ANALYSIS OF THE CRITICAL CR GJMS OPERATOR

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ABSTRACT. The critical CR GJMS operator on a strictly pseudoconvex CR manifold is a non-hypoelliptic CR invariant differential operator. We prove that, under the embeddability assumption, it is essentially self-adjoint and has closed range. Moreover, its spectrum is discrete, and the eigenspace corresponding to each non-zero eigenvalue is a finite-dimensional subspace of the space of smooth functions. As an application, we obtain a necessary and sufficient condition for the existence of a contact form with zero CR Q-curvature.

CONTENTS

1. INTRODUCTION

It is one of the most important topics in both conformal and CR geometries to study invariant differential operators. Analytic properties of such operators are deeply connected to geometric problems, such as the Yamabe problem and the constant Q-curvature problem.

In conformal geometry, Graham, Jenne, Mason, and Sparling [GJMS92] have constructed a family of conformally invariant differential operators, called GJMS operators. Let (N, g) be a Riemannian manifold of dimension n. For $k \in \mathbb{N}$ and $k \leq n/2$ if n is even, the k-th GJMS operator P_k is a differential operator acting on $C^\infty(N)$ such that its principal part coincides with the k-th power of the Laplacian, and it has the following transformation

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law under the conformal change $\hat{g} = e^{2\Upsilon} g$:

$$e^{(n/2+k)\Upsilon}\widehat{P}_k = P_k e^{(n/2-k)\Upsilon},$$

where $\widehat{P}_k$ is defined in terms of $\hat{g}$. Analytic properties of P_k on closed manifolds are quite simple. It follows from standard elliptic theory that P_k is essentially self-adjoint and has closed range. Moreover, its spectrum is a discrete subset of $\mathbb{R}$, and the eigenspace corresponding to each eigenvalue is a finite-dimensional subspace of $C^\infty(N)$.

In CR geometry, Gover and Graham [GG05] have introduced a family of CR invariant differential operators, called CR GJMS operators, via Fefferman construction. Let $(M, T^{1,0}M, \theta)$ be a $(2n+1)$-dimensional pseudo-Hermitian manifold and $k \in \mathbb{N}$ with $k \leq n+1$. The *k-th CR GJMS operator* P_k is a differential operator acting on $C^\infty(M)$ such that its principal part is the k-th power of the sub-Laplacian, and its transformation rule under the conformal change $\hat{\theta} = e^{\Upsilon}\theta$ is given by

$$e^{(n+1+k)\Upsilon/2}\widehat{P}_k = P_k e^{(n+1-k)\Upsilon/2},$$

where $\widehat{P}_k$ is defined in terms of $\hat{\theta}$. Although P_k is not elliptic, it is known to be *subelliptic* for $1 \leq k \leq n$ [Pon08a]; in particular, the same statements as in the previous paragraph also hold for P_k on closed manifolds. However, the kernel of the *critical CR GJMS operator* P_{n+1} contains the space of CR pluriharmonic functions, which is infinite-dimensional on closed embeddable CR manifolds (Remark 2.1). Moreover, there exist L^2 non-smooth CR pluriharmonic functions, which implies that P_{n+1} is not even hypoelliptic (Remark 2.2). In this paper, nevertheless, we will prove that similar results to the above are true for P_{n+1} on the orthogonal complement of $\operatorname{Ker} P_{n+1}$. In what follows, we simply write P for the critical CR GJMS operator.

In the remainder of this section, let $(M, T^{1,0}M, \theta)$ be a closed embeddable pseudo-Hermitian manifold of dimension $2n+1$. Here, “embeddable” means that $(M, T^{1,0}M)$ can be CR embedded into some $\mathbb{C}^N$. Note that the embeddability automatically holds if $n \geq 2$ [BdM75]. We consider P as an unbounded operator on $L^2(M)$ with domain

$$\operatorname{Dom} P = \left\{ u \in L^2(M) \;\middle|\; Pu \text{ in the weak sense is in } L^2(M) \right\}.$$

We will first prove

Theorem 1.1. *The operator P is self-adjoint and has closed range.*

Moreover, we obtain the following theorem on the spectrum of P:

Theorem 1.2. *The spectrum of P is a discrete subset in $\mathbb{R}$ and consists only of eigenvalues. Moreover, the eigenspace corresponding to each non-zero eigenvalue of P is a finite-dimensional subspace of $C^\infty(M)$. Furthermore,* $\operatorname{Ker} P \cap C^\infty(M)$ *is dense in* $\operatorname{Ker} P$.

In dimension three, Hsiao [Hsi15] has shown Theorems 1.1 and 1.2 by using Fourier integral operators with complex phase. Our proofs are similar to Hsiao's ones, but based on the *Heisenberg calculus*, the theory of Heisenberg pseudodifferential operators. The use of these operators simplifies some proofs and gives better regularity results.

We will also give some applications of these theorems and their proofs. Let $\mathscr{P}$ and $\overline{\mathscr{P}}$ be the space of CR pluriharmonic functions and its L^2-closure respectively. Then $\operatorname{Ker} P$ contains $\overline{\mathscr{P}}$, and the *supplementary space* $\mathscr{W}$ is defined by

$$\mathscr{W} := \operatorname{Ker} P \cap \overline{\mathscr{P}}^{\perp}.$$

Proposition 1.3. *The supplementary space $\mathscr{W}$ is a finite dimensional subspace of $C^\infty(M)$.*

In dimension three, Proposition 1.3 has been already proved by Hsiao [Hsi15]. However, in this case, the author [Tak20] has shown that $\mathscr{W}$ is equal to zero. On the other hand, for each $n \geq 2$, there exists a closed pseudo-Hermitian manifold $(M, T^{1,0}M, \theta)$ of dimension $2n+1$ such that $\mathscr{W} \neq 0$; see the proof of [Tak18, Theorem 1.6].

We will also tackle the zero CR Q-curvature problem. The *CR Q-curvature* Q, introduced by Fefferman and Hirachi [FH03], is a smooth function on M such that it transforms as follows under the conformal change $\hat{\theta} = e^{\Upsilon}\theta$:

$$\widehat{Q} = e^{-(n+1)\Upsilon}(Q + P\Upsilon), \tag{1.1}$$

where $\widehat{Q}$ is defined in terms of $\hat{\theta}$. Marugame [Mar18] has proved that the total CR Q-curvature

$$\overline{Q} := \int_M Q\,\theta \wedge (d\theta)^n$$

is always equal to zero. Moreover, the CR Q-curvature itself is identically zero for pseudo-Einstein contact forms [FH03]. Hence it is natural to ask whether $(M, T^{1,0}M)$ admits a contact form whose CR Q-curvature vanishes identically; this is the *zero CR Q-curvature problem*. This problem has been solved affirmatively for embeddable CR three-manifolds by the author [Tak20]. However, it is still open in general. By the transformation law (1.1), it is necessary that

$$\int_M fQ\,\theta \wedge (d\theta)^n = 0$$

holds for any $f \in \operatorname{Ker} P \cap C^\infty(M)$. Note that this condition is independent of the choice of θ. The following proposition states that it is also a sufficient condition for embeddable CR manifolds:

Proposition 1.4. *There exists a contact form $\hat{\theta}$ on M such that the CR Q-curvature $\widehat{Q}$ vanishes identically if and only if $Q \perp (\operatorname{Ker} P \cap C^\infty(M))$.*

This paper is organized as follows. In Section 2, we recall basic facts on CR manifolds. Section 3 deals with convolution operators on the Heisenberg group, which is a "model" of the Heisenberg calculus. In Section 4, we give a brief exposition of the Heisenberg calculus. Section 5 is devoted to proofs of the main results in this paper.

2. CR manifolds

Let M be an orientable smooth $(2n+1)$-dimensional manifold without boundary. A *CR structure* is a rank n complex subbundle $T^{1,0}M$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ such that

$$T^{1,0}M \cap T^{0,1}M = 0, \qquad \left[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)\right] \subset \Gamma(T^{1,0}M),$$

where $T^{0,1}M$ is the complex conjugate of $T^{1,0}M$ in $TM \otimes \mathbb{C}$. Define a hyperplane bundle HM of TM by $HM \coloneqq \operatorname{Re} T^{1,0}M$. A typical example of CR manifolds is a real hypersurface M in an $(n+1)$-dimensional complex manifold X; this M has the canonical CR structure

$$T^{1,0}M \coloneqq T^{1,0}X|_M \cap (TM \otimes \mathbb{C}).$$

Take a nowhere-vanishing real one-form θ on M such that θ annihilates $T^{1,0}M$. The *Levi form* $\mathcal{L}_\theta$ with respect to θ is the Hermitian form on $T^{1,0}M$ defined by

$$\mathcal{L}_\theta(Z, W) \coloneqq -\sqrt{-1}\, d\theta(Z, \overline{W}), \qquad Z, W \in T^{1,0}M.$$

A CR structure $T^{1,0}M$ is said to be *strictly pseudoconvex* if the Levi form is positive definite for some θ; such a θ is called a *contact form*. The triple $(M, T^{1,0}M, \theta)$ is called a *pseudo-Hermitian manifold*. Denote by T the *Reeb vector field* with respect to θ; that is, the unique vector field satisfying

$$\theta(T) = 1, \qquad T \lrcorner\, d\theta = 0.$$

Define an operator $\overline{\partial}_b \colon C^\infty(M) \to \Gamma((T^{0,1}M)^*)$ by

$$\overline{\partial}_b f \coloneqq df|_{T^{0,1}M}.$$

A smooth function f is called a *CR holomorphic function* if $\overline{\partial}_b f = 0$. A *CR pluriharmonic function* is a real-valued smooth function that is locally the real part of a CR holomorphic function. We denote by $\mathscr{P}$ the space of CR pluriharmonic functions.

Remark 2.1. It is known that the spaces of CR holomorphic functions and CR pluriharmonic functions are infinite-dimensional if there exists a not locally constant CR holomorphic function f. Suppose to the contrary that the space of CR holomorphic functions is finite-dimensional. Then f is algebraically dependent over $\mathbb{C}$ since f^k is also CR holomorphic for $k \in \mathbb{N}$. This implies that f is locally constant, which is a contradiction. Taking the real part yields that the space of CR pluriharmonic functions is also infinite-dimensional.

Remark 2.2. Let

$$S^{2n+1} = \left\{\, z = (z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; |z|^2 = 1 \,\right\}$$

be the unit sphere in $\mathbb{C}^{n+1}$ with the standard CR structure. The function $u = \log\left|1 - z^1\right|^2$ is L^2 but not continuous. Moreover, $u_\varepsilon = \log\left|1 + \varepsilon - z^1\right|^2$

is CR pluriharmonic and $u_\varepsilon \to u$ as $\varepsilon \to +0$ in $L^2(S^{2n+1})$. Hence u is an example of L^2 non-smooth CR pluriharmonic functions.

The Levi form induces a Hermitian metric on $(T^{0,1}M)^*$. By using this Hermitian metric and the volume form $\theta \wedge (d\theta)^n$, we obtain the formal adjoint $\overline{\partial}_b^* \colon \Gamma((T^{0,1}M)^*) \to C^\infty(M)$ of $\overline{\partial}_b$. The *Kohn Laplacian* $\Box_b$ and the *sub-Laplacian* Δ_b are defined by

$$\Box_b \coloneqq \overline{\partial}_b^* \overline{\partial}_b, \qquad \Delta_b \coloneqq \Box_b + \overline{\Box}_b.$$

Note that

$$\Box_b = \frac{1}{2}\Delta_b + \frac{\sqrt{-1}}{2} nT;$$

see [Lee86, Theorem 2.3] for example. The Gaffney extension of the Kohn Laplacian, also denoted by $\Box_b$, is a self-adjoint operator on $L^2(M)$. The kernel $\operatorname{Ker} \Box_b$ is the space of L^2 CR holomorphic functions.

The *critical CR GJMS operator* P is a real differential operator of order $2n+2$ acting on $C^\infty(M)$. It is known to be formally self-adjoint [GG05, Proposition 5.1]. Moreover, it annihilates CR pluriharmonic functions [Hir14, Section 3.2].

A CR manifold $(M, T^{1,0}M)$ is said to be *embeddable* if there exists a smooth embedding of M to some $\mathbb{C}^N$ such that $T^{1,0}M = T^{1,0}\mathbb{C}^N|_M \cap (TM \otimes \mathbb{C})$. It is known that a closed strictly pseudoconvex CR manifold $(M, T^{1,0}M)$ is embeddable if and only if $\Box_b$ has closed range [BdM75, Koh86].

3. Model operators on the Heisenberg group

The Heisenberg group G is the Lie group with the underlying manifold $\mathbb{R} \times \mathbb{C}^n$ and the multiplication

$$(t, z) \cdot (t', z') \coloneqq (t + t' + 2\operatorname{Im}(z \cdot \overline{z}'), z + z').$$

The left translation by (t, z) and the inversion on G are denoted by $l_{(t,z)}$ and ι respectively.

For $\alpha = 1, \dots, n$, we introduce a left-invariant complex vector field Z_α^0 by

$$Z_\alpha^0 \coloneqq \frac{\partial}{\partial z^\alpha} + \sqrt{-1}\overline{z}^\alpha \frac{\partial}{\partial t}.$$

The canonical CR structure $T^{1,0}G$ is spanned by $Z_1^0, \dots, Z_n^0$. Define a left-invariant one-form θ^0 on G by

$$\theta^0 \coloneqq dt + \sqrt{-1} \sum_{\alpha=1}^n (z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha).$$

Then θ^0 annihilates $T^{1,0}G$ and the Levi form $\mathcal{L}_{\theta^0}$ satisfies $\mathcal{L}_{\theta^0}(Z_\alpha^0, Z_\beta^0) = 2\delta_{\alpha\beta}$; in particular, θ^0 is a contact form on G. The Reeb vector field T^0 coincides with $\partial/\partial t$.

The Lie algebra $\mathfrak{g}$ of G is isomorphic to $\mathbb{R} \times \mathbb{C}^n$ as a linear space via

$$\mathfrak{g} \to \mathbb{R} \times \mathbb{C}^n; \qquad tT^0 + 2\sum_{\alpha=1}^n \operatorname{Re}(z^\alpha Z_\alpha^0) \mapsto (t, z).$$

Under this identification, the Lie bracket on $\mathfrak{g}$ is given by

$$[(t,z),(t',z')] = (4\operatorname{Im}(z\cdot\overline{z}'),0).$$

Moreover, the exponential map $\mathfrak{g}\to G$ coincides with the identity map on $\mathbb{R}\times\mathbb{C}^n$. Furthermore, the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ is also canonically isomorphic to $\mathbb{R}\times\mathbb{C}^n$ as a linear space. We write this linear coordinate as (τ,ζ).

For $r\in\mathbb{R}_+$, the parabolic dilation δ_r on $\mathbb{R}\times\mathbb{C}^n$ is defined by

$$\delta_r(t,z) = (r^2t, rz).$$

This dilation defines automorphisms on G, $\mathfrak{g}$, and $\mathfrak{g}^*$, for which we will use the same letter δ_r by abuse of notation. In what follows, the term "homogeneous" is defined in terms of δ_r. We will sometime write v for a point of G. Denote by dv the Lebesgue measure on G, which is a Haar measure on G.

Let $\mathscr{S}(G)$ (resp. $\mathscr{S}(\mathfrak{g}^*)$) be the space of rapidly decreasing functions on G (resp. $\mathfrak{g}^*$), and $\mathscr{S}'(G)$ (resp. $\mathscr{S}'(\mathfrak{g}^*)$) be that of tempered distributions on G (resp. $\mathfrak{g}^*$). The coupling of $f\in\mathscr{S}(G)$ and $k\in\mathscr{S}'(G)$ is written as $\langle k,f\rangle$. The pull-back by δ_r induces endomorphisms on $\mathscr{S}(G)$ and $\mathscr{S}(\mathfrak{g}^*)$, and these extend to those on $\mathscr{S}'(G)$ and $\mathscr{S}'(\mathfrak{g}^*)$. The Fourier transform $\mathcal{F}$ defines isomorphisms

$$\mathscr{S}(G)\xrightarrow{\cong}\mathscr{S}(\mathfrak{g}^*),\qquad \mathscr{S}'(G)\xrightarrow{\cong}\mathscr{S}'(\mathfrak{g}^*);$$

in our convention, the Fourier transform $\mathcal{F}(f)$ of $f\in\mathscr{S}(G)$ is defined by

$$\mathcal{F}(f)(\tau,\zeta) := \int_G e^{-\sqrt{-1}(t\tau+\operatorname{Re}(z\cdot\overline{\zeta}))} f(t,z)dv.$$

Now we consider "model operators" of the Heisenberg calculus. For $m\in\mathbb{R}$, set

$$S^m_H := \{\, a\in C^\infty(\mathfrak{g}^*\setminus\{0\}) \mid \delta_r^*a = r^m a \,\},$$

which is the space of *Heisenberg symbols* of order m. Let $\mathscr{G}^m$ be the space of $g\in\mathscr{S}'(\mathfrak{g}^*)$ such that g is smooth on $\mathfrak{g}^*\setminus\{0\}$ and satisfies

$$\delta_r^*g = r^m g + (r^m\log r)h,$$

where $h\in\mathscr{S}'(\mathfrak{g}^*)$ with $\operatorname{supp}h\subset\{0\}$ and $\delta_r^*h = r^m h$. The restriction map $\mathscr{G}^m\to S^m_H$ is known to be surjective [BG88, Proposition 15.8]. Moreover, the inverse Fourier transform gives an isomorphism

$$\mathcal{F}^{-1}\colon \mathscr{G}^m\xrightarrow{\cong}\mathscr{K}_{-m-2n-2},$$

where $\mathscr{K}_l$ is the space of $k\in\mathscr{S}'(G)$ such that k is smooth on $G\setminus\{0\}$ and satisfies

$$\delta_r^*k = r^l k + (r^l\log r)\psi$$

for a homogeneous polynomial ψ of degree l [BG88, Proposition 15.24]. We also introduce a function space on which Heisenberg symbols act. Let $\mathscr{S}_0(G)$ be the space of $f\in\mathscr{S}(G)$ such that

$$\int_G \psi(v)f(v)dv = 0$$

for any polynomial ψ on G. This condition is equivalent to that $\mathcal{F}(f) \in \mathscr{S}(\mathfrak{g}^*)$ vanishes to infinite order at the origin.

We denote by Ψ_H^m the space of endomorphisms A on $\mathscr{S}_0(G)$ commuting with left translation and admitting its formal adjoint A^* of homogeneous degree m; that is,

$$A^* \circ \delta_r^* = r^m \delta_r^* \circ A^*.$$

We would like to define a canonical isomorphism between S_H^m and Ψ_H^m.

Proposition 3.1. *Let $a \in S_H^m$ and take $g \in \mathscr{G}^m$ with $g|_{\mathfrak{g}^* \setminus \{0\}} = a$. Then the convolution operator*

$$f \mapsto [\mathcal{F}^{-1}(g) * f](v) := \left\langle \mathcal{F}^{-1}(g), f \circ l_v \circ \iota \right\rangle \tag{3.1}$$

defines an endomorphism on $\mathscr{S}_0(G)$ and is independent of the choice of g. Moreover, this operator commutes with left translation and is homogeneous of degree m. Furthermore, it is equal to zero if and only if $a = 0$.

Definition 3.2. For $a \in S_H^m$, an operator $O^0(a) \colon \mathscr{S}_0(G) \to \mathscr{S}_0(G)$ is defined by (3.1).

Proof of Proposition 3.1. It follows from [CGGP92, Proposition 2.2] that (3.1) defines an endomorphism on $\mathscr{S}_0(G)$ commuting with left translation and homogeneous of degree m. Assume that g' also satisfies $g'|_{\mathfrak{g}^* \setminus \{0\}} = a$. Then the support of $g' - g$ is contained in $\{0\} \subset \mathfrak{g}^*$. Hence $\mathcal{F}^{-1}(g' - g)$ is a polynomial on G, and so $\mathcal{F}^{-1}(g' - g) * f = 0$ for any $f \in \mathscr{S}_0(G)$. This implies the independence of the choice of g. Next, suppose that the operator (3.1) is equal to zero. For any $f \in \mathscr{S}_0(G)$, we have $\langle \mathcal{F}^{-1}(g), f \circ \iota \rangle = 0$. Hence g annihilates $\mathcal{F}(\mathscr{S}_0(G))$. Since $C_c^\infty(\mathfrak{g}^* \setminus \{0\})$ is a subspace of $\mathcal{F}(\mathscr{S}_0(G))$, the support of g is contained in $\{0\} \subset \mathfrak{g}^*$. Therefore $a = g|_{\mathfrak{g}^* \setminus \{0\}} = 0$. □

The operator $O^0(a)$ is well-behaved under formal adjoint and composition.

Theorem 3.3. *(i) The formal adjoint of $O^0(a)$, $a \in S_H^m$, is given by $O^0(\overline{a})$. In particular, $O^0(a)$ is formally self-adjoint if and only if a is real-valued.*

(ii) There exists a bilinear product

$$*^0 \colon S_H^{m_1} \times S_H^{m_2} \to S_H^{m_1+m_2}$$

such that $O^0(a_1)O^0(a_2) = O^0(a_1 *^0 a_2)$ for any $a_1 \in S_H^{m_1}$ and $a_2 \in S_H^{m_2}$.

Proof. (i) Take $g \in \mathscr{G}^m$ with $g|_{\mathfrak{g}^* \setminus \{0\}} = a$ The formal adjoint of $O^0(a)$ is given by the convolution with respect to

$$\overline{\mathcal{F}^{-1}(g)} \circ \iota = \mathcal{F}^{-1}(\overline{g});$$

see [CGGP92, Section 3]. Thus we have $(O^0(a))^* = O^0(\overline{a})$.

(ii) See [Pon08a, Proposition 3.1.3(2)]. □

In particular, O^0 defines an injective map from S_H^m to Ψ_H^m. In fact, this is an isomorphism.

Proposition 3.4. *For any $A \in \Psi_H^m$, there exists the unique $a \in S_H^m$ such that $A = O^0(a)$.*

Proof. Let $A \in \Psi_H^m$. By [CGGP92, Proposition 3.2], we have $k \in \mathscr{K}_{-m-2n-2}$ such that $Af = k * f$ for any $f \in \mathscr{S}_0(G)$. If we define $a \in S_H^m$ by $a \coloneqq \mathcal{F}(k)|_{\mathfrak{g}^* \setminus \{0\}}$, then $O^0(a)$ coincides with A by definition. □

Definition 3.5. The *Heisenberg symbol*

$$\sigma_m^0 \colon \Psi_H^m \to S_H^m$$

is defined by the inverse map of O^0.

It follows from Theorem 3.3 that

$$\sigma_m^0(A^*) = \overline{\sigma_m^0(A)}, \qquad \sigma_{m_1+m_2}^0(A_1 A_2) = \sigma_{m_1}^0(A_1) *^0 \sigma_{m_2}^0(A_2)$$

for $A \in \Psi_H^m$, $A_1 \in \Psi_H^{m_1}$, and $A_2 \in \Psi_H^{m_2}$. In particular, A is formally self-adjoint if and only if $\sigma_m^0(A)$ is real-valued.

Before the end of this section, we note a relation between the Reeb vector field and Ψ_H^m.

Lemma 3.6. *The Reeb vector field T^0 commutes with any $A \in \Psi_H^m$.*

Proof. The vector field T^0 generates the flow $l_{(t,0)}$. Since $A \in \Psi_H^m$ commutes with left translation, we have $[T^0, A] = 0$. □

4. Heisenberg calculus

In this section, we recall basic properties of Heisenberg pseudodifferential operators; see [BG88, Pon08a] for a comprehensive introduction to the Heisenberg calculus.

Throughout this section, we fix a closed pseudo-Hermitian manifold $(M, T^{1,0}M, \theta)$ of dimension $2n+1$. Let

$$\mathfrak{g}M \coloneqq (TM/HM) \oplus HM.$$

The Reeb vector field T defines a nowhere-vanishing section $[T]$ of TM/HM. For sections X_0 and Y_0 of TM/HM and X' and Y' of HM, the Lie bracket $[X_0 + X', Y_0 + Y']$ is defined by

$$[X_0 + X', Y_0 + Y'] \coloneqq -d\theta(X', Y')[T].$$

This bracket makes $\mathfrak{g}M$ a bundle of two-step nilpotent Lie algebras. The dilation δ_r on $\mathfrak{g}M$ is defined by

$$\delta_r|_{TM/HM} \coloneqq r^2, \qquad \delta_r|_{HM} \coloneqq r.$$

It follow from the definition of the Lie bracket that δ_r is a fiberwise Lie algebra isomorphism. Set $GM \coloneqq \mathfrak{g}M$ as a smooth fiber bundle with the fiberwise group structure defined via the Baker-Campbell-Hausdorff formula. The dilation δ_r on $\mathfrak{g}M$ induces that on GM, which we write as δ_r for abbreviation.

Take a local frame (Z_α) of $T^{1,0}M$ on an open set $U \subset M$ such that

$$\mathcal{L}_\theta(Z_\alpha, Z_\beta) = 2\delta_{\alpha\beta}.$$

Then the map

$$\mathfrak{g}M|_U \to U \times \mathfrak{g}; \qquad \left(p, t[T] + 2\operatorname{Re}\sum_{\alpha=1}^{n} z^\alpha Z_\alpha\right) \mapsto (p, t, z) \tag{4.1}$$

gives an isomorphism between fiber bundles of Lie algebras. This isomorphism is compatible with the dilation. The identification (4.1) induces those on GM and the dual bundle $\mathfrak{g}^*M := (\mathfrak{g}M)^*$ of $\mathfrak{g}M$:

$$GM|_U \to U \times G, \qquad \mathfrak{g}^*M|_U \to U \times \mathfrak{g}^*. \tag{4.2}$$

These are also compatible with the dilation. Let (Z'_α) be another local frame of $T^{1,0}M$ on U satisfying $\mathcal{L}_\theta(Z'_\alpha, Z'_\beta) = 2\delta_{\alpha\beta}$. This gives another identification $\mathfrak{g}M|_U \to U \times \mathfrak{g}$. These two identifications relate with each other by a smooth family $(U(p))_{p\in U}$ of unitary matrices; that is,

$$U \times \mathfrak{g} \to U \times \mathfrak{g}; \qquad (p, t, z) \mapsto (p, t, U(p) \cdot z).$$

The same is true for GM and $\mathfrak{g}^*M$.

For $m \in \mathbb{R}$, the space $S^m_H(M)$ consists of functions in $C^\infty(\mathfrak{g}^*M \setminus \{0\})$ that are homogeneous of degree m on each fiber. Under the identification (4.2), the fiberwise product $*^0$ induces a well-defined bilinear product

$$*\colon S^{m_1}_H(M) \times S^{m_2}_H(M) \to S^{m_1+m_2}_H(M).$$

Now we consider Heisenberg pseudodifferential operators. For $m \in \mathbb{R}$, denote by $\Psi^m_H(M)$ the space of *Heisenberg pseudodifferential operators* $A\colon C^\infty(M) \to C^\infty(M)$ *of order* m. This space is closed under complex conjugate, transpose, and formal adjoint [Pon08a, Proposition 3.1.23]. In particular, any $A \in \Psi^m_H$ extends to a linear operator

$$A\colon \mathscr{D}'(M) \to \mathscr{D}'(M),$$

where $\mathscr{D}'(M)$ is the space of distributions on M. For example, $V \in \Gamma(HM)$ is an element of $\Psi^1_H(M)$ and $T \in \Psi^2_H(M)$. Note that $\Psi^{-\infty}_H(M) := \bigcap_{m\in\mathbb{Z}} \Psi^m_H(M)$ coincides with the space of smoothing operators on M. As in the usual pseudodifferential calculus, there exists the *Heisenberg principal symbol*

$$\sigma_m\colon \Psi^m_H(M) \to S^m_H(M),$$

which has the following properties:

Proposition 4.1 ([Pon08a, Propositions 3.2.6 and 3.2.9])**.** *(i) The Heisenberg principal symbol σ_m gives the following exact sequence:*

$$0 \to \Psi^{m-1}_H(M) \to \Psi^m_H(M) \xrightarrow{\sigma_m} S^m_H(M) \to 0.$$

(ii) For $A_1 \in \Psi^{m_1}_H(M)$ and $A_2 \in \Psi^{m_2}_H(M)$, the operator A_1A_2 is a Heisenberg pseudodifferential operator of order $m_1 + m_2$, and

$$\sigma_{m_1+m_2}(A_1A_2) = \sigma_{m_1}(A_1) * \sigma_{m_2}(A_2).$$

On the other hand, there exists a crucial difference between the usual pseudodifferential calculus and the Heisenberg one. Since the product $*$ is non-commutative, the commutator $[A_1, A_2]$ of $A_1 \in \Psi_H^{m_1}(M)$ and $A_2 \in \Psi_H^{m_2}(M)$ is not an element of $\Psi_H^{m_1+m_2-1}(M)$ in general. However, we have the following

Lemma 4.2. *Let $A \in \Psi_H^m(M)$. Then $[T, A] \in \Psi_H^{m+1}(M)$.*

Proof. It is enough to show that $\sigma_{m+2}([T, A]) = 0$, or equivalently,

$$\sigma_2(T) * \sigma_m(A) = \sigma_m(A) * \sigma_2(T).$$

Fix an identification (4.2). Then $\sigma_2(T) \in S_H^2(M)$ is given by

$$\sigma_2(T)(p, \tau, \zeta) = \sqrt{-1}\tau = \sigma_2^0(T^0)(\tau, \zeta);$$

see [Pon08a, Example 3.2.5]. Hence it suffices to prove that $\sigma_2^0(T^0) *^0 a = a *^0 \sigma_2^0(T^0)$ holds for any $a \in S_H^m$. From Lemma 3.6, we obtain

$$O^0(\sigma_2^0(T^0) *^0 a) = T^0 O^0(a) = O^0(a) T^0 = O^0(a *^0 \sigma_2^0(T^0)),$$

which is equivalent to $\sigma_2^0(T^0) *^0 a = a *^0 \sigma_2^0(T^0)$. □

Next, consider approximate inverses of Heisenberg pseudodifferential operators. We write $A \sim B$ if $A - B$ is a smoothing operator.

Definition 4.3. Let $A \in \Psi_H^m(M)$. An operator $B \in \Psi_H^{-m}(M)$ is called a *parametrix* of A if $AB \sim I$ and $BA \sim I$.

The existence of a parametrix of a Heisenberg pseudodifferential operator is determined only by its Heisenberg principal symbol.

Proposition 4.4 ([Pon08a, Proposition 3.3.1])**.** *Let $A \in \Psi_H^m(M)$ with Heisenberg principal symbol $a \in S_H^m(M)$. Then A has a parametrix if and only if there exists $b \in S_H^{-m}(M)$ such that $a * b = b * a = 1$.*

Now consider the Heisenberg differential operator $\Delta_b + 1$ of order 2. It is known that this operator has a parametrix; see the proof of [Pon08a, Proposition 3.5.7] for example. Since $\Delta_b + 1$ is positive and self-adjoint, the s-th power $(\Delta_b + 1)^s$ of $\Delta_b + 1$, $s \in \mathbb{R}$, is a Heisenberg pseudodifferential operator of order $2s$ [Pon08a, Theorems 5.3.1 and 5.4.10]. Using this operator, we define

$$W_H^s(M) := \left\{ u \in \mathscr{D}'(M) \mid (\Delta_b + 1)^{s/2} u \in L^2(M) \right\}.$$

This space is a Hilbert space with the inner product

$$(u, v)_s = \left((\Delta_b + 1)^{s/2} u, (\Delta_b + 1)^{s/2} v \right)_{L^2(M)};$$

write $\|\cdot\|_s$ for the norm determined by $(\cdot, \cdot)_s$. The space $C^\infty(M)$ is dense in $W_H^s(M)$, and $C^\infty(M) = \bigcap_{s \in \mathbb{R}} W_H^s(M)$ [Pon08a, Proposition 5.5.3]. Note that, for $k \in \mathbb{N}$, the Hilbert space $W_H^k(M)$ coincides with the Folland-Stein space $S^{k,2}(M)$ as a topological vector space [Pon08a, Proposition 5.5.5]. Similar to the usual L^2-Sobolev space theory, we obtain the following

Lemma 4.5. *For $s_1 < s_2$, the embedding $W_H^{s_2}(M) \hookrightarrow W_H^{s_1}(M)$ is compact.*

Proof. The operator $(\Delta_b + 1)^{s'/2}$, $s' \in \mathbb{R}$, gives an isometry $W_H^{s+s'}(M) \to W_H^{s}(M)$, and so we may assume that $s_1 = 0$. From [Pon08a, Proposition 5.5.7], we derive that the embedding $W_H^{s_2}(M) \hookrightarrow W_H^0(M) = L^2(M)$ is the composition of the two embeddings $W_H^{s_2}(M) \hookrightarrow H^{s_2/2}(M)$ and $H^{s_2/2}(M) \hookrightarrow L^2(M)$, where $H^s(M)$ is the usual L^2-Sobolev space on M of order s. Thus the compactness of $W_H^{s_2}(M) \hookrightarrow L^2(M)$ follows from Rellich's lemma. □

Heisenberg pseudodifferential operators act on these Hilbert spaces as follows:

Proposition 4.6. *Any $A \in \Psi_H^m(M)$ extends to a continuous linear operator*

$$A \colon W_H^{s+m}(M) \to W_H^s(M)$$

for every $s \in \mathbb{R}$. In particular if $m < 0$, the operator $A \colon L^2(M) \to L^2(M)$ is compact.

Proof. The former statement follows from [Pon08a, Propositions 5.5.8]. The latter one is a consequence of the former one and Lemma 4.5. □

5. Proofs of the main results

In this section, we prove the main results in this paper. In what follows, we fix a closed embeddable pseudo-Hermitian manifold $(M, T^{1,0}M, \theta)$ of dimension $2n + 1$.

For $\mu \in \mathbb{R}$, we define a formally self-adjoint Heisenberg differential operator L_μ of order 2 by

$$L_\mu \coloneqq \frac{1}{2}\Delta_b + \frac{\sqrt{-1}}{2}\mu T.$$

It is known that L_μ has a parametrix $N_\mu \in \Psi_H^{-2}(M)$ if and only if $\mu \notin \pm(n + 2\mathbb{N})$; see the proof of [Pon08a, Proposition 3.5.7] for example. On the other hand, the embeddability of M implies that there exist the partial inverse $N_n \in \Psi_H^{-2}(M)$ of $L_n = \Box_b$ and the orthogonal projection $S \in \Psi_H^0(M)$ to $\operatorname{Ker} \Box_b$, called the *Szegő projection* [BG88, Theorem 24.20 and Corollary 25.67]. Note that $\sigma_0(S) \neq 0$; see [Pon08b, Section 5] for example. Taking the complex conjugate gives the partial inverse $N_{-n} \in \Psi_H^{-2}(M)$ of $L_{-n} = \overline{\Box}_b$ and the orthogonal projection $\overline{S} \in \Psi_H^0(M)$ to $\operatorname{Ker} \overline{\Box}_b$

Lemma 5.1. *For any $\mu \in \mathbb{R}$, one has $[L_\mu, S] \in \Psi_H^1(M)$.*

Proof. We have

$$[L_\mu, S] = [L_n, S] + \frac{\sqrt{-1}}{2}(\mu - n)[T, S] = \frac{\sqrt{-1}}{2}(\mu - n)[T, S] \in \Psi_H^1(M)$$

by Lemma 4.2. □

On the other hand, Hsiao [Hsi10, Chapter 7] has studied the distribution kernel of the Szegő projection. A similar discussion to [Hsi15, Lemma 4.2] yields

Lemma 5.2. *The operators $\overline{S}S$ and $S\overline{S}$ are smoothing operators.*

The critical CR GJMS operator P on $(M, T^{1,0}M, \theta)$ coincides with

$$L_n L_{n-2} \cdots L_{-n+2} L_{-n}$$

modulo $\Psi_H^{2n+1}(M)$; see [Pon08a, Proposition 3.5.7]. Set

$$G_0 \coloneqq N_{-n} N_{-n+2} \cdots N_{n-2} N_n \in \Psi_H^{-2n-2}(M), \qquad \Pi_0 \coloneqq S + \overline{S} \in \Psi_H^0(M)$$

Then modulo $\Psi_H^{-1}(M)$,

$$\begin{aligned} G_0 P &\equiv N_{-n} N_{-n+2} \cdots N_{n-2} N_n L_n L_{n-2} \cdots L_{-n+2} L_{-n} \\ &= N_{-n} N_{-n+2} \cdots N_{n-2} (I - S) L_{n-2} \cdots L_{-n+2} L_{-n} \\ &\equiv N_{-n} N_{-n+2} \cdots N_{n-2} L_{n-2} \cdots L_{-n+2} L_{-n} (I - S) \qquad (\because \text{Lemma 5.1}) \\ &\equiv (I - \overline{S})(I - S) \\ &= I - S - \overline{S} + \overline{S}S \\ &\equiv I - \Pi_0. \end{aligned}$$

Thus we have

$$R_0 \coloneqq G_0 P + \Pi_0 - I \in \Psi_H^{-1}(M).$$

This G_0 gives an approximation of the partial inverse of P.

Proposition 5.3. *There exists $G_\infty \in \Psi_H^{-2n-2}(M)$ such that*

$$G_\infty P + \Pi_0 - I \in \Psi_H^{-\infty}(M).$$

Proof. Since the critical GJMS operator annihilates CR pluriharmonic functions, we have $P\Pi_0 = 0$. Hence

$$(\Pi_0)^2 = (G_0 P + \Pi_0)\Pi_0 = \Pi_0 + R_0 \Pi_0.$$

On the other hand, $(\Pi_0)^2$ is equal to Π_0 modulo a smoothing operator by Lemma 5.2. Thus we have $R_0 \Pi_0 \in \Psi_H^{-\infty}(M)$. Take $G_\infty \in \Psi_H^{-2n-2}(M)$ such that

$$G_\infty - \sum_{l=0}^{k} (-R_0)^l G_0 \in \Psi_H^{-2n-k-3}(M)$$

for any $k \in \mathbb{N}$. Then modulo $\Psi_H^{-k-1}(M)$,

$$\begin{aligned} G_\infty P + \Pi_0 &\equiv \sum_{l=0}^{k} (-R_0)^l G_0 P + \Pi_0 \\ &= \sum_{l=0}^{k} (-R_0)^l (I - \Pi_0 + R_0) + \Pi_0 \\ &\equiv \sum_{l=0}^{k} (-R_0)^l (I + R_0) \\ &\equiv I. \end{aligned}$$

Therefore $G_\infty P + \Pi_0 - I$ is a smoothing operator. $\square$

Consider P as an unbounded closed operator on $L^2(M)$ with domain

$$\operatorname{Dom} P = \left\{ u \in L^2(M) \;\middle|\; Pu \text{ in the weak sense is in } L^2(M) \right\}.$$

This domain contains $W_H^{2n+2}(M)$ by Proposition 4.6. Conversely, any $u \in \operatorname{Dom} P$ is an element of $W_H^{2n+2}(M)$ modulo $\operatorname{Ker} P$ by the lemma below.

Lemma 5.4. *For $u \in \operatorname{Dom} P$, one has $u - \Pi_0 u \in W_H^{2n+2}(M)$. In particular,* $\operatorname{Dom} P = \operatorname{Ker} P + W_H^{2n+2}(M)$.

Proof. Set

$$R_\infty := G_\infty P + \Pi_0 - I \in \Psi_H^{-\infty}(M). \tag{5.1}$$

If $v = Pu \in L^2(M)$, then

$$u - \Pi_0 u = G_\infty v - R_\infty u \in W_H^{2n+2}(M).$$

In particular, $u \in \operatorname{Ker} P + W_H^{2n+2}(M)$ since $\Pi_0 u \in \operatorname{Ker} P$. □

Lemma 5.5. *The range* $\operatorname{Ran} P$ *of P is orthogonal to* $\operatorname{Ran} \Pi_0$ *in $L^2(M)$.*

Proof. Assume that $u \in \operatorname{Dom} P$ and $v \in L^2(M)$. Take a sequence $(v_j) \in C^\infty(M)$ such that v_j converges to v in $L^2(M)$ as $j \to +\infty$. Since $\Pi_0 \in \Psi_H^0(M)$, the function $\Pi_0 v_j$ is smooth and converges to $\Pi_0 v$ in $L^2(M)$ as $j \to +\infty$ also. Hence

$$(Pu, \Pi_0 v)_0 = \lim_{j\to\infty} (Pu, \Pi_0 v_j)_0 = \lim_{j\to\infty} (u, P\Pi_0 v_j)_0 = 0,$$

which completes the proof. □

Proof of Theorem 1.1. We first prove that P is self-adjoint. To this end, it is enough to show that P is symmetric. Let $u, v \in \operatorname{Dom} P$. It follows from Lemma 5.4 that $v' := v - \Pi_0 v$ is in $W_H^{2n+2}(M)$. Take a sequence (v_j) in $C^\infty(M)$ such that v_j converges to v' in $W_H^{2n+2}(M)$ as $j \to +\infty$. Then Pv_j converges to $Pv' = Pv$ in $L^2(M)$ as $j \to +\infty$ by the continuity of $P\colon W_H^{2n+2}(M) \to L^2(M)$. We derive from Lemma 5.5 that

$$(Pu, v)_0 = (Pu, v')_0 + (Pu, \Pi_0 v)_0 = \lim_{j\to\infty} (Pu, v_j)_0 = \lim_{j\to\infty} (u, Pv_j)_0 = (u, Pv)_0,$$

which means that P is symmetric.

We next prove that $P\colon \operatorname{Dom} P \to L^2(M)$ has closed range. It suffices to show that there exists $\varepsilon > 0$ such that

$$\|Pu\|_0 \geq \varepsilon \|u\|_0$$

for any $u \in \operatorname{Dom} P \cap (\operatorname{Ker} P)^\perp$. Note that $(\operatorname{Ker} P)^\perp \subset \operatorname{Ker} \Pi_0$ since $\operatorname{Ran} \Pi_0 \subset \operatorname{Ker} P$. Suppose to the contrary that we can take a sequence (u_j) in $\operatorname{Dom} P \cap (\operatorname{Ker} P)^\perp$ such that

$$\|u_j\|_0 = 1, \qquad \|Pu_j\|_0 \leq \frac{1}{j}.$$

Let R_∞ be as in (5.1). Then

$$u_j = G_\infty (Pu_j) - R_\infty u_j$$

is uniformly bounded in $W_H^{2n+2}(M)$. By Lemma 4.5, we may assume that u_j converges to some $u \in L^2(M)$ as $j \to +\infty$. We derive from the definition of u_j that u is in $(\operatorname{Ker} P)^\perp$ and $\|u\|_0 = 1$. However, since $\|Pu_j\|_0 \leq 1/j$, we have $u \in \operatorname{Dom} P$ and $Pu = 0$. This is a contradiction. □

Since P is a range-closed operator, there exist the partial inverse G of P and the orthogonal projection Π to $\operatorname{Ker} P$. Next, we show that these operators are Heisenberg pseudodifferential operators.

Theorem 5.6. *The operators G and Π are Heisenberg pseudodifferential operators of order $-2n-2$ and 0 respectively. Moreover, Π coincides with Π_0 modulo $\Psi_H^{-\infty}(M)$.*

Proof. First note that

$$\Pi\Pi_0 = \Pi_0\Pi = \Pi_0.$$

since $\operatorname{Ran} \Pi_0 \subset \operatorname{Ker} P$. Composing Π to (5.1) from the right and taking its adjoint, we have

$$\Pi_0 = \Pi + R_\infty \Pi, \qquad \Pi_0 = \Pi + \Pi(R_\infty)^*.$$

Hence

$$\Pi - \Pi_0 = -\Pi(R_\infty)^* = R_\infty \Pi (R_\infty)^* - \Pi_0 (R_\infty)^*,$$

which is a smoothing operator. In particular, Π is a Heisenberg pseudodifferential operator of order 0 and coincides with Π_0 modulo a smoothing operator.

Next consider G. Composing G to (5.1) from the right and taking its adjoint give that

$$G_\infty(I - \Pi) = G + R_\infty G, \qquad (I - \Pi)(G_\infty)^* = G + G(R_\infty)^*.$$

Hence

$$G - G_\infty(I - \Pi) = -R_\infty G = -R_\infty(I - \Pi)(G_\infty)^* + R_\infty G (R_\infty)^*,$$

which is a smoothing operator. Therefore G is a Heisenberg pseudodifferential operator of order $-2n-2$. □

This theorem proves Theorem 1.2.

Proof of Theorem 1.2. From Proposition 4.6 and Theorem 5.6, we derive that the partial inverse $G\colon L^2(M) \to L^2(M)$ is a compact self-adjoint operator. Hence the spectrum $\sigma(G)$ of G is bounded and consists only of eigenvalues, and 0 is the only accumulation point of $\sigma(G)$. Moreover, for any non-zero eigenvalue λ, the eigenspace $H_\lambda := \operatorname{Ker}(G - \lambda)$ is finite-dimensional, and there exists the following orthogonal decomposition:

$$L^2(M) = \operatorname{Ker} G \oplus \bigoplus_{\lambda \in \sigma(G) \setminus \{0\}} H_\lambda.$$

Furthermore, since G maps $W_H^s(M)$ to $W_H^{s+2n+2}(M)$, the eigenspace H_λ is a linear subspace of $C^\infty(M)$. By the definition of the partial inverse, H_λ is the eigenspace of P with eigenvalue $1/\lambda$, and $\operatorname{Ker} G = \operatorname{Ker} P$. Hence the spectrum $\sigma(P)$ is discrete and consists only of eigenvalues, and the eigenspace corresponding to each non-zero eigenvalue is a finite-dimensional subspace of $C^\infty(M)$. Moreover, $\operatorname{Ker} P \cap C^\infty(M)$ is dense in $\operatorname{Ker} P$ since the orthogonal projection Π to $\operatorname{Ker} P$ is a Heisenberg pseudodifferential operator of order 0. □

An argument similar to the proof of Theorem 5.6 also gives Proposition 1.3.

Proof of Proposition 1.3. Let π be the orthogonal projection to $\overline{\mathscr{P}}$. Note that $\Pi - \pi$ is the orthogonal projection to $\mathscr{W}$. Hence it is enough to prove that $\Pi - \pi$ is a smoothing operator. Since $\Pi \sim \Pi_0$, it suffices to show that $\pi - \Pi_0$ is a smoothing operator. Since $\operatorname{Ran} \Pi_0 \subset \operatorname{Ran} \pi$,

$$\pi \Pi_0 = \Pi_0 \pi = \Pi_0.$$

It follows from (5.1) that

$$\Pi_0 = \pi + R_\infty \pi, \qquad \Pi_0 = \pi + \pi (R_\infty)^*.$$

Therefore we have

$$\pi - \Pi_0 = -\pi (R_\infty)^* = R_\infty \pi (R_\infty)^* - \Pi_0 (R_\infty)^*,$$

which is a smoothing operator. □

As an application of results in this section, we give a necessary and sufficient condition for the zero CR Q-curvature problem.

Proof of Proposition 1.4. As we saw in the introduction, $Q \perp (\operatorname{Ker} P \cap C^\infty(M))$ if there exists a contact form with zero Q-curvature. Conversely, assume that Q is orthogonal to $\operatorname{Ker} P \cap C^\infty(M)$. It follows from Theorem 1.2 that Q is in fact orthogonal to $\operatorname{Ker} P$. Then $\Upsilon := -GQ \in C^\infty(M)$ and $P\Upsilon = -Q$. Hence $\hat{\theta} := e^\Upsilon \theta$ satisfies $\widehat{Q} = 0$. □

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