\colon A \to A$ to be $\rho := \rho_1 \otimes \rho_2$. Then $\rho$ together with any proper tracial weight $\tau_A$ on $A$ satisfies the foregoing equality.)

Consider the Hilbert space $\mathfrak{H} := \bigoplus_{n\in\mathbb{N}} \ell^2(\Gamma)^{\otimes n}$. We define a unitary representation $u\colon \Gamma \curvearrowright \mathfrak{H}$ to be

$$u_g := \bigoplus_{n\in\mathbb{N}} \lambda_g^{\otimes n}, \quad g \in \Gamma.$$

Here $\lambda\colon \Gamma \curvearrowright \ell^2(\Gamma)$ denotes the left regular representation of $\Gamma$. Next, for each $g \in \Gamma$, define a map $v_g\colon \mathfrak{H} \to \mathfrak{H}$ to be

$$v_g(\xi) := \delta_g \otimes \xi, \quad \xi \in \mathfrak{H}.$$

This gives a family $(v_g)_{g\in\Gamma}$ of isometric operators with pairwise orthogonal ranges. (Cf. the full Fock space and the left creation operators.)

Put $B := c_0(\Gamma) \otimes \mathbb{K}(\mathfrak{H}) \otimes A$. Let $\beta\colon \Gamma \curvearrowright B$ be the action given by

$$\beta_g := \mathrm{L}_g \otimes \mathrm{Ad}_{u_g \otimes 1_A}, \quad g \in \Gamma.$$

Here $\mathrm{L}\colon \Gamma \curvearrowright c_0(\Gamma)$ denotes the left translation action and $1_A$ denotes the unit of $\mathcal{M}(A)$. For each $g \in \Gamma$, define an endomorphism $\nu_g$ on $\mathbb{K}(\mathfrak{H}) \otimes A$ to be $\nu_g := \mathrm{Ad}_{v_g} \otimes \rho$. Since the isometric elements $(v_g)_{g\in\Gamma}$ have pairwise orthogonal ranges, one can define an injective endomorphism $\sigma\colon B \to B$ by the formula

$$\sigma(b) := \sum_{g\in\Gamma} \chi_{gS} \otimes \nu_g(b_g), \quad b \in B.$$

Here we identify $B$ with $c_0(\Gamma, \mathbb{K}(\mathfrak{H}) \otimes A)$ in the obvious way.

Let $\tau_\Gamma$ denote the proper tracial weight on $c_0(\Gamma)$ defined by the counting measure on $\Gamma$. Fix a proper tracial weight $\tau_{\mathfrak{H}}$ on $\mathbb{K}(\mathfrak{H})$. On the C*-algebra $B$, consider the proper tracial weight

$$\tau_B := \tau_\Gamma \otimes \tau_{\mathfrak{H}} \otimes \tau_A.$$

Since $\mathbb{K}(\mathfrak{H}) \otimes A$ is simple, $\tau_{\mathfrak{H}} \otimes \tau_A$, hence $\tau_B$, is faithful.

Since

$$(\tau_{\mathfrak{H}} \otimes \tau_A) \circ \nu_g = \frac{1}{|S|}\tau_{\mathfrak{H}} \otimes \tau_A$$

for all $g \in \Gamma$, one has $\tau_B \circ \sigma = \tau_B$.

Next we define a C*-algebra $\mathfrak{B}$ to be the inductive limit of the inductive system

$$B \xrightarrow{\ \sigma\ } B \xrightarrow{\ \sigma\ } B \xrightarrow{\ \sigma\ } \cdots$$

of C*-algebras. For each $n \in \mathbb{N}$, let

$$\theta_n\colon B \to \mathfrak{B}$$

denote the canonical embedding from the $n$-th $B$ into $\mathfrak{B}$. Since $\sigma$ preserves $\tau_B$, one has a proper tracial weight $\tau_{\mathfrak{B}}$ on $\mathfrak{B}$ satisfying

$$\tau_{\mathfrak{B}} \circ \theta_n = \tau_B$$

for all $n \in \mathbb{N}$. Since $\tau_B$ is faithful, so is $\tau_{\mathfrak{B}}$.

We next show that $\sigma\colon B \to B$ is $\Gamma$-equivariant. To see this, note that

$$u_g v_h = v_{gh} u_g$$

for all $g, h \in \Gamma$. This implies

$$\mathrm{Ad}_{u_g \otimes 1_A} \circ \nu_h = \nu_{gh} \circ \mathrm{Ad}_{u_g \otimes 1_A}.$$

Then, for any $g \in \Gamma$ and any $b \in B$, we have

$$\begin{aligned} \beta_g(\sigma(b)) &= \sum_{h \in \Gamma} \chi_{ghS} \otimes \mathrm{Ad}_{u_g \otimes 1_A}(\nu_h(b_h)) \\ &= \sum_{h \in \Gamma} \chi_{ghS} \otimes \nu_{gh}(\mathrm{Ad}_{u_g \otimes 1_A}(b_h)) \\ &= \sum_{h \in \Gamma} \chi_{hS} \otimes \nu_h(\mathrm{Ad}_{u_g \otimes 1_A}(b_{g^{-1}h})) \\ &= \sigma(\beta_g(b)). \end{aligned}$$

Hence $\sigma$ is $\Gamma$-equivariant. Therefore one has an action $\gamma\colon \Gamma \curvearrowright \mathfrak{B}$ satisfying

$$\gamma_g(\theta_n(b)) = \theta_n(\beta_g(b))$$

for all $g \in \Gamma$, $n \in \mathbb{N}$, $b \in B$.

We next define an automorphism $\alpha\colon \mathfrak{B} \to \mathfrak{B}$ to be

$$\alpha(\theta_n(b)) := \theta_{n+1}(b), \quad n \in \mathbb{N}, b \in B.$$

It is not hard to see that $\alpha$ is indeed a (well-defined) automorphism of $\mathfrak{B}$. Clearly $\alpha$ is $\Gamma$-equivariant and preserves $\tau_{\mathfrak{B}}$. Observe that for any $n \in \mathbb{N}$, as $\theta_{n+1} \circ \sigma = \theta_n$, one has $\alpha^{-1} \circ \theta_n = \theta_n \circ \sigma$.

Consider the reduced crossed product C$^*$-algebra

$$C := \mathfrak{B} \rtimes_{\mathrm{r},\alpha} \mathbb{Z}.$$

Let $w \in \mathcal{M}(C)$ denote the canonical implementing unitary element of $\alpha$. Let

$$E\colon C = \mathfrak{B} \rtimes_{\mathrm{r},\alpha} \mathbb{Z} \to \mathfrak{B}$$

denote the canonical conditional expectation. Since $\alpha$ preserves $\tau_{\mathfrak{B}}$, one has a faithful proper tracial weight $\tau_C := \tau_{\mathfrak{B}} \circ E$ on $C$. In particular $C$ is stably finite. Since the two actions $\gamma$ and $\alpha$ commute, one has an action $\eta\colon \Gamma \curvearrowright C$ satisfying

$$\eta_g(xw^n) = \gamma_g(x)w^n \quad \text{for } x \in \mathfrak{B}, n \in \mathbb{Z}, g \in \Gamma.$$

We show that $(C, \eta)$ possesses the desired properties.

To see the amenability of $\eta$, let us consider the gauge action $\zeta\colon \mathbb{T} \curvearrowright C$. (That is, the action given by $\zeta_z(xw^n) := z^n x w^n$ for $z \in \mathbb{T} := \{s \in \mathbb{C} : |s| = 1\}$, $x \in \mathfrak{B}$, $n \in \mathbb{Z}$.) Note that $\zeta$ commutes with $\eta$ and that the fixed point algebra $C^{\mathbb{T}}$ is equal to $\mathfrak{B}$. Since $(B, \beta)$ is (strongly) amenable, the action $\gamma$ is amenable. Now Theorem 5.1 of [17] implies the amenability of $\eta$. (An alternative proof: One can directly show the nuclearity of $C \rtimes_{\mathrm{r},\eta} \Gamma \cong (\mathfrak{B} \rtimes_{\mathrm{r},\gamma} \Gamma) \rtimes_{\mathrm{r}} \mathbb{Z}$. Then Theorem 4.5 of [2] implies the amenability of $\eta$.)

We now prove that the C*-algebra $C$ has the desired properties. In fact all properties other than the simplicity of $C$ are clear from the construction. To show the simplicity of $C$, we first show that $\mathfrak{B}$ has no proper (two-sided, closed) ideal $I$ which is $\alpha$-invariant, that is, $\alpha(I) = I$. Let $I \subset \mathfrak{B}$ be a nonzero $\alpha$-invariant ideal. Choose $n \in \mathbb{N}$ satisfying $I \cap \theta_n(B) \neq \{0\}$. Observe that any ideal $J$ of $B$ is of the form $c_0(X) \otimes \mathbb{K}(\mathfrak{H}) \otimes A$ for some subset $X \subset \Gamma$. Here we identify $c_0(X)$ with an ideal of $c_0(\Gamma)$ in the obvious way. Choose a (non-empty) subset $X \subset \Gamma$ satisfying

$$I \cap \theta_n(B) = \theta_n(c_0(X) \otimes \mathbb{K}(\mathfrak{H}) \otimes A).$$

We claim that $X = \Gamma$. Observe that for any $b \in B$, one has

$$\mathrm{supp}(\sigma(b)) = \mathrm{supp}(b)S$$

by the definition of $\sigma$. Since $\alpha(I) = I$, we obtain

$$\theta_n(\sigma(c_0(X) \otimes \mathbb{K}(\mathfrak{H}) \otimes A)) = \alpha^{-1}(I \cap \theta_n(B)) = \alpha^{-1}(I) \cap \theta_n(\sigma(B)) \subset I \cap \theta_n(B).$$

This proves $XS \subset X$. By the choice of $S$, we have $X = \Gamma$. This yields $I = \mathfrak{B}$.

Now let $I \subset C$ be a nonzero ideal. As $\mathfrak{B}$ has no proper $\alpha$-invariant ideal, it suffices to show that $I \cap \mathfrak{B} \neq \{0\}$. Since $E$ is a faithful conditional expectation, the image $J := E(I)$ is a nonzero algebraic ideal of $\mathfrak{B}$ with $\alpha(J) = E(wIw^*) = J$. Hence $J$ is norm dense in $\mathfrak{B}$. By Theorem 5.6.1 in [18], we have $\mathrm{Ped}(\mathfrak{B}) \subset J$. Since $\theta_1(\sigma(\mathrm{Ped}(B))) = \mathrm{Ped}(\theta_1(\sigma(B))) \subset \mathrm{Ped}(\mathfrak{B})$, we conclude $\theta_1(\sigma(\mathrm{Ped}(B))) \subset J$. Then, notice that the algebraic tensor product $c_c(\Gamma) \odot \mathbb{F}(\mathfrak{H}) \odot \mathrm{Ped}(A)$ is contained in $\mathrm{Ped}(B)$, where $\mathbb{F}(\mathfrak{H}) = \mathrm{Ped}(\mathbb{K}(\mathfrak{H}))$ denotes the $*$-algebra consisting of all bounded operators of finite rank on $\mathfrak{H}$. Hence one can find a self-adjoint element $c \in I$ with

$$E(c) = \theta_1(\sigma(r \otimes a))$$

for some positive elements $r \in c_0(\Gamma) \otimes \mathbb{K}(\mathfrak{H})$ and $a \in A$ with $\|r\| = \|a\| = 1$. Note that for any $n \in \mathbb{N}$, by the definition of $\sigma$, there is a positive element $r_n \in c_0(\Gamma) \otimes \mathbb{K}(\mathfrak{H})$ of norm one satisfying $\sigma^n(r \otimes d) = r_n \otimes \rho^n(d)$ for all $d \in A$. We next choose a self-adjoint element $c_0 \in C$ of the form

$$c_0 = \sum_{k=-N}^{N} \theta_M(b_k) w^k \quad \text{where } N, M \in \mathbb{N}, \quad b_k \in B, k = -N, -N+1, \ldots, N$$

satisfying

$$\theta_M(b_0) = \theta_1(\sigma(r \otimes a)) \text{ (that is, } E(c) = E(c_0)), \quad \|c - c_0\| < \frac{1}{2}.$$

Observe that

$$\theta_1(\sigma(r \otimes a)) = \theta_M(\sigma^M(r \otimes a)) = \theta_M(r_M \otimes \rho^M(a))$$

hence $b_0 = r_M \otimes \rho^M(a)$.

Note that the nonzero powers of $\rho$ are outer, because they do not preserve the proper tracial weight $\tau_A$. Therefore, by Kishimoto's theorem [12] (see Lemma 3.2 therein), one can find a sequence $(x_n)_{n=1}^{\infty}$ in $A$ satisfying

(1) $\|x_n\| = 1$ for all $n \in \mathbb{N}$,

(2) $\lim_{n\to\infty} \|x_n d\rho^k(x_n^*)\| = 0$ for all $d \in A$ and $k = 1, \ldots, N$,
(3) $\lim_{n\to\infty} \|x_n \rho^M(a) x_n^*\| = 1$.

Put $y_n := \theta_M(r_M \otimes x_n)$ for $n \in \mathbb{N}$. Then for any $k = 1, 2, \ldots, N$, by conditions (1) and (2), one has

$$\begin{aligned}\lim_{n\to\infty} \|y_n \theta_M(b_{-k}) w^{-k} y_n^*\| &= \lim_{n\to\infty} \|(r_M \otimes x_n) b_{-k}(r_{M+k} \otimes \rho^k(x_n^*))\| \\ &= 0.\end{aligned}$$

Since $c_0$ is self-adjoint, this proves

$$\lim_{n\to\infty} y_n(c_0 - E(c_0))y_n^* = 0.$$

Condition (3) yields

$$\lim_{n\to\infty} \|y_n \theta_M(b_0) y_n^*\| = \lim_{n\to\infty} \|r_M^3 \otimes x_n \rho^M(a) x_n^*\| = 1.$$

Therefore for a sufficiently large $n \in \mathbb{N}$, we have

$$\operatorname{dist}(y_n E(c) y_n^*, I) \leq \|y_n(c - E(c))y_n^*\| < \frac{1}{2} < \|y_n E(c) y_n^*\|.$$

This shows that the canonical map $\mathfrak{B} \to C/I$ is not isometric. Thus $\mathfrak{B} \cap I \neq \{0\}$. $\square$

**Remark 2.1.** Here we record a few observations on the foregoing construction. Keep the same notations as in the proof of Theorem A.

(1): When $A$ satisfies the universal coefficient theorem of Rosenberg–Schochet [20], so does the resulting C$^*$-algebra $C$.

(2): The tracial weight $\tau_C$ on $C$ is $\Gamma$-invariant. Hence the reduced crossed product C$^*$-algebra $C \rtimes_{\mathrm{r},\eta} \Gamma$ has a faithful tracial weight. In particular $C \rtimes_{\mathrm{r},\eta} \Gamma$ is stably finite. We also note that $C \rtimes_{\mathrm{r},\eta} \Gamma$ is simple. To see this, observe that $B \rtimes_{\mathrm{r},\beta} \Gamma$ is isomorphic to $\mathbb{K}(\ell^2(\Gamma)) \otimes \mathbb{K}(\mathfrak{H}) \otimes A$. Since $\mathfrak{B} \rtimes_{\mathrm{r},\gamma} \Gamma$ is isomorphism to an inductive limit of copies of $B \rtimes_{\mathrm{r},\beta} \Gamma$, we conclude the simplicity of $\mathfrak{B} \rtimes_{\mathrm{r},\gamma} \Gamma$. Now the same proof as the simplicity of $C$ proves the simplicity of $C \rtimes_{\mathrm{r},\eta} \Gamma \cong (\mathfrak{B} \rtimes_{\mathrm{r},\gamma} \Gamma) \rtimes_{\mathrm{r}} \mathbb{Z}$.

(3): When $\Gamma$ is non-amenable, the C$^*$-algebra $C$ has proper tracial weights other than the scalar multiples of $\tau_C$. This follows from the following general observation. By [1], non-amenable groups cannot act amenably on a factor (in the von Neumann algebra sense). Thus, when a non-amenable group has an amenable action on a C$^*$-algebra, its space of all proper tracial weights cannot be a ray.

Alternatively, one can give different proper tracial weights on $C$ as follows. Let $\nu$ denote the averaging measure on $S^{-1} \subset \Gamma$. Then we replace the counting measure in the definition of $\tau_B$ by another (nonzero, $\sigma$-finite) right $\nu$-harmonic measure $\mu$ on $\Gamma$ (that is, $\mu * \nu = \mu$). This gives a $\sigma$-invariant proper tracial weight $\tau_{B,\mu}$ on $B$. Then, by the same way as in the original case, $\tau_{B,\mu}$ gives an $\alpha$-invariant proper tracial weight on $\mathfrak{B}$. As a result we obtain a proper tracial weight on $C$. (The new proper tracial weight is no longer $\Gamma$-invariant, unlike the original $\tau_C$.) Note that $\Gamma$ has many right $\nu$-harmonic measures. In fact, the space $\mathrm{H}^\infty_{\mathrm{r}}(\Gamma, \nu)$ of bounded right $\nu$-harmonic functions is already huge. To see this, define $P_\nu \colon \ell^\infty(\Gamma) \to \ell^\infty(\Gamma)$ to be

$P_\nu(f) := f * \nu$, $f \in \ell^\infty(\Gamma)$. Then any cluster point of the sequence

$$\Phi_n := \frac{1}{n}\sum_{k=1}^{n} P_\nu^k \colon \ell^\infty(\Gamma) \to \ell^\infty(\Gamma), \quad n \in \mathbb{N},$$

in the point-ultraweak topology gives a positive projection $\Phi\colon \ell^\infty(\Gamma) \to \mathrm{H}^\infty_{\mathrm{r}}(\Gamma, \nu)$. Moreover $\Phi$ commutes with the left translation $\Gamma$-action. Since $\Gamma$ is non-amenable, the Choi–Effros product makes $\mathrm{H}^\infty_{\mathrm{r}}(\Gamma, \nu)$ an infinite-dimensional injective C*-algebra.

(4): Consider the special case that one has a (continuous) flow $\Theta\colon \mathbb{R} \curvearrowright A$ which commutes with $\rho$ and satisfies $\tau_A \circ \Theta_t = \kappa^t \tau_A$, $t \in \mathbb{R}$, for some constant $\kappa > 1$. (See e.g., [13].) Then clearly $\Xi_t := \mathrm{id}_{c_0(\Gamma)} \otimes \mathrm{id}_{\mathbb{K}(\mathfrak{H})} \otimes \Theta_t$ and $\sigma$ commute for all $t \in \mathbb{R}$. Hence one has a flow $\Psi\colon \mathbb{R} \curvearrowright \mathfrak{B}$ satisfying $\Psi_t \circ \theta_n = \theta_n \circ \Xi_t$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. The flow $\Psi$ and the automorphism $\alpha$ commute hence $\Psi$ induces a flow $\Omega$ on $C$. By the definitions of $\Psi$ and $\tau_C$, one has $\tau_C \circ \Omega_t = \kappa^t \tau_C$ for all $t \in \mathbb{R}$. In particular, the resulting C*-algebra $C$ is stably projectionless.

*Proof of Theorem B.* Take an amenable action $\beta\colon G \curvearrowright X$ on a compact metrizable space [16], [4]. Let $\gamma\colon G \curvearrowright \mathrm{Prob}(X)$ denote the action induced from $\beta$. Here we equip $\mathrm{Prob}(X) \subset C(X)^*$ with the weak-$*$ topology. Then it is known that $\gamma$ is again amenable (see e.g., Exercise 15.2.1 of [5] or Proposition 3.5 of [17]). Note that $\mathrm{Prob}(X)$ is a metrizable compact convex subset of $C(X)^*$ in the weak-$*$ topology. Therefore any closed amenable subgroup $L$ of $G$ has a fixed point in $\mathrm{Prob}(X)$. Hence $\mathrm{Prob}(X)$ is $L$-equivariantly contractible. In particular the unital inclusion $\mathbb{C} \subset C(\mathrm{Prob}(X))$ is a weak equivalence in $\mathrm{KK}^G$ in the sense of [14].

We now apply Theorem 6.1 in [17] to $(C(\mathrm{Prob}(X)), \gamma)$. This gives an ambient $G$-C*-algebra $(A, \alpha)$ of $(C(\mathrm{Prob}(X)), \gamma)$ satisfying the following conditions:

(1) $A$ is a Kirchberg algebra,
(2) $(A, \alpha)$ is amenable,
(3) the inclusion $C(\mathrm{Prob}(X)) \subset A$ is unital and $\mathrm{KK}^G$-equivalent.

From conditions (1) and (3), one has $A \cong \mathcal{O}_\infty$ by the Kirchberg–Phillips classification theorem [11], [19].

When $G$ satisfies the strong Baum-Connes conjecture (in the sense of [15], page 304), the unital inclusion $\mathbb{C} \subset C(\mathrm{Prob}(X))$ and hence also the unital inclusion $\mathbb{C} \subset A$ is $\mathrm{KK}^G$-equivalent (cf. Proposition 4.3 and Corollary 7.3 in [14]). ☐

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