A PROOF OF CONJECTURED PARTITION IDENTITIES OF NANDI

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Abstract. We generalize the theory of linked partition ideals due to Andrews using finite automata in formal language theory and apply it to prove three Rogers–Ramanujan type identities for modulus 14 that were posed by Nandi through a vertex operator theoretic construction of the level 4 standard modules of the affine Lie algebra $A_2^{(2)}$.

1. Introduction

1.1. **Rogers–Ramanujan type identities.** A partition of a nonnegative integer n is a weakly decreasing sequence of positive integers (called parts) whose sum is n. We denote the set of all partitions by Par. Let $i = 1$ or 2. Then the celebrated Rogers–Ramanujan identities may be stated as follows.

The number of partitions of n such that parts are at least i and such that consecutive parts differ by at least 2 is equal to the number of partitions of n into parts congruent to $\pm i$ modulo 5. (1.1)

As q-series identities the Rogers–Ramanujan identities are stated as

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty},\quad \sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty},\tag{1.2}$$

where for $n\in\mathbb{Z}_{\geq 0}\sqcup\{\infty\}$,

$$(a;q)_n:=\prod_{0\leq j<n}(1-aq^j),\quad (a_1,\ldots,a_k;q)_n:=(a_1;q)_n\cdots(a_k;q)_n.$$

The identities (1.1) and (1.1) have a number of generalizations, often called *Rogers–Ramanujan type* (*RR type* for short) identities, arising from various motivations (see e.g., [4, §7], [35]). In particular, generalizations of (1.1) are called RR type partition identities, which are theorems of the form $\mathcal{C}\overset{\mathrm{PT}}{\sim}\mathcal{D}$ for $\mathcal{C},\mathcal{D}\subseteq\mathsf{Par}$ ([3, Definition 3]), meaning that partitions of n in $\mathcal{C}$ are equinumerous to those in $\mathcal{D}$ for all $n\geq 0$, where most commonly $\mathcal{C}$ (resp. $\mathcal{D}$) is given by "difference conditions" (resp. "congruence conditions") on parts.

1.2. **Algorithmic derivation of q-difference equations.** A common strategy to prove a RR type partition identity is to follow the three steps below (cf. [3, p. 1037]), starting with the set $\mathcal{C}\subseteq\mathsf{Par}$ given by "difference conditions" in the statement of the identity.

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(Step 1) Find a q-difference equation for the generating function

$$f_{\mathcal{C}}(x,q) := \sum_{\lambda \in \mathcal{C}} x^{\ell(\lambda)} q^{|\lambda|}, \tag{1.3}$$

where $|\lambda| := \sum_{i=1}^{\ell} \lambda_i \,(= \sum_{i\geq 1} i\, m_i(\lambda))$ and $\ell(\lambda) := \ell \,(= \sum_{i\geq 1} m_i(\lambda))$ for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \in \mathsf{Par}$ and $m_i(\lambda) := \#\{j \mid \lambda_j = i\}$ for $i \geq 1$.

(Step 2) Solve the equation and find a q-series expression for $f_{\mathcal{C}}(1,q)$.

(Step 3) Use q-series formulas to show that $f_{\mathcal{C}}(1,q)$ is equal to the desired infinite product corresponding to $\mathcal{D}$.

The aim of this paper is to give a proof (following these steps) for three conjectural RR type partition identities (see §1.3) posed by Nandi [30]. For that purpose, we give an extention (by using *finite automata* in *formal language theory*) of the theory of *linked partition ideals* introduced by Andrews [3],[4, §8], which provides in many cases an algorithmic derivation in the Step 1 above (see §1.4 for more details).

1.3. **Nandi's conjectures.** The Rogers–Ramanujan identities were one of the motivations for inventing vertex operators. It started from Lepowsky–Milne's observation [24], which led to Lepowsky–Wilson's proof for Rogers–Ramanujan identities [25–27] by constructing bases of the vacuum spaces $\Omega(V(\lambda))$ for the standard modules $V(\lambda)$ of the affine Lie algebra $A_1^{(1)}$ associated with the level 3 dominant integral weights λ, using certain vertex operators called *Z-operators*. Moreover, Andrews–Gordon's [2,20] and Andrews–Bressoud's [6,7] generalizations of the Rogers–Ramanujan identities can be interpreted and proved via similar constructions for the level ≥ 4 standard modules of $A_1^{(1)}$[27–29].

It is therefore natural to expect that there should exist a RR type identity corresponding to any given affine Lie type and a dominant integral weight (see [13,16,17,21,31] on recent progress). As a first step beyond the case $A_1^{(1)}$, Capparelli [9] investigated the structure of the level 3 standard modules of the affine Lie algebra $A_2^{(2)}$ via Z-operators, yielding some conjectural partition identities (which were later proved in [5,10,32,39,40] etc.). As a next step, Nandi [30] studied the level 4 standard modules of $A_2^{(2)}$ via Z-operators and conjectured some partition identities (Conjecture 1.2). For higher levels, see e.g., [34],[39, §1.4].

Definition 1.1. For a finite sequence $\boldsymbol{j} = (j_1, \dots, j_n)$ (which we assume to be nonempty for simplicity, i.e., $n > 0$) and a (finite or infinite) sequence $\boldsymbol{i} = (i_1, \dots, i_N)$ or $\boldsymbol{i} = (i_1, i_2, \dots)$, we say that, letting $\mathsf{len}(\boldsymbol{i}) := N (\geq 0)$ or ∞ respectively,

- $\boldsymbol{i}$ *matches* $\boldsymbol{j}$ if $(i_{k+1}, i_{k+2}, \dots, i_{k+n}) = (j_1, j_2, \dots, j_n)$ for some $0 \leq k \leq \mathsf{len}(\boldsymbol{i}) - n$,
- $\boldsymbol{i}$ *begins with* $\boldsymbol{j}$ if $n \leq \mathsf{len}(\boldsymbol{i})$ and $(i_1, i_2, \dots, i_n) = (j_1, j_2, \dots, j_n)$.

Conjecture 1.2 (Nandi [30, §8.1]. See also [35, Conjecture 5.5, 5.6, 5.7])**.** *Let $\mathcal{N}$ denote the set of partitions λ satisfying the conditions* (N1)-(N6)*:*

(N1) *For all $1 \leq i \leq \ell(\lambda) - 1$, $\lambda_i - \lambda_{i+1} \neq 1$.*

(N2) *For all* $1 \le i \le \ell(\lambda) - 2$, $\lambda_i - \lambda_{i+2} \ge 3$.
(N3) *For all* $1 \le i \le \ell(\lambda) - 2$, $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \ne \lambda_{i+1}$.
(N4) *For all* $1 \le i \le \ell(\lambda) - 2$, $\lambda_i - \lambda_{i+2} = 3$ *and* $2 \nmid \lambda_i \implies \lambda_{i+1} \ne \lambda_{i+2}$.
(N5) *For all* $1 \le i \le \ell(\lambda) - 2$,
$\lambda_i - \lambda_{i+2} = 4$ *and* $2 \nmid \lambda_i \implies \lambda_i \ne \lambda_{i+1}$ *and* $\lambda_{i+1} \ne \lambda_{i+2}$.
(N6) $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{\ell(\lambda)-1} - \lambda_{\ell(\lambda)})$ *does not match* $(3, 2^*, 3, 0)$. *Here* 2^* *denotes any number (possibly zero) of repetitions of* 2.

Define $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \subseteq \mathcal{N}$ *by*

$$\mathcal{N}_1 = \{\lambda \in \mathcal{N} \mid m_1(\lambda) = 0\},$$
$$\mathcal{N}_2 = \{\lambda \in \mathcal{N} \mid m_i(\lambda) \le 1 \text{ for } i = 1, 2, 3\},$$
$$\mathcal{N}_3 = \left\{ \lambda \in \mathcal{N} \;\middle|\; \begin{array}{c} m_1(\lambda) = m_3(\lambda) = 0,\ m_2(\lambda) \le 1, \\ \lambda \text{ does not match } (2k+3, 2k, 2k-2, \ldots, 4, 2) \text{ for any } k \ge 1 \end{array} \right\}.$$

Then

$$\mathcal{N}_1 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,3,4,10,11,12}, \quad \mathcal{N}_2 \overset{\mathrm{PT}}{\sim} T^{(14)}_{1,4,6,8,10,13}, \quad \mathcal{N}_3 \overset{\mathrm{PT}}{\sim} T^{(14)}_{2,5,6,8,9,12}.$$

Here $T^{(N)}_{a_1,\ldots,a_k}$ *denotes the set of partitions with parts congruent to* $a_1, \ldots, a_k$ *modulo* N.

In the present article we prove Conjecture 1.2. We also give the corresponding q-series identities (like (1.1), although not manifestly positive), which are missing in Conjecture 1.2. For $a = 1, 2, 3$ we consider the double sum

$$N_a := \sum_{i,j \ge 0} \frac{(-1)^j q^{\binom{i}{2} + 2\binom{j}{2} + 2ij + A_a(i,j)}}{(q;q)_i (q^2;q^2)_j}, \tag{1.4}$$

where $A_1(i,j) = i + j$, $A_2(i,j) = i + 3j$ and $A_3(i,j) = 2i + 3j$.

Theorem 1.3. *We have*

$$\sum_{\lambda \in \mathcal{N}_1} q^{|\lambda|} = \frac{1}{(q^2, q^3, q^4, q^{10}, q^{11}, q^{12}; q^{14})_\infty} = N_1,$$
$$\sum_{\lambda \in \mathcal{N}_2} q^{|\lambda|} = \frac{1}{(q, q^4, q^6, q^8, q^{10}, q^{13}; q^{14})_\infty} = N_2,$$
$$\sum_{\lambda \in \mathcal{N}_3} q^{|\lambda|} = \frac{1}{(q^2, q^5, q^6, q^8, q^9, q^{12}; q^{14})_\infty} = N_3.$$

Obviously, the first equality in each of the statements of Theorem 1.3 implies one of the claims of Conjecture 1.2.

1.4. **Linked partition ideals and regularly linked sets.** As mentioned above, a common technique for achieving Step 1 (in §1.2) is to use *linked partition ideals* (*LPI* for short) of Andrews [3],[4, §8] (see also [11, 12, 33]), which we review in Appendix E. Roughly speaking, a linked partition ideal is a subset $\mathcal{C} \subseteq \mathsf{Par}$ whose elements can be encoded as infinite sequences (on a certain finite set) in which certain (finite length) patterns are forbidden to appear. Theorem 1.4 below is a main result of [3], and this is applicable for most of known RR type identities.

Theorem 1.4 ([3, Theorem 4.1], [4, Theorem 8.11])**.** *If $\mathcal{C} \subseteq \mathsf{Par}$ is an LPI, then one can algorithmically obtain a q-difference equation for $f_{\mathcal{C}}(x,q)$.*

It is natural to hope to apply this to Nandi's conjectures, but unfortunately it can be shown that the set $\mathcal{N}$ (and $\mathcal{N}_a$ for $a = 1, 2, 3$) is not an LPI. Roughly speaking, this is because while elements of $\mathcal{N}$ can be encoded as certain infinite sequences (Proposition 2.6), there are arbitrarily long forbidden patterns which originally come from the condition (N6). Hence the theory of LPIs is not applicable in an obvious way. However, we can still derive a q-difference equation for $f_{\mathcal{N}}(x,q)$ (and $f_{\mathcal{N}_a}(x,q)$ for $a = 1, 2, 3$ as well). We show this in a generalized and algorithmic manner in §3, and apply it to Nandi's conjectures in §4.1.

In §3 we extend the theory of LPIs using *finite automata*. We consider a class of subsets $\mathcal{C} \subseteq \mathsf{Par}$ such that, roughly speaking, the elements of $\mathcal{C}$ can be encoded as infinite sequences (on a certain finite set) in which certain patterns given by a *regular language* (in the sense of *formal language theory*; see Definition 3.1) are forbidden to appear, and we say such $\mathcal{C}$ is *regularly linked* (Definition 3.8). This notion generalizes LPIs (Proposition E.4), and we show that $\mathcal{N}$ and $\mathcal{N}_a$ ($a = 1, 2, 3$) are regularly linked (Example 3.9).

Theorem 1.5 (Theorem 3.14 + Appendix B)**.** *If a subset $\mathcal{C} \subseteq \mathsf{Par}$ is regularly linked, then one can algorithmically obtain a q-difference equation for $f_{\mathcal{C}}(x,q)$.*

As an application of the main result above, in §4.1 we automatically obtain a q-difference equation for $f_{\mathcal{N}_a}(x,q)$ (for $a = 1, 2, 3$), finishing Step 1 for Nandi's conjectures (Proposition 4.2). We solve these equations in §4.2, finishing Step 2. The technique used there seems to be common in dealing with such equations. Indeed, the flow of §4.2 is similar to [1], [8, Proposition 2.2, Proposition 2.3], etc. Finally, Step 3 is done (also in §4.2) by employing three identities of Slater [37].

As we see in §3.3, once Theorem 1.5 is expressed in terms of finite automata its key part (Theorem 3.14) is proved immediately from an almost trivial lemma (Lemma 3.13). Nevertheless, its application to a concrete problem can be nontrivial (such as Proposition 4.2) and it seems worthwhile presenting the details of this generalization of LPIs as it works well in solving Nandi's conjectures. We hope that the regulary linked sets would be widely used as a method of algorithmic derivation of q-difference equations in the theory of partitions like the WZ method in hypergeometric summations.

Organization of the paper. In §2 we rephrase the defining conditions for $\mathcal{N}$ as certain forbidden patterns and prefixes on a certain finite set. In §3.1 we recall standard definitions and facts in formal language theory (some details are put in Appendix A). In §3.2 we define regularly linked sets and in §3.3 show Theorem 1.5. In §4.1 we obtain q-difference equations for $f_{\mathcal{N}_a}(x,q)$ ($a = 1, 2, 3$) using the results in §3.3 (we also need the Modified Murray–Miller Theorem reviewed in Appendix B, which is given in [3] and constitutes the final step in Theorem 1.4 (and Theorem 1.5). We apply it explicitly in Appendix C). In §4.2 we solve these equations, proving Theorem

1.3. In Appendix D we give supplementary results regarding Theorem 1.5. In Appendix E we review LPIs and compare it to our results.

2. Nandi's partitions $\mathcal{N}$

2.1. **Multiplicity sequences.** A partition $\lambda \in \mathsf{Par}$ can be identified with its multiplicity sequence $(f_i)_{i\geq 1}$, where $f_i = m_i(\lambda)$. By this, we have a bijection

$$\widehat{\ }: \mathsf{Par} \xrightarrow{\sim} \widehat{\mathsf{Par}} := \left\{ (f_i)_{i\geq 1} \in (\mathbb{Z}_{\geq 0})^{\mathbb{Z}_{\geq 1}} \;\middle|\; \#\{i \geq 1 \mid f_i > 0\} < \infty \right\},$$

and we denote the image in $\widehat{\mathsf{Par}}$ (via this bijection) of $\lambda \in \mathsf{Par}$ and $\mathcal{C} \subseteq \mathsf{Par}$, say, by $\widehat{\lambda}$ and $\widehat{\mathcal{C}}$. It is easy to see for any $\lambda \in \mathsf{Par}$ and $k, d \geq 1$ that

$$1 \leq \forall i \leq \ell(\lambda) - k,\ \lambda_i - \lambda_{i+k} \geq d \iff \forall j \geq 1,\ f_j + \cdots + f_{j+d-1} \leq k. \tag{2.1}$$

Lemma 2.1. *The set $\widehat{\mathcal{N}}$ consists of $(f_i)_{i\geq 1}$ satisfying* $(\mathsf{N1}')$-$(\mathsf{N4}')$, $(\mathsf{N5a}')$, $(\mathsf{N5b}')$ *and* $(\mathsf{N6}'_k)$ *(for all $k \geq 0$). Here, for $i = 1, \ldots, 4, 5\mathrm{a}, 5\mathrm{b}$, the condition* $(\mathsf{N}i')$ *is given by*

$(\mathsf{N}i')$: *there are no $j \geq 1$ such that* $(\mathsf{P}i_j)$, *where*

- $(\mathsf{P1}_j)$: $(f_j, f_{j+1}) = (\geq 1, \geq 1)$,
- $(\mathsf{P2}_j)$: $f_j + f_{j+1} + f_{j+2} \geq 3$,
- $(\mathsf{P3}_j)$: $(f_j, f_{j+1}, f_{j+2}, f_{j+3}) = (\geq 1, 0, 0, \geq 2)$,
- $(\mathsf{P4}_j)$: $(f_{2j}, f_{2j+1}, f_{2j+2}, f_{2j+3}) = (\geq 2, 0, 0, \geq 1)$,
- $(\mathsf{P5a}_j)$: $(f_{2j-1}, f_{2j}, f_{2j+1}, f_{2j+2}, f_{2j+3}) = (\geq 2, 0, 0, 0, \geq 1)$,
- $(\mathsf{P5b}_j)$: $(f_{2j-1}, f_{2j}, f_{2j+1}, f_{2j+2}, f_{2j+3}) = (\geq 1, 0, 0, 0, \geq 2)$,

and the condition $(\mathsf{N6}'_k)$ $(k \geq 0)$ *is given by*

$(\mathsf{N6}'_k)$: *there are no $j \geq 1$ such that*

$$(f_j, f_{j+1}, \ldots, f_{j+2k+6}) = (\geq 2, 0, 0, 1, \underbrace{0, 1, 0, \ldots, 1, 0, 1}_{2k}, 0, 0, \geq 1).$$

Here, for $n \geq 2$, we write $(x_1, x_2, \ldots, x_{n-1}, x_n) = (\geq y_1, y_2, \ldots, y_{n-1}, \geq y_n)$ to mean $x_1 \geq y_1$, $x_i = y_i$ (for $2 \leq i \leq n-1$) and $x_n \geq y_n$.

Proof. It is clear that $(\mathsf{N1})\Longleftrightarrow(\mathsf{N1}')$. That $(\mathsf{N2})\Longleftrightarrow(\mathsf{N2}')$ is a special case of (2.1). The condition $(\mathsf{N3})$ is equivalent to that λ does not match (in the sense of Definition 1.1) $(j+3, j+3, j)$ for $j \geq 1$, which is precisely $(\mathsf{N3}')$. Similarly we have $(\mathsf{N4}) \iff (\mathsf{N4}')$ and $(\mathsf{N5}) \iff (\mathsf{N5a}'), (\mathsf{N5b}')$. For $(\mathsf{N6})$, the condition $(\mathsf{N6}'_k)$ is equivalent to that $(\lambda_1 - \lambda_2, \ldots, \lambda_{\ell(\lambda)-1} - \lambda_{\ell(\lambda)})$ does not match $(3, 2^k, 3, 0)$. □

2.2. **Encoding $\mathcal{N}$ as infinite sequences.** We write

$$f_{\leq m} := (f_1, \ldots, f_m, 0, 0, \ldots), \quad \lambda_{\leq m} := (\lambda_{\ell'+1}, \ldots, \lambda_{\ell(\lambda)})$$

for $m > 0$ and $f = (f_i)_{i\geq 1} \in \widehat{\mathsf{Par}}$, $\lambda \in \mathsf{Par}$, where $\ell' := \#\{i \geq 1 \mid \lambda_i > m\}$. We clearly have $\widehat{\lambda_{\leq m}} = f_{\leq m}$ when $\widehat{\lambda} = f$. Furthermore, for $\mathcal{C} \subseteq \mathsf{Par}$ we write

$$\mathcal{C}_{\leq m} := \{\lambda \in \mathcal{C} \mid \lambda = \lambda_{\leq m}\}.$$

Definition 2.2. The maps $\phi_+, \phi_- \colon \widehat{\mathsf{Par}} \longrightarrow \widehat{\mathsf{Par}}$ are given by

$$\phi_+((f_1, f_2, \dots)) = (0, f_1, f_2, \dots), \quad \phi_-((f_1, f_2, \dots)) = (f_2, f_3, \dots).$$

By abuse of notation, we also regard ϕ_+, ϕ_- as maps from Par to Par:

$$\phi_+((\lambda_1, \dots, \lambda_\ell)) = (\lambda_1 + 1, \dots, \lambda_\ell + 1),$$
$$\phi_-((\lambda_1, \dots, \lambda_\ell)) = (\lambda_1 - 1, \dots, \lambda_{\ell'} - 1), \text{ where } \ell' := \#\{i \geq 1 \mid \lambda_i > 1\} (= f_2 + f_3 + \cdots).$$

The following lemma is essentially [4, Lemma 8.9]. We include the proof since we require weaker conditions on $\mathcal{C}$. For a comparison with the original arguments in [3],[4, §8] we refer the reader to Appendix E.1. For $\lambda, \mu \in \mathsf{Par}$, let $\lambda \oplus \mu$ be the partition obtained by reordering $(\lambda_1, \dots, \lambda_{\ell(\lambda)}, \mu_1, \dots, \mu_{\ell(\mu)})$ in non-increasing order. In terms of $\widehat{\mathsf{Par}}$, it means $(f_i)_{i \geq 1} \oplus (g_i)_{i \geq 1} = (f_i + g_i)_{i \geq 1}$.

Lemma 2.3. *If a subset $\mathcal{C} \subseteq \mathsf{Par}$ and an integer $m \in \mathbb{Z}_{>0}$ satisfy*

$$\lambda \in \mathcal{C} \implies \lambda_{\leq m} \in \mathcal{C} \qquad \text{and} \qquad \phi_-^m(\mathcal{C}) \subseteq \mathcal{C}, \tag{2.2}$$

then for each $\lambda \in \mathcal{C}$ there exists a unique sequence $\lambda^{(1)}, \lambda^{(2)}, \dots$ in $\mathcal{C}_{\leq m}$ such that

$$\lambda = \lambda^{(1)} \oplus \phi_+^m(\lambda^{(2)}) \oplus \phi_+^{2m}(\lambda^{(3)}) \oplus \cdots.$$

Proof. Let $f = (f_i)_{i \geq 1} := \widehat{\lambda}$. Obviously $\widehat{\lambda^{(i)}}$ must be $(f_{1+m(i-1)}, \dots, f_{mi}, 0, 0, \dots)$ $(= (\phi_-^{m(i-1)}(f))_{\leq m})$ and hence is unique. On the other hand, by the assumption we have $\phi_-^{m(i-1)}(\lambda) \in \mathcal{C}$ and hence $(\phi_-^{m(i-1)}(\lambda))_{\leq m} \in \mathcal{C}_{\leq m}$. □

Lemma 2.4. *The set $\mathcal{N}$ satisfies* (2.3) *with $m = 2$.*

Proof. The conditions $(\mathsf{N1'})$-$(\mathsf{N3'})$, $(\mathsf{N6}'_k)$ (resp. $(\mathsf{N4'})$, $(\mathsf{N5a'})$, $(\mathsf{N5b'})$) are stable under ϕ_- (resp. ϕ_-^2) and all the conditions $(\mathsf{N1'})$-$(\mathsf{N6}'_k)$ are stable under $(\widehat{\mathsf{Par}} \ni) f \mapsto f_{\leq m}$ for any $m > 0$. □

2.3. **Forbidden patterns and prefixes.** To avoid confusion we denote the empty partition by $\emptyset \in \mathsf{Par}$.

Definition 2.5. (1) For a nonempty set I, we write the set of infinite sequences of I as

$$\mathsf{Seq}(I) := \{(i_1, i_2, \dots) \mid i_j \in I \text{ for } j \geq 1\} \ (= I^{\mathbb{Z}_{\geq 1}}).$$

(2) For a triple (I, m, π) where I is a nonempty set, $m \in \mathbb{Z}_{>0}$ and $\pi \colon I \longrightarrow \mathsf{Par}_{\leq m}$ is a map, we define

$$\mathsf{Seq}(I, \pi) := \{(i_1, i_2, \dots) \in \mathsf{Seq}(I) \mid \#\{j \geq 1 \mid \pi(i_j) \neq \emptyset\} < \infty\}$$

and $\pi^\bullet \colon \mathsf{Seq}(I, \pi) \longrightarrow \mathsf{Par}$ by

$$\pi^\bullet(i_1, i_2, i_3, \dots) := \pi(i_1) \oplus \phi_+^m(\pi(i_2)) \oplus \phi_+^{2m}(\pi(i_3)) \oplus \cdots.$$

Now $\mathcal{N}_{\le 2} = \{\lambda \in \mathcal{N} \mid \lambda_1 \le 2\} = \{\pi_i \mid i \in I\}$ where $I = \{0,1,2,3,4\}$ and

$$\pi_0 = \emptyset, \quad \pi_1 = (2), \quad \pi_2 = (2,2), \quad \pi_3 = (1), \quad \pi_4 = (1,1), \tag{2.3}$$

and $\widehat{\mathcal{N}}_{\le 2} = \{\widehat{\pi}_i \mid i \in I\}$ is given by

$$\widehat{\pi}_0 = (0,0), \quad \widehat{\pi}_1 = (0,1), \quad \widehat{\pi}_2 = (0,2), \quad \widehat{\pi}_3 = (1,0), \quad \widehat{\pi}_4 = (2,0).$$

Here we simply write $\widehat{\pi}_i = (f_1, f_2)$ instead of $\widehat{\pi}_i = (f_1, f_2, 0, 0, \dots)$. Moreover we write $\pi\colon I \longrightarrow \mathsf{Par}_{\le 2}\,;\, i \mapsto \pi_i$ (using the same symbol).

By Lemma 2.3 and Lemma 2.4, we see $\mathcal{N}$ (and hence $\mathcal{N}_a$, $a = 1,2,3$) is in bijection with a subset of $\mathsf{Seq}(I,\pi)$, and the condition that $(i_1, i_2, \dots) \in \mathsf{Seq}(I,\pi)$ is in the image of $\mathcal{N}$ (resp. $\mathcal{N}_a$) is as follows.

Proposition 2.6. *Let $\boldsymbol{i} = (i_1, i_2, \dots) \in \mathsf{Seq}(I,\pi)$. Then $\pi^\bullet(\boldsymbol{i}) \in \mathcal{N}$ if and only if $\boldsymbol{i}$ does not match any of*

$$\begin{gathered} (1,\{2,3,4\}), \quad (2,\{1,2,3,4\}), \quad (3,\{2,4\}), \quad (4,\{2,3,4\}), \\ (1,0,4), \quad (2,0,\{3,4\}), \quad (3,0,4), \quad (4,0,4), \quad (4,1^*,0,3). \end{gathered} \tag{2.4}$$

Here, for $x, y, \dots \in I$, $\{x, y, \dots\}$ means exactly one occurence of one of $x, y, \dots$, and x^* means zero or more repetitions of x (see also (3)).

Proof. It is straightforward to check the conditions $(\mathsf{N1}')$-$(\mathsf{N5}')$ and $(\mathsf{N6}'_k)$ $(k \ge 0)$ correspond to forbidding the patterns in Table 1 and 2.

	(j: odd)	(j: even)
$(\mathsf{N1}')$		$(1,3),(1,4),(2,3),(2,4)$
$(\mathsf{N2}')$	$(1,4),(2,3),(2,4)$ $(3,4),(4,3),(4,4)$	$(1,2),(1,4),(2,1),$ $(2,2),(2,3),(2,4)$
$(\mathsf{N3}')$	$(3,2),(4,2)$	$(1,0,4),(2,0,4)$
$(\mathsf{N6}'_k)$ $(k \ge 0)$	$(4,1,1^k,0,3),(4,1,1^k,0,4)$	$(2,0,3,3^k,1),(2,0,3,3^k,2)$

Table 1. Forbidden patterns corresponding to $(\mathsf{N1}')$-$(\mathsf{N3}')$, $(\mathsf{N6}')$.

$(\mathsf{N4}')$	$(2,0,3)$, $(2,0,4)$
$(\mathsf{N5a}')$	$(4,0,3)$, $(4,0,4)$
$(\mathsf{N5b}')$	$(3,0,4)$, $(4,0,4)$

Table 2. Forbidden patterns corresponding to $(\mathsf{N4}')$, $(\mathsf{N5a}')$, $(\mathsf{N5b}')$.

It is also easy to see that forbidding all the patterns in Table 1 and 2 is equivalent to forbidding the patterns in (2.6). (For example, $(2,0,3,3^k,1)$ in $(\mathsf{N6}'_k)$ is redundant since $(2,0,3)$ is in (2.6).) □

Proposition 2.7. *Let $\boldsymbol{i} = (i_1, i_2, \dots) \in \mathsf{Seq}(I,\pi)$ satisfy $\pi^\bullet(\boldsymbol{i}) \in \mathcal{N}$. Then $\pi^\bullet(\boldsymbol{i}) \in \mathcal{N}_a$ $(a = 1,2,3)$ if and only if $\boldsymbol{i}$ does not start with any of*

$$\begin{aligned} (3),\ (4), &\qquad (a=1), \\ (2),\ (4),\ (0,4), &\qquad (a=2), \\ (2),\ (3),\ (4),\ (0,4),\ (1^*,0,3), &\qquad (a=3). \end{aligned} \tag{2.5}$$

Proof. For $\mathcal{N}_1$, the additional condition $m_1(\lambda) = 0$ is equivalent to the condition that $\boldsymbol{i}$ does not start with any of $(3), (4)$.

For $\mathcal{N}_2$, the additional condition $m_i(\lambda) \leq 1$ for $i = 1, 2, 3$ is equivalent to the condition that $\boldsymbol{i}$ does not start with any of $(2), (4), (\{0, 1, 2, 3, 4\}, 4)$, which is equivalent to, after reducing redundancy, the condition that $\boldsymbol{i}$ does start with any of $(2), (4), (0, 4)$.

For $\mathcal{N}_3$, the additional condition that $m_1(\lambda) = m_3(\lambda) = 0$, $m_2(\lambda) \leq 1$ and λ does not match $(2k+3, 2k, 2k-2, \dots, 4, 2)$ (for $k \geq 1$) is equivalent to the condition that $\boldsymbol{i}$ does not start with any of $(2), (3), (4), (\{0, 1, 2, 3, 4\}, \{3, 4\}), (1^k, 0, \{3, 4\})$ (for $k \geq 1$). That is equivalent to, after reducing redundancy, the condition that $\boldsymbol{i}$ does not start with any of $(2), (3), (4), (0, 4), (1^*, 0, 3)$. □

3. A formal language theoretic approach

In this section we assume that Σ is a nonempty finite set. Let $\Sigma^* = \bigsqcup_{n\geq 0} \Sigma^n$ be the monoid of words on Σ. An element of Σ^* is written like $i_1 \cdots i_n$ (where $i_j \in \Sigma$), and the monoid multiplication is concatenation. Let ε be the empty word (i.e., $\Sigma^0 = \{\varepsilon\}$) and put $\Sigma^+ := \Sigma^* \setminus \{\varepsilon\}$. A *language* (over Σ) is a subset of Σ^*. We write the *product* of $X, Y \subseteq \Sigma^*$ and the *Kleene star* of $X \subseteq \Sigma^*$ as

$$XY := \{ab \mid a \in X, b \in Y\}, \qquad X^* := \bigcup_{n\geq 0} X^n. \tag{3.1}$$

3.1. **Regular languages and finite automata.**

Definition 3.1 ([36, Definition 1.5])**.** A *deterministic finite automaton* (or *DFA* for short) over Σ is a 5-tuple $M = (Q, \Sigma, \delta, s, F)$ where Q is a finite set (the set of *states*), $\delta \colon Q \times \Sigma \longrightarrow Q$ (the *transition function*), $s \in Q$ (the *start state*) and $F \subseteq Q$ (the set of *accept states*).

Definition 3.2. For a DFA $(Q, \Sigma, \delta, s, F)$, we define $\widehat{\delta} \colon Q \times \Sigma^* \longrightarrow Q$ inductively by $\widehat{\delta}(q, \varepsilon) := q$ and $\widehat{\delta}(q, wa) := \delta(\widehat{\delta}(q, w), a)$ ($q \in Q$, $a \in \Sigma$, $w \in \Sigma^*$). For M a DFA, let $L(M)$ denote the language that M *recognizes* (or *accepts*), i.e., $L(M) = \{w \in \Sigma^* \mid \widehat{\delta}(s, w) \in F\}$.

Definition 3.3 ([36, Definition 1.16])**.** A language $X \subseteq \Sigma^*$ is called *regular* (or *rational*) if there exists a DFA recognizing X.

Example 3.4. The empty set $\emptyset$, singletons $\{i\}$ ($i \in \Sigma$) and Σ are regular.

Proposition 3.5 (See e.g., [36, Theorem 1.25, 1.47, 1.49])**.** *If $Y, Z \subseteq \Sigma^*$ are regular, so are $Y \cap Z$, YZ, Y^c and Y^* (and thus $Y \cup Z$ and $Y \setminus Z$ as well).*

We review an algorithmic proof of Proposition 3.5 in Appendix A.

For any regular language $X \subseteq \Sigma^*$, there exists a unique DFA recognizing X with the fewest states (up to isomorphism of DFAs, i.e., renaming of states), called the *minimal DFA* recognizing X. For a DFA M, we denote by $M_{\min}$ the minimal DFA such that $L(M_{\min}) = L(M)$. There is an algorithm to compute $M_{\min}$ from a given DFA M (see Remark A.4).

Corollary 3.6. $L(M) = L(N) \iff M_{\min} \simeq N_{\min}$ *for DFAs M and N.*

3.2. **Regularly linked sets.** Recall Definition 2.5. Recall also that we assume $n > 0$ in Definition 1.1 for simplicity.

Definition 3.7. For $S \subseteq \mathsf{Seq}(\Sigma)$ and $X, X' \subseteq \Sigma^+$, we write

$$\mathsf{avoid}(S, X, X') := \left\{ \boldsymbol{i} \in S \;\middle|\; \begin{array}{l} \forall \boldsymbol{j} \in X,\ \boldsymbol{i} \text{ does not match } \boldsymbol{j}, \text{ and} \\ \forall \boldsymbol{j} \in X',\ \boldsymbol{i} \text{ does not begin with } \boldsymbol{j} \end{array} \right\}.$$

In other words,

$$\mathsf{avoid}(S, X, X') = S \cap \left(\Sigma^* \setminus (\Sigma^* X \Sigma^* \cup X' \Sigma^*)\right)^{\wedge} \tag{3.2}$$

where we write for $A \subseteq \Sigma^*$

$$A^{\wedge} := \{(i_j)_{j \geq 1} \in \mathsf{Seq}(\Sigma) \mid \forall n \geq 1,\ i_1 \cdots i_n \in A\}.$$

Definition 3.8. We say that a subset $\mathcal{C} \subseteq \mathsf{Par}$ is *regularly linked* if there exists a 5-tuple (m, I, π, X, X') consisting of a positive integer m, a nonempty finite set I, an injective map $\pi \colon I \longrightarrow \mathsf{Par}_{\leq m}$ (hence $\pi^\bullet \colon \mathsf{Seq}(I, \pi) \longrightarrow \mathsf{Par}$ is injective) and regular languages $X, X' \subseteq I^+$ such that

$$\pi^\bullet(\mathsf{avoid}(\mathsf{Seq}(I, \pi), X, X')) = \mathcal{C}.$$

Example 3.9. The sets $\mathcal{N}$ and $\mathcal{N}_a$ ($a = 1, 2, 3$) (see Conjecture 1.2) are regularly linked. Indeed, Proposition 2.6 and 2.7 translate as

$$\begin{aligned} \pi^\bullet(\mathsf{avoid}(\mathsf{Seq}(I, \pi), X_{\mathcal{N}}, \emptyset)) &= \mathcal{N}, \\ \pi^\bullet(\mathsf{avoid}(\mathsf{Seq}(I, \pi), X_{\mathcal{N}}, X_{\mathcal{N}_a})) &= \mathcal{N}_a \end{aligned} \tag{3.3}$$

where $I = \{0, 1, 2, 3, 4\}$, $\pi \colon I \longrightarrow \mathsf{Par}_{\leq 2}\,;\ i \mapsto \pi_i$ is given by (2.3), and

$$X_{\mathcal{N}} = \left\{ \begin{array}{c} 12, 13, 14, 21, 22, 23, 24, 32, 34, 42, 43, 44, \\ 104, 203, 204, 304, 404 \end{array} \right\} \cup \{4\}\{1\}^*\{03\}, \tag{3.4}$$

$$X_{\mathcal{N}_1} = \{3, 4\},\ X_{\mathcal{N}_2} = \{2, 4, 04\},\ X_{\mathcal{N}_3} = \{2, 3, 4, 04\} \cup \{1\}^*\{03\} \tag{3.5}$$

are the regular languages over I consisting of the patterns in (2.6) and (2.7).

Remark 3.10. In Definition 3.8, X is superfluous since we can write

$$\mathsf{avoid}(\mathsf{Seq}(I, \pi), X, X') = \mathsf{avoid}(\mathsf{Seq}(I, \pi), \emptyset, X' \cup \Sigma^* X).$$

Note that if $X (\subseteq \Sigma^+)$ is regular then so is $\Sigma^* X (\subseteq \Sigma^+)$ (see Proposition 3.5). Nevertheless, it seems more consistent with intuition to separate some forbidden patterns from forbidden prefixes as seen in Example 3.9 (see also Proposition E.4).

3.3. **The main construction.**

Definition 3.11. For a DFA $M = (Q, \Sigma, \delta, s, F)$ and $v \in Q$ we write $M_v := (Q, \Sigma, \delta, v, F)$. That is, M_v is identical to M except that its start state is v.

Definition 3.12. For a nonempty set I, we write

$$\begin{aligned} j \cdot \boldsymbol{i} &:= (j, i_1, i_2, \dots) \in \mathsf{Seq}(I) && \text{for } j \in I,\ \boldsymbol{i} = (i_1, i_2, \dots) \in \mathsf{Seq}(I), \\ j \cdot S &:= \{j \cdot \boldsymbol{i} \mid \boldsymbol{i} \in S\} \subseteq \mathsf{Seq}(I) && \text{for } j \in I,\ S \subseteq \mathsf{Seq}(I). \end{aligned}$$

Lemma 3.13. *Let $M = (Q, I, \delta, s, F)$ be a DFA. For $v \in Q$ we have*

$$(L(M_v)^c)^\wedge = \bigsqcup_{\substack{a \in I \\ \delta(v,a) \notin F}} a \cdot (L(M_{\delta(v,a)})^c)^\wedge.$$

Proof. By $(L(M_v)^c)^\wedge = \{(a_i)_{i\geq 1} \in \mathsf{Seq}(I) \mid \forall n \geq 1, \widehat{\delta}(v, a_1 \cdots a_n) \notin F\}$. □

For $\lambda \in \mathsf{Par}$ we write $\mathrm{wt}(\lambda) := x^{\ell(\lambda)} q^{|\lambda|}$. Assume a map $\pi \colon I \longrightarrow \mathsf{Par}_{\leq m}$ is given. For $\boldsymbol{i} \in \mathsf{Seq}(I, \pi)$ and $j \in I$, we have

$$\mathrm{wt}(\pi^\bullet(j \cdot \boldsymbol{i})) = \mathrm{wt}(\pi(j)) \cdot \big(\mathrm{wt}(\pi^\bullet(\boldsymbol{i}))|_{x \mapsto xq^m}\big) \tag{3.6}$$

by $\pi^\bullet(j \cdot \boldsymbol{i}) = \pi(j) \oplus \phi_+^m(\pi^\bullet(\boldsymbol{i}))$ (and $\mathrm{wt}(\phi_+(\lambda)) = \mathrm{wt}(\lambda)|_{x \mapsto xq}$).

Theorem 3.14. *Assume $\mathcal{C} \subseteq \mathsf{Par}$ is regularly linked and let m, I, π, X, X' be as in Definition 3.8. Let $M = (Q, I, \delta, s, F)$ be a DFA recognizing $I^* X I^* \cup X' I^*$. Define*

$$\mathcal{C}^{(v)} := \pi^\bullet\big(\mathsf{Seq}(I, \pi) \cap (L(M_v)^c)^\wedge\big) \tag{3.7}$$

for $v \in Q \setminus F$. Then $s \in Q \setminus F$ and $\mathcal{C}^{(s)} = \mathcal{C}$. Moreover, we have the system of q-difference equations

$$f_{\mathcal{C}^{(v)}}(x, q) = \sum_{u \in Q \setminus F} \Bigg(\sum_{\substack{a \in I \\ u = \delta(v,a)}} x^{\ell(\pi_a)} q^{|\pi_a|} \Bigg) f_{\mathcal{C}^{(u)}}(xq^m, q) \quad (v \in Q \setminus F). \tag{3.8}$$

Proof. We have $s \in Q \setminus F$ since $\varepsilon \notin I^* X I^* \cup X' I^* = L(M)$. The fact $\mathcal{C}^{(s)} = \mathcal{C}$ is obvious by $M_s = M$ and (3.7). Put $S_v := \mathsf{Seq}(I, \pi) \cap (L(M_v)^c)^\wedge$. Then by Lemma 3.13 we have

$$S_v = \bigsqcup_{\substack{a \in I \\ \delta(v,a) \notin F}} a \cdot S_{\delta(v,a)}.$$

Apply the map $\mathsf{Seq}(I, \pi) \supseteq S \mapsto \sum_{\boldsymbol{i} \in S} \mathrm{wt}(\pi^\bullet(\boldsymbol{i}))$. Since $\pi^\bullet$ is injective and $f_{\mathcal{C}}(x, q) = \sum_{\lambda \in \mathcal{C}} \mathrm{wt}(\lambda)$ for $\mathcal{C} \subseteq \mathsf{Par}$ (see (1.2)), S_v is then mapped to $f_{\mathcal{C}^{(v)}}(x, q)$. Hence by (3.3) we get (3.14). □

Remark 3.15. In Theorem 3.14, we can explicitly determine $\mathcal{C}^{(v)}$ if $v \in Q \setminus F$ is reachable, i.e., $v = \widehat{\delta}(s, w)$ for some $w \in \Sigma^*$. (For example, every state in a minimal DFA is reachable.) In Appendix D we show

$$L(M_v) = I^* X I^* \cup X'' I^* \quad \text{for some } X'' \subseteq I^+,$$

and explicitly find the minimum such X'', namely, the regular language X_v given in (D.4) (with $\Sigma := I$). Hence for $v \in Q \setminus F$ we have by (3.7)

$$(L(M_v)^c)^\wedge = \mathsf{avoid}(\mathsf{Seq}(I), X, X_v).$$

Thus, $\mathcal{C}^{(v)}$ is regularly linked with forbidden patterns X and prefixes X_v:

$$\mathcal{C}^{(v)} = \pi^\bullet(\mathsf{avoid}(\mathsf{Seq}(I, \pi), X, X_v)) \quad (\subseteq \mathcal{C}).$$

Proof of Theorem 1.5. Since $|Q \setminus F|$ is finite in Theorem 3.14, from the system (3.14) we can deduce for any $v \in Q \setminus F$ a single q-difference equation for $f_{\mathcal{C}^{(v)}}(x, q)$ by the algorithm given in [3, p. 1040] (called *Modified Murray–Miller Theorem* therein). We review it in Appendix B for completeness. □

4. A proof of Nandi's conjectures

4.1. **Algorithmic derivation of q-difference equations.** We apply Theorem 3.14 to $\mathcal{N}$ (recall Example 3.9). The resulting system (3.14) depends on the choice of a DFA M in Theorem 3.14, and in this case we can complete the proof by taking M minimal.

Let $I = \{0,1,2,3,4\}$ and let $X = X_{\mathcal{N}} \subseteq I^+$ be the regular language given in (3.9). Since the proof of Proposition 3.5 and Remark A.4 are constructive, we can algorithmically find (see Remark 4.3) the minimal DFA that recognizes I^*XI^* is $M = (Q, I, \delta, s, F)$ where $Q = \{q_0, \dots, q_7\}$, $s = q_0$, $F = \{q_6\}$ and $\delta\colon Q \times I \longrightarrow Q$ is given by Table 3, in which we display $\delta'(v,j)$ such that $\delta(q_v, j) = q_{\delta'(v,j)}$ ($v \in \{0,\dots,7\}$, $j \in I$). See also Figure 1.

$v\backslash j$	0	1	2	3	4
0	0	1	2	3	4
1	5	1	6	6	6
2	7	6	6	6	6
3	5	1	6	3	6
4	7	4	6	6	6
5	0	1	2	3	6
6	6	6	6	6	6
7	0	1	2	6	6

TABLE 3. $\delta'(v,j)$

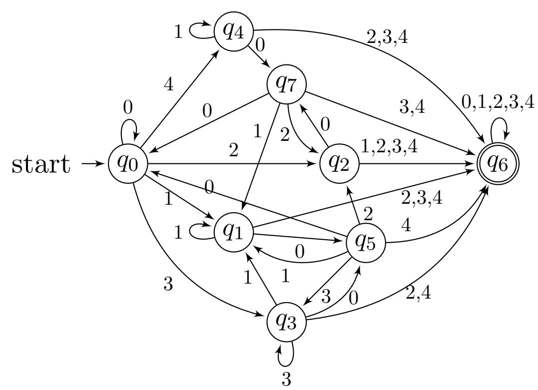

FIGURE 1

Writing $F_i(x) := f_{\mathcal{N}^{(q_i)}}(x,q)$ for $q_i \in Q \setminus F$ (i.e., $i \in \{0,\dots,5,7\}$), by Theorem 3.14 we obtain a system of q-difference equations

$$\begin{pmatrix} F_0(x) \\ F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \\ F_7(x) \end{pmatrix} = \begin{pmatrix} 1 & xq^2 & x^2q^4 & xq & x^2q^2 & 0 & 0 \\ 0 & xq^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & xq^2 & 0 & xq & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & xq^2 & 0 & 1 \\ 1 & xq^2 & x^2q^4 & xq & 0 & 0 & 0 \\ 1 & xq^2 & x^2q^4 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_0(xq^2) \\ F_1(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \\ F_4(xq^2) \\ F_5(xq^2) \\ F_7(xq^2) \end{pmatrix}. \tag{4.1}$$

Moreover, it can be algorithmically proved (see Remark 4.3) that

$$\begin{aligned} L(M_{q_7}) &= I^*XI^* \cup X_{\mathcal{N}_1}I^*, \\ L(M_{q_3}) &= I^*XI^* \cup X_{\mathcal{N}_2}I^*, \\ L(M_{q_4}) &= I^*XI^* \cup X_{\mathcal{N}_3}I^*, \end{aligned} \tag{4.2}$$

where $X_{\mathcal{N}_1}, X_{\mathcal{N}_2}, X_{\mathcal{N}_3} \subseteq I^+$ are as in (3.9). Actually, one can construct DFAs recognizing the right-hand sides via Proposition 3.5 and then use Corollary 3.6. Now by (3.7), (3.9), (3.14) and (4.1) we have

$$\mathcal{N}^{(q_7)} = \mathcal{N}_1, \quad \mathcal{N}^{(q_3)} = \mathcal{N}_2, \quad \mathcal{N}^{(q_4)} = \mathcal{N}_3. \tag{4.3}$$

Remark 4.1. Alternatively, one can show (4.1) by computing DFAs recognizing $X_{\mathcal{N}_a}$ ($a = 1, 2, 3$) and X_{q_i} (given by (D.4); see also Remark 3.15) for $q_i \in Q \setminus F$ and check $X_{q_7} = X_{\mathcal{N}_1}$, $X_{q_3} = X_{\mathcal{N}_2}$, $X_{q_4} = X_{\mathcal{N}_3}$ by Corollary 3.6.

Hence, we can apply the algorithm described in Appendix B to obtain q-difference equations for $f_{\mathcal{N}_1}(x, q) = F_7(x)$, $f_{\mathcal{N}_2}(x, q) = F_3(x)$ and $f_{\mathcal{N}_3}(x, q) = F_4(x)$ (the explicit calculation is given in Appendix C).

Proposition 4.2. *For $a = 1, 2, 3$, the series $f_{\mathcal{N}_a}(x, q)$ satisfies the q-difference equation*

$$0 = \sum_{i=0}^{5} p_{2i}^{(a)}(x, q) f_{\mathcal{N}_a}(xq^{2i}, q), \tag{4.4}$$

where $p_{2i}^{(a)} = p_{2i}^{(a)}(x, q)$ are given in the following table.

	$a = 1$	$a = 2$	$a = 3$
$p_0^{(a)}$	1	1	1
$p_2^{(a)}$	$-1 - x(q^2 + q^3 + q^4)$	$-1 - x(q + q^2 + q^4)$	$-1 - x(q^2 + q^4 + q^5)$
$p_4^{(a)}$	$xq^4(1 - x + xq^3 + xq^4 + xq^5)$	$xq^4(1 + xq + xq^3)$	$xq^4(1 + xq^5 + xq^7)$
$p_6^{(a)}$	$x^2q^6(-1 + xq^4(1 + q + q^2 - q^5))$	$x^2q^{10}(-1 + xq^4 + xq^6)$	$x^2q^{10}(-1 + xq^4 + xq^6)$
$p_8^{(a)}$	$x^3q^{13}(1 + q + q^2)(1 - xq^6)$	$x^3q^{15}(1 + q^2 + q^3)(1 - xq^6)$	$x^3q^{18}(1 + q + q^3)(1 - xq^6)$
$p_{10}^{(a)}$	$x^3q^{17}(1 - xq^6)(1 - xq^8)$	$x^3q^{19}(1 - xq^6)(1 - xq^8)$	$x^3q^{23}(1 - xq^6)(1 - xq^8)$

Remark 4.3. We can use computer algebra in these constructions. For example, using a GAP package Automata [15, 22] we can compute M (up to renaming of states) as follows.

```
gap> LoadPackage("automata");
gap> Xn:=RationalExpression("12U13U14U21U22U23U24U32U34U42U43U44U104U203U204U304U404U41*03","01234");
gap> Is:=RationalExpression("(0U1U2U3U4)*","01234");
gap> r:=ProductRatExp(Is,ProductRatExp(Xn,Is));
gap> M:=RatExpToAut(r);
gap> Display(M);
```

We can also check (4.1) for $\mathcal{N}_1$ ($\mathcal{N}_2$ and $\mathcal{N}_3$ are similar).

```
gap> Xn1:=RationalExpression("3U4","01234");
gap> r1:=UnionRatExp(r,ProductRatExp(Xn1,Is));
gap> SetInitialStatesOfAutomaton(M,5);
gap> AreEquivAut(M,RatExpToAut(r1));
```

Here, the state 5 (in the third line) corresponds to q_7 in our notation.

4.2. **Solving the equation (4.2).** Recall *Euler's identities* [19, (II.1),(II.2)]

$$\sum_{n\geq 0} \frac{x^n}{(q;q)_n} \overset{\text{(A)}}{=} \frac{1}{(x;q)_\infty}, \qquad \sum_{n\geq 0} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} \overset{\text{(B)}}{=} (-x;q)_\infty.$$

The following lemma is a formal series version of Appell's comparison theorem [18, p. 101].

Lemma 4.4. *For formal series*

$$A(x) = \sum_{m\geq 0} a_m x^m, \qquad B(x) = \frac{A(x)}{(1-x)} = \sum_{n\geq 0} b_n x^n,$$

if $\lim_{n\to\infty} b_n$ exists then $(A(1) =) \sum_{m\geq 0} a_m = \lim_{n\to\infty} b_n$.

Within the proof below we freely use the fact the q-difference equation $\sum_{i,j,k} a_{ijk} x^i q^j F(xq^k) = 0$ for a formal series $F(x) = \sum_{M\in\mathbb{Z}} f_M x^M$ is equivalent to the recurrence $\sum_{i,j,k} a_{ijk} q^{k(M-i)+j} f_{M-i} = 0$ for all $M \in \mathbb{Z}$.

Proof of Theorem 1.3. We simply write $F_a(x) = f_{\mathcal{N}_a}(x, q)$. First we consider the case $a = 1$. Define $G_1(x)$ and $\{g_M^{(1)}\}_{M\in\mathbb{Z}}$ by

$$G_1(x) = \sum_{M\in\mathbb{Z}} g_M^{(1)} x^M := \frac{F_1(x)}{(x;q^2)_\infty}. \tag{4.5}$$

Note that $g_M^{(1)} = 0$ if $M < 0$. Dividing (4.2) by $(xq^6;q^2)_\infty$ yields

$$\begin{aligned}
0 = &(1-x)(1-xq^2)(1-xq^4)G_1(x)\\
&-(1-xq^2)(1-xq^4)(1+xq^2+xq^3+xq^4)G_1(xq^2)\\
&+xq^4(1-xq^4)(1-x+xq^3+xq^4+xq^5)G_1(xq^4)\\
&-x^2q^6(1-xq^4-xq^5-xq^6+xq^9)G_1(xq^6)\\
&+x^3q^{13}(1+q+q^2)G_1(xq^8)+x^3q^{17}G_1(xq^{10}),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 = &(1-q^{2M})g_M^{(1)} + (-1-q^2-q^4-q^{2M+1}+q^{4M})g_{M-1}^{(1)}\\
&+q^2(1+q^2+q^4-q^{2M-3})(1+q^{2M-3})(1+q^{2M-2})g_{M-2}^{(1)}\\
&-q^6(1-q^{2M-5})(1+q^{2M-5})(1+q^{2M-4})(1+q^{2M-3})(1+q^{2M-2})g_{M-3}^{(1)}
\end{aligned} \tag{4.6}$$

for all $M \in \mathbb{Z}$. Letting

$$h_M^{(1)} := \frac{g_M^{(1)}}{(-q;q)_{2M}} \quad \text{and} \quad H_1(x) := \sum_{M\in\mathbb{Z}} h_M^{(1)} x^M \tag{4.7}$$

(note that $h_M^{(1)} = 0$ if $M < 0$) and dividing (4.2) by $q^{-1}(-q;q)_{2M-2}$, we have

$$\begin{aligned}
0 = &q(1-q^{2M})(1+q^{2M-1})(1+q^{2M})h_M^{(1)} + q(-1-q^2-q^4-q^{2M+1}+q^{4M})h_{M-1}^{(1)}\\
&+q^3(1+q^2+q^4-q^{2M-3})h_{M-2}^{(1)} - q^7(1-q^{2M-5})h_{M-3}^{(1)}
\end{aligned}$$

for all $M \in \mathbb{Z}$, which is equivalent to

$$\begin{aligned}
0 = &q(1-x)(1-xq^2)(1-xq^4)H_1(x)\\
&+(1-xq^2)(1-xq^4)(1+xq^2)H_1(xq^2) - q(1-xq^4)H_1(xq^4) - H_1(xq^6).
\end{aligned} \tag{4.8}$$

Finally we define $I_1(x)$ and $i_M^{(1)}$ $(M \in \mathbb{Z})$ by

$$I_1(x) = \sum_{M\in\mathbb{Z}} i_M^{(1)} x^M := H_1(x)(x;q^2)_\infty \tag{4.9}$$

(note that $i_M^{(1)} = 0$ if $M < 0$) and multiply (4.2) by $(xq^6;q^2)_\infty$ to obtain

$$0 = qI_1(x) + (1+xq^2)I_1(xq^2) - qI_1(xq^4) - I_1(xq^6)$$

which is equivalent to

$$0 = q(1-q^{2M})(1+q^{2M})(1+q^{2M-1})i_M^{(1)} + q^{2M} i_{M-1}^{(1)}$$

for all $M \in \mathbb{Z}$. Since $i_0^{(1)} = h_0^{(1)} = g_0^{(1)} = F_1(0) = f_{\mathcal{N}_1}(0,q) = 1$, we have

$$i_M^{(1)} = \frac{(-1)^M q^{M^2}}{(-q;q)_{2M}(q^2;q^2)_M}, \text{ i.e., } I_1(x) = \sum_{M\geq 0} \frac{(-1)^M q^{M^2}}{(-q;q)_{2M}(q^2;q^2)_M} x^M.$$

The cases $a = 2, 3$ can be treated in parallel. Defining $G_a(x) = \sum_M g_M^{(a)} x^M$, $H_a(x) = \sum_M h_M^{(a)} x^M$, $I_a(x) = \sum_M i_M^{(a)} x^M$ by transformations shown below,

$a=1$	$a=2$	$a=3$
(4.2)	$G_2(x) = F_2(x)/(x;q^2)_\infty$	$G_3(x) = F_3(x)/(x;q^2)_\infty$
(4.2)	$h_M^{(2)} = g_M^{(2)}/(-q;q)_{2M}$	$h_M^{(3)} = g_M^{(3)}/(-q^2;q)_{2M}$
(4.2)	$I_2(x) = H_2(x)(x;q^2)_\infty$	$I_3(x) = H_3(x)(x;q^2)_\infty$

we can get

$$I_a(x) = \sum_{M\geq 0} \frac{(-1)^M q^{M(M+2t)}}{(-q^{1+s};q)_{2M}(q^2;q^2)_M} x^M,$$

where $(s,t) := (0,0), (0,1), (1,1)$ for $a = 1,2,3$ respectively.

For each $a = 1,2,3$, by (A) we have

$$H_a(x) = \frac{I_a(x)}{(x;q^2)_\infty} = \sum_{N\geq 0} \frac{x^N}{(q^2;q^2)_N} \sum_{M\geq 0} \frac{(-1)^M q^{M(M+2t)}}{(-q^{1+s};q)_{2M}(q^2;q^2)_M} x^M.$$

Hence by (4.2)

$$g_L^{(a)} = \sum_{0\leq M\leq L} \frac{(-1)^M q^{M(M+2t)} (-q^{1+s};q)_{2L}}{(q^2;q^2)_{L-M}(-q^{1+s};q)_{2M}(q^2;q^2)_M}$$

for $L \geq 0$, which implies

$$\lim_{L\to\infty} g_L^{(a)} = \frac{(-q;q)_\infty}{(q^2;q^2)_\infty} \sum_{M\geq 0} \frac{(-1)^M q^{M(M+2t)}}{(-q;q)_{2M+s}(q^2;q^2)_M}.$$

Since $F_a(x) = (x;q^2)_\infty G_a(x) = (1-x)(xq^2;q^2)_\infty G_a(x)$, by Lemma 4.4

$$F_a(1) = (q^2;q^2)_\infty \lim_{L\to\infty} g_L^{(a)} = (-q;q)_\infty \sum_{M\geq 0} \frac{(-1)^M q^{M(M+2t)}}{(-q;q)_{2M+s}(q^2;q^2)_M} \tag{4.10}$$

Now the first equality in each of the statements of Theorem 1.3 follows from three identities due to Slater ([37, (117),(118),(119)]=[35, (A.187),(A.186),(A.188)] with $q \mapsto -q$)

$$\sum_{n\geq 0} \frac{(-1)^n q^{n(n+2t)}}{(-q;q)_{2n+s}(q^2;q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \frac{(q^{2b}, q^{14-2b}, q^{14}; q^{14})_\infty}{(q^b, q^{14-b}; q^{14})_\infty},$$

where $(b,s,t) = (3,0,0), (1,0,1), (5,1,1)$.

Also, using (B) for $(-q;q)_\infty/(-q;q)_{2M+s} = (-q^{2M+1+s};q)_\infty$, we have

$$(4.2) = \sum_{M\geq 0} \frac{(-1)^M q^{M(M+2t)}}{(q^2;q^2)_M} \sum_{K\geq 0} \frac{q^{\binom{K}{2}+(2M+1+s)K}}{(q;q)_K} = N_a,$$

proving the second equality in each of the statements of Theorem 1.3. $\square$

Remark 4.5. By eliminating the summation over j in (1.3) using (B), we see

$$N_a = \sum_{i\geq 0} (q^{1+2i+2t};q^2)_\infty \frac{q^{\binom{i}{2}+(1+s)i}}{(q;q)_i} = (q;q^2)_\infty \sum_{i\geq 0} \frac{q^{\binom{i}{2}+(1+s)i}}{(q;q)_i (q;q^2)_{i+t}}$$

for $(a,s,t) = (1,0,0),(2,0,1),(3,1,1)$. Hence, as Step 3 in §1.2 for Nandi's conjectures, we can employ another three identities due to Slater

$$\sum_{i\geq 0} \frac{q^{\binom{i}{2}+(1+s)i}}{(q;q)_i (q;q^2)_{i+t}} = \frac{(q^a,q^{7-a},q^7;q^7)_\infty (q^{7-2a},q^{7+2a};q^{14})_\infty}{(q;q)_\infty (q;q^2)_\infty},$$

where $(a,s,t) = (1,0,0),(2,0,1),(3,1,1)$ (see [37, (81),(80),(82)]=[35, (A.124),(A.125),(A.126)]).

Appendix A. Textbook constructions for finite automata

To recall the proof of Proposition 3.5 we need *ε-NFAs*.

Definition A.1 ([36, Definition 1.37])**.** A *nondeterministic finite automaton with ε-transitions* (or *ε-NFA* for short) over Σ is a 5-tuple $M = (Q,\Sigma,\Delta,s,F)$ where Q is a finite set, $\Delta\colon Q\times(\Sigma\sqcup\{\varepsilon\}) \longrightarrow 2^Q$, $s\in Q$ and $F\subseteq Q$.

Definition A.2. Let $M = (Q,\Sigma,\Delta,s,F)$ be an ε-NFA.
(1) For $A\subseteq Q$, its *ε-closure* $\mathcal{E}(A)$ is the set of states that are reachable from a state in A via successive ε-transitions, i.e., $\mathcal{E}(A) := \bigcup_{n\geq 0}\Delta_\varepsilon^n(A)$ where $\Delta_\varepsilon(B) := \bigcup_{q\in B}\Delta(q,\varepsilon)$ for $B\subseteq Q$.
(2) We define $\widehat{\Delta}\colon Q\times\Sigma^* \longrightarrow 2^Q$ inductively by $\widehat{\Delta}(q,\varepsilon) = \mathcal{E}(\{q\})$ and $\widehat{\Delta}(q,wa) = \mathcal{E}\big(\bigcup_{q'\in\widehat{\Delta}(q,w)}\Delta(q',a)\big)$ ($q\in Q$, $a\in\Sigma$, $w\in\Sigma^*$). We write $L(M) = \{w\in\Sigma^* \mid \widehat{\Delta}(s,w)\cap F\neq\emptyset\}$, the language that M *recognizes*.

Proposition A.3 (See e.g., [36, Corollary 1.40] for the details)**.** *A language $X\subseteq\Sigma^*$ is regular if and only if there exists an ε-NFA recognizing X.*

Proof. Every DFA can be seen as an ε-NFA (with no ε-transitions). Conversely, an ε-NFA (Q,Σ,Δ,s,F) can be converted into an equivalent DFA $(Q',\Sigma,\delta',s',F')$ via the *subset construction*: $Q' = 2^Q$, $\delta'\colon Q'\times\Sigma\longrightarrow Q'$; $(A,a)\mapsto\mathcal{E}(\bigcup_{q\in A}\Delta(q,a))$, $s' = \mathcal{E}(\{s\})$ and $F' = \{A\subseteq Q \mid A\cap F\neq\emptyset\}$. $\square$

Proof of Proposition 3.5. Assume DFAs $(Q_1,\Sigma,\delta_1,s_1,F_1)$ and $(Q_2,\Sigma,\delta_2,s_2,F_2)$ recognize Y and Z respectively. By Proposition A.3 it suffices to give a DFA or an ε-NFA recognizing (1) $Y\cap Z$, (2) YZ, (3) Y^c, and (4) Y^*.
(1) The DFA $(Q_1\times Q_2,\Sigma,\delta,(s_1,s_2),F_1\times F_2)$ recognizes $Y\cap Z$, where $\delta((q_1,q_2),a) = (\delta_1(q_1,a),\delta_2(q_2,a))$.
(2) The ε-NFA (Q,Σ,Δ,s,F) recognizes YZ, where $Q = Q_1\sqcup Q_2$, $s = s_1$, $F = F_2$, $\Delta(q,a) = \{\delta_i(q,a)\}$ ($i=1,2$, $q\in Q_i$, $a\in\Sigma$) and $\Delta(q,\varepsilon) = \{s_2\}$ if $q\in F_1$ and $\Delta(q,\varepsilon) = \emptyset$ if $q\in(Q_1\setminus F_1)\sqcup Q_2$.

(3) The DFA $(Q_1, \Sigma, \delta_1, s_1, Q_1 \setminus F_1)$ recognizes Y^c.
(4) The ε-NFA $(Q, \Sigma, \Delta, s, F)$ recognizes Y^*, where $Q = Q_1 \sqcup \{s\}$, $F = \{s\} \sqcup F_1$, $\Delta(q, a) = \{\delta_1(q, a)\}$ $(q \in Q_1,\ a \in \Sigma)$, $\Delta(s, a) = \emptyset$ $(a \in \Sigma)$, $\Delta(s, \varepsilon) = \{s_1\}$, $\Delta(q, \varepsilon) = \{s_1\}$ if $q \in F_1$ and $\Delta(q, \varepsilon) = \emptyset$ if $q \in Q_1 \setminus F_1$. $\square$

Remark A.4 (DFA minimization. See e.g., [23, Lecture 14]). Given a DFA $M = (Q, \Sigma, \delta, s, F)$, one can compute $M_{\min}$ by the following algorithm.

1. Remove all unreachable states.
2. Mark all (unordered) pairs $\{q, q'\}$ with $q \in F$, $q' \in Q \setminus F$.
3. Repeat until no more changes occur:
   if there exists an unmarked pair $\{q, q'\} \subseteq Q$ such that $\{\delta(q, a), \delta(q', a)\}$ is marked for some $a \in \Sigma$, then mark $\{q, q'\}$.
4. The relation "$q \sim q' :\Longleftrightarrow \{q, q'\}$ is unmarked" is then an equivalence relation. Writing $[q] := \{q' \in Q \mid q \sim q'\}$, we have a new DFA $M_{\min} = (Q', \Sigma, \delta', s', F')$ where $Q' := \{[q] \mid q \in Q\}$, $\delta'([q], a) := [\delta(q, a)]$, $s' := [s]$, $F' := \{[q] \mid q \in F\}$.

Appendix B. Modified Murray–Miller Theorem

We review an algorithm given in [3, p. 1040], [4, Lemma 8.10] (see also [12, §3] for an exposition), which outputs a (nontrivial) q-difference equation for $F_1(x)$ from a given system of q-difference equations

$$F_i(x) = \sum_{j=1}^{\ell} p_{ij}(x) F_j(xq^m) \qquad (i = 1, \dots, \ell), \tag{B.1}$$

where $p_{ij}(x) = p_{ij}(x, q) \in \mathbb{Q}(x, q)$.
<u>Step 1</u>: We obtain from (B) another system

$$F'_i(x) = \sum_{j=1}^{\ell'} p'_{ij}(x) F'_j(xq^m) \quad (i = 1, \dots, \ell'), \tag{B.2}$$

where $1 \le \ell' \le \ell$, $F'_1(x) = F_1(x)$, and $(p'_{ij})_{i,j=1}^{\ell'} \in \mathrm{Mat}_{\ell'}(\mathbb{Q}(x, q))$ is of the form (B) with (s, ℓ) replaced by (ℓ', ℓ').

Step 1 is done in Algorithm 1, which receives $(p_{ij}(x))_{i,j=1}^{\ell}$ as the input and returns $(p'_{ij}(x))_{i,j=1}^{\ell'}$ as the output. The following are supplementary explanations on the s-th iteration of the **for** loop in Algorithm 1.

- In the line 3, i.e., at the beginning of the iteration, it is ensured that (a) the matrix $P^{(s)}$ is defined and is of the form

$$\begin{array}{cc} & \begin{array}{cccccccc} 1 & 2 & & \dots & s-1 & s & \dots & \ell \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ \\ s-1 \\ s \\ \vdots \\ \ell \end{array} & \begin{pmatrix} \star & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \star & \star & 1 & \dots & 0 & 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \star & \star & \star & \dots & 1 & 0 & \dots & 0 \\ \star & \star & \star & \dots & \star & 1 & \dots & 0 \\ \star & \star & \star & \dots & \star & \star & \dots & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \star & \star & \star & \dots & \star & \star & \dots & \star \end{pmatrix} \end{array}, \tag{B.3}$$

i.e., $P^{(s)}_{1,2} = \cdots = P^{(s)}_{s-1,s} = 1$ and $P^{(s)}_{i,j} = 0$ if $i < s$ and $j > i+1$; and (b) $F'_1(x), \dots, F'_s(x)$ are (implicitly) defined (we let $F'_1(x) := F_1(x)$ when $s = 1$) and satisfy

$$\begin{aligned} &{}^t\big(F'_1(x), \dots, F'_s(x), F_{s+1}(x), \dots, F_\ell(x)\big) \\ &= P^{(s)} \cdot {}^t\big(F'_1(xq^m), \dots, F'_s(xq^m), F_{s+1}(xq^m), \dots, F_\ell(xq^m)\big). \end{aligned} \tag{B.4}$$

These assertions are obvious when $s = 1$ by putting $P^{(1)} := (p_{ij}(x))_{i,j=1}^{\ell}$.

- The **if** statement in the line 4 is always true if $s = \ell$.
- If the algorithm reaches the line 5, we can see by (B) and (B) that $F'_1(x), \dots, F'_{\ell'}(x)$ satisfy the system of q-difference equations (B) with $(p'_{ij})_{i,j=1}^{\ell'} := (P^{(s)}_{ij})_{i,j=1}^{s}$.
- The lines 9 and 10 correspond to switching $F'_{s+1}(x)$ and $F'_t(x)$. To make the algorithm deterministic, one should choose the smallest t in line 8, for example.
- In line 13, it can be checked (see [12, Claim 3.1]) that $P^{(s+1)}$ ($\in \mathrm{Mat}_\ell(\mathbb{Q}(x,q))$) is again of the form (B) with (s,ℓ) replaced by $(s+1,\ell)$. Moreover, for $F'_{s+1}(x) := \sum_{j=s+1}^{\ell} P^{(s)}_{s,j}(xq^{-m})F_j(x)$ we can check (B) with s replaced by $s+1$ (see [12, (3.7)]).

Algorithm 1 Obtain (B) from (B) ([4, Lemma 8.10], see also [12, §3])

Input: $(p_{ij}(x))_{i,j=1}^{\ell}$ // the coefficients in (B)
Output: ℓ', $(p'_{ij}(x))_{i,j=1}^{\ell'}$ // the coefficients in (B)

1: $P^{(1)} \leftarrow (p_{ij})_{i,j=1}^{\ell}$
2: **for** $s = 1$ **to** ℓ **do**
3:     // assert that $P^{(s)}$ is of the form (B)
4:     **if** $P^{(s)}_{s,s+1} = P^{(s)}_{s,s+2} = \cdots = P^{(s)}_{s,\ell} = 0$ **then**
5:         **return** s, $\big(P^{(s)}_{ij}(x)\big)_{i,j=1}^{s}$
6:     **end if**
7:     **if** $P^{(s)}_{s,s+1} = 0$ **then**
8:         choose any t such that $s+1 < t \leq \ell$ and $P^{(s)}_{s,t} \neq 0$
9:         swap $s+1$-th and t-th rows of $P^{(s)}$
10:         swap $s+1$-th and t-th columns of $P^{(s)}$
11:     **end if**
12:     $T_s(x) \leftarrow \begin{array}{c c} & \begin{array}{ccccccc} 1 & & s & s+1 & & & \ell \end{array} \\ \begin{array}{c} 1 \\ \\ s \\ s+1 \\ \\ \\ \ell \end{array} & \begin{pmatrix} 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & P^{(s)}_{s,s+1}(x) & P^{(s)}_{s,s+2}(x) & \dots & P^{(s)}_{s,\ell}(x) \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \end{array}$
13:     $P^{(s+1)} \leftarrow T_s(xq^{-m})P^{(s)}T_s(x)^{-1}$
14: **end for**

This completes the algorithm to obtain a new system (B).

Step 2: Now the i-th equation (for $i = 1, \ldots, \ell' - 1$) in (B) is of the form $0 = -F'_i(x) + F'_{i+1}(xq^m) + \sum_{j<i+1} p'_{ij}(x) F'_j(xq^m)$. We can eliminate $F'_{\ell'}, \ldots, F'_2$ from the system (in this order) to transform the final equation in (B) into a q-difference equation for $F'_1(x) = F_1(x)$, which is nontrivial (see [4, Lemma 8.10] for more details).

Appendix C. Proof of Proposition 4.2

We apply the algorithm in Appendix B to (4.1).

C.1. **The case $\mathcal{N}_1$.** To find a q-difference equaiton for $F_7(x)$, first we permute the positions of $F_0, \ldots, F_5, F_7$ in (4.1) as follows.

$$\begin{pmatrix} F_7(x) \\ F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \\ F_0(x) \end{pmatrix} = \begin{pmatrix} 0 & xq^2 & x^2q^4 & 0 & 0 & 0 & 1 \\ 0 & xq^2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & xq^2 & 0 & xq & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & xq^2 & 0 & 0 \\ 0 & xq^2 & x^2q^4 & xq & 0 & 0 & 1 \\ 0 & xq^2 & x^2q^4 & xq & x^2q^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_7(xq^2) \\ F_1(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \\ F_4(xq^2) \\ F_5(xq^2) \\ F_0(xq^2) \end{pmatrix}.$$

Next we apply Algorithm 1 (note that it is deterministic). It stops at the 5-th iteration of the **for** loop and we get

$$\begin{pmatrix} G_1(x) \\ G_2(x) \\ G_3(x) \\ G_4(x) \\ G_5(x) \\ G_6(x) \\ G_7(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x^2 & x+1 & 1 & 0 & 0 & 0 & 0 \\ \frac{-x^3+x^2}{q^2} & \frac{x}{q} & \frac{1}{q} & 1 & 0 & 0 & 0 \\ \frac{-x^2}{q^3} & \frac{-xq^2+x^2}{q^4} & \frac{-q^2+x}{q^4} & \frac{-q^2+x}{q^3} & 1 & 0 & 0 \\ \frac{-x^3}{q^7} & 0 & 0 & 0 & \frac{x}{q^4} & 0 & 0 \\ 0 & 1 & 0 & \frac{q}{x-1} & \frac{q^3}{-x^2+x} & 0 & 0 \\ 0 & 1 & 0 & \frac{q}{x-1} & \frac{q^3}{-x+1} & 0 & 0 \end{pmatrix} \begin{pmatrix} G_1(xq^2) \\ G_2(xq^2) \\ G_3(xq^2) \\ G_4(xq^2) \\ G_5(xq^2) \\ G_6(xq^2) \\ G_7(xq^2) \end{pmatrix},$$

where the square matrix displayed just above is $P^{(5)}$ in the notation of Algorithm 1 and each G_i is a certain $\mathbb{Q}(x,q)$-linear combination of F_j ($j \in \{0, \ldots, 5, 7\}$) with $G_1 = F_7$. The equation given in the i-th row ($i = 1, \ldots, 4$) is

$$0 = -G_i(x) + G_{i+1}(xq^2) + \sum_{j \le i} P^{(5)}_{i,j}(x) G_j(xq^2),$$

by which each $G_{i+1}(x)$ is written in terms of $G_j(x)$ ($j \le i$) and $G_i(xq^{-2})$. Thus we can eliminate $G_5, \ldots, G_2$ and then the equation in the 5-th row

$$0 = -G_5(x) - \frac{x^3}{q^7} G_1(xq^2) + \frac{x}{q^4} G_5(xq^2)$$

turns into an equation for $\{G_1(xq^{2k}) \mid k \in \mathbb{Z}\}$:

$$\begin{aligned} 0 = &- G_1(xq^{-8}) + \frac{q^6 + x(q^2+q+1)}{q^6} G_1(xq^{-6}) - \frac{xq^8 + x^2(q^5+q^4+q^3-1)}{q^{12}} G_1(xq^{-4}) \\ &- \frac{-x^2q^4 - x^3(q^5+q^2+q+1)}{q^{14}} G_1(xq^{-2}) + \frac{x^3(x-q^2)(1+q+q^2)}{q^{13}} G_1(x) \\ &- \frac{x^3(x-1)(x-q^2)}{q^9} G_1(xq^2). \end{aligned} \tag{C.1}$$

By letting $x \mapsto xq^8$ in (C.1), we obtain (4.2) for $G_1(x) = F_7(x) = f_{\mathcal{N}_1}(x,q)$.

C.2. **The case $\mathcal{N}_2$.** The proof of Proposition 4.2 for $\mathcal{N}_2$ (and $\mathcal{N}_3$) proceeds almost the same. We start by rewriting (4.1) as

$$\begin{pmatrix} F_3(x) \\ F_1(x) \\ F_2(x) \\ F_4(x) \\ F_7(x) \\ F_5(x) \\ F_0(x) \end{pmatrix} = \begin{pmatrix} xq & xq^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & xq^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & xq^2 & 1 & 0 & 0 \\ 0 & xq^2 & x^2q^4 & 0 & 0 & 0 & 1 \\ xq & xq^2 & x^2q^4 & 0 & 0 & 0 & 1 \\ xq & xq^2 & x^2q^4 & x^2q^2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_3(xq^2) \\ F_1(xq^2) \\ F_2(xq^2) \\ F_4(xq^2) \\ F_7(xq^2) \\ F_5(xq^2) \\ F_0(xq^2) \end{pmatrix}.$$

Here we permuted the positions of $F_0, \ldots, F_5, F_7$ so that further row (and column) swapping (in the lines 9 and 10 of Algorithm 1) will not happen. Then Algorithm 1 stops at the 5-th iteration with

$$P^{(5)} = \begin{pmatrix} xq & 1 & 0 & 0 & 0 & 0 & 0 \\ xq & x+1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{x^2}{q^4} & \frac{x^2}{q^4} & \frac{x}{q^2} & 1 & 0 & 0 \\ 0 & \frac{x^2q^2-x^3}{q^8} & \frac{x^2q^2-x^3}{q^8} & 0 & 0 & 0 & 0 \\ xq & 1 & 1 & 0 & 0 & 0 & 0 \\ xq & 1 & 1 & 1 & \frac{q^2}{x-1} & 0 & 0 \end{pmatrix},$$

and by the same procedure we obtain

$$\begin{aligned} 0 = &- G_1(xq^{-8}) + \frac{q^7 + x(1+q+q^3)}{q^7} G_1(xq^{-6}) - \frac{xq^7 + x^2q^2 + x^2}{q^{11}} G_1(xq^{-4}) \\ &+ \frac{x^2q^4 - x^3q^2 - x^3}{q^{10}} G_1(xq^{-2}) + \frac{x^3(x-q^2)(1+q^2+q^3)}{q^{11}} G_1(x) \\ &- \frac{x^3(x-1)(x-q^2)}{q^7} G_1(xq^2), \end{aligned} \tag{C.2}$$

where $G_1(x) = F_3(x) = f_{\mathcal{N}_2}(x,q)$. Now, by letting $x \mapsto xq^8$ in (C.2) we obtain (4.2) for $f_{\mathcal{N}_2}(x,q)$.

C.3. **The case $\mathcal{N}_3$.** Similarly, we start the algorithm by writing

$$\begin{pmatrix} F_4(x) \\ F_7(x) \\ F_2(x) \\ F_3(x) \\ F_5(x) \\ F_1(x) \\ F_0(x) \end{pmatrix} = \begin{pmatrix} xq^2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^2q^4 & 0 & 0 & xq^2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & xq & 1 & xq^2 & 0 \\ 0 & 0 & x^2q^4 & xq & 0 & xq^2 & 1 \\ 0 & 0 & 0 & 0 & 1 & xq^2 & 0 \\ x^2q^2 & 0 & x^2q^4 & xq & 0 & xq^2 & 1 \end{pmatrix} \begin{pmatrix} F_4(xq^2) \\ F_7(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \\ F_5(xq^2) \\ F_1(xq^2) \\ F_0(xq^2) \end{pmatrix}.$$

Then Algorithm 1 stops at the 5-th iteration with

$$P^{(5)} = \begin{pmatrix} xq^2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ x^2q^2 & x^2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x}{q^2} & \frac{xq+x}{q^2} & 1 & 0 & 0 \\ 0 & 0 & \frac{xq^2-x^2}{q^5} & \frac{xq^2-x^2}{q^5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{q}{-x^2+x} & 0 & 0 \\ x^2q^2 & 0 & 1 & 1 & \frac{-q}{-x+1} & 0 & 0 \end{pmatrix}.$$

By the same procedure we obtain

$$\begin{aligned}0 =& -G_1(xq^{-8}) + \frac{q^6 + x(q^3+q^2+1)}{q^6} G_1(xq^{-6}) - \frac{x^2q^2 + xq^3 + x^2}{q^7} G_1(xq^{-4}) \\ &+ \frac{x^2q^4 - x^3q^2 - x^3}{q^{10}} G_1(xq^{-2}) + \frac{x^3(x-q^2)(1+q+q^3)}{q^8} G_1(x) \\ &- \frac{x^3(x-1)(x-q^2)}{q^3} G_1(xq^2), \end{aligned} \tag{C.3}$$

where $G_1(x) = F_4(x) = f_{\mathcal{N}_3}(x,q)$. Now, by letting $x \mapsto xq^8$ in (C.3) we obtain (4.2) for $f_{\mathcal{N}_3}(x,q)$.

Appendix D. Minimal forbidden patterns and prefixes

Let Σ be a nonempty finite set. Recall that for $B \subseteq \Sigma^*$, the language $\Sigma^* B \Sigma^*$ (resp. $B\Sigma^*$) is the set of words matching (resp. beginning with) some $w \in B$ in the sense of Definition 1.1 (allowing $n = 0$ in Definition 1.1). Then, a language $A \subseteq \Sigma^*$ is of the form $A = \Sigma^* B \Sigma^*$ (resp. $A = B\Sigma^*$) for some $B \subseteq \Sigma^*$ if and only if $A = \Sigma^* A \Sigma^*$ (resp. $A = A\Sigma^*$). In [14] they gave an algorithm to find from given $A = \Sigma^* A \Sigma^* \subseteq \Sigma^*$ the minimum $B \subseteq \Sigma^*$ such that $A = \Sigma^* B \Sigma^*$. By a slight generalization it can also be used to find the minimum B for which $A = B\Sigma^*$, given $A \subseteq \Sigma^*$ such that $A = A\Sigma^*$ (Proposition D.1 and D.2).

In general, for a poset $(P, \leq)$ and a subset $A \subseteq P$ we write

$$\begin{aligned} \mathsf{minimal}\, A = \mathsf{minimal}_{\leq} A &:= \{w \in A \mid \forall v \in A, (v \leq w \Longrightarrow v = w)\}, \\ V(A) = V_{\leq}(A) &:= \{w \in P \mid \exists v \in A,\, v \leq w\}. \end{aligned}$$

Let us say a poset $(P, \leq)$ is *good* if $A \subseteq V(\mathsf{minimal}\, A)$ for any $A \subseteq P$.

Proposition D.1. *Let $(P, \leq)$ be a good poset and assume that $A, B \subseteq P$ satisfy $A = V(A)$. Then $A = V(B)$ if and only if $\mathsf{minimal}\, A \subseteq B \subseteq A$.*

Proof. ($\Longrightarrow$): Assume $A = V(B)$. Then $B \subseteq A$ is obvious. For any $w \in A$ we have $u \leq w$ for some $u \in B$, and if $w \in \mathsf{minimal}\, A$ then $w = u$. ($\Longleftarrow$): $\mathsf{minimal}\, A \subseteq B \subseteq A$ implies $A \subseteq V(\mathsf{minimal}\, A) \subseteq V(B) \subseteq V(A) = A$. $\square$

Let us consider partial orders $\leq$ and $\leq_r$ on Σ^* defined by

$$\begin{aligned} v \leq w &:\Longleftrightarrow \exists u \in \Sigma^*, \exists u' \in \Sigma^*,\, w = uvu', \\ v \leq_r w &:\Longleftrightarrow \exists u \in \Sigma^*,\, w = vu. \end{aligned}$$

Clearly $V_{\leq}(B) = \Sigma^* B \Sigma^*$ and $V_{\leq_r}(B) = B\Sigma^*$ for $B \subseteq \Sigma^*$. It is easy to see that $(\Sigma^*, \leq)$ and $(\Sigma^*, \leq_r)$ are good.

Proposition D.2. *Let $A \subseteq \Sigma^+ \, (= \Sigma^* \setminus \{\varepsilon\})$.*
(1) If $A = \Sigma^* A \Sigma^* \, (= V_{\leq}(A))$ then $\mathsf{minimal}_{\leq} A = A \cap A^c\Sigma \cap \Sigma A^c$.
(2) If $A = A\Sigma^* \, (= V_{\leq_r}(A))$ then $\mathsf{minimal}_{\leq_r} A = A \cap A^c \Sigma$.

Proof. (1) is [14, Eq. (2)] with A replaced by A^c. (2) is proved in the same way as (1), but for completeness we duplicate the proof. $\overline{(\subseteq)}$: Clearly $\mathsf{minimal}_{\leq_r} A \subseteq A$. For any $w \in \mathsf{minimal}_{\leq_r} A$, since $w \neq \varepsilon$ (otherwise we get $A = \Sigma^*$) we can write $w = w'a$ with $w' \in \Sigma^*$, $a \in \Sigma$. Then $w' \notin A$ by $w \in \mathsf{minimal}_{\leq_r} A$. Hence $w = w'a \in A^c\Sigma$. $\underline{(\supseteq)}$: For any

$w = a_1 \cdots a_n \in A^c\Sigma$, we have $n \geq 1$ and $a_1 \cdots a_{n-1} \notin A$. For $v \in \Sigma^*$, if $v < w$ then $v = a_1 \cdots a_i$ for some $0 \leq i < n$, and hence $v \notin A$ since $A = A\Sigma^*$. Therefore $w \in \mathsf{minimal}_{\leq_r} A$ if $w \in A$. $\square$

Lemma D.3. *Let $A, X \subseteq \Sigma^+$ and assume $\Sigma^* X \Sigma^* \subseteq A = A\Sigma^*$. Then*

$$X' := \mathsf{minimal}_{\leq_r}(A) \setminus \Sigma^* X = \big(A \cap (A^c\, \Sigma)\big) \setminus \Sigma^* X \quad (\subseteq \Sigma^+) \tag{D.1}$$

is the minimum set (with respect to inclusion) such that $A = \Sigma^* X \Sigma^* \cup X' \Sigma^*$.

Proof. The equality in (D.3) follows from Proposition D.2 (2). We apply Proposition D.1. Writing $A' = \Sigma^* X$, we have $A = A'\Sigma^* \cup B\Sigma^*$ ($= V_{\leq_r}(A' \cup B)$) $\iff \mathsf{minimal}_{\leq_r} A \subseteq (A' \cup B) \subseteq A \iff (\mathsf{minimal}_{\leq_r} A) \setminus A' \subseteq B \subseteq A$ for $B \subseteq \Sigma^*$. Thus, X' is the desired one. $\square$

We apply this to DFAs. Recall Definition 3.11.

Proposition D.4. *Let $M = (Q, \Sigma, \delta, s, F)$ be a DFA and assume $L(M) = \Sigma^* X \Sigma^* \cup X' \Sigma^*$ for some $X, X' \subseteq \Sigma^*$. For any reachable state $v \in Q \setminus F$ we have $L(M_v) = \Sigma^* X \Sigma^* \cup X_v \Sigma^*$, where*

$$X_v := \big(L(M_v) \cap (L(M_v)^c\, \Sigma)\big) \setminus \Sigma^* X \quad (\subseteq \Sigma^+). \tag{D.2}$$

Moreover, X_v is the minimum such set (with respect to inclusion).

Proof. By the reachability, $\widehat{\delta}(s, b) = v$ for some $b \in \Sigma^*$. Then $a \in L(M_v) \iff ba \in L(M) = \Sigma^* X \Sigma^* \cup X' \Sigma^*$ for any $a \in \Sigma^*$, by which $\Sigma^* X \Sigma^* \subseteq L(M_v) = L(M_v)\Sigma^*$ follows. Now the proposition follows from Lemma D.3 (note that $v \notin F$ implies $\varepsilon \notin L(M_v)$). $\square$

Appendix E. Connection to linked partition ideals

E.1. **On the definition of partition ideals.** Consider a partial order $\leq$ on $\mathsf{Par}\,(\simeq \widehat{\mathsf{Par}})$ defined by $(f_i)_{i\geq 1} \leq (g_i)_{i \geq 1} :\iff \forall i \geq 1, f_i \leq g_i$. In [3, Definition 1] a subset $\mathcal{C}$ of Par is called a *partition ideal* (*PI* for short) if it is an order ideal ([38, p. 282]) with respect to $\leq$, i.e.,

$$\forall f \in \widehat{\mathcal{C}}, \forall g \in \widehat{\mathsf{Par}}, (g \leq f \implies g \in \widehat{\mathcal{C}}). \tag{E.1}$$

For $m > 0$ and $\lambda \in \mathsf{Par}$ we write $\lambda_{>m} := (\lambda_1, \ldots, \lambda_{\ell'})$ where $\ell' := \#\{i \geq 1 \mid \lambda_i > m\}$. In [3, Definition 7] a PI $\mathcal{C}$ is defined to have *modulus* $m > 0$ if $\phi_+^m(\mathcal{C}) = \mathcal{C}_{>m} := \{\lambda \in \mathcal{C} \mid \lambda = \lambda_{>m}\}$. As we see below, this is equivalent to adding an extra condition $\phi_+^m(\mathcal{C}) \subseteq \mathcal{C}$ to $\phi_-^m(\mathcal{C}) \subseteq \mathcal{C}$ (cf. (2.3)) under the assumption

$$\lambda \in \mathcal{C} \implies \lambda_{>m} \in \mathcal{C}. \tag{E.2}$$

Proposition E.1. *Let a subset $\mathcal{C} \subseteq \mathsf{Par}$ satisfy* (E.1). *Then $\phi_+^m(\mathcal{C}) = \mathcal{C}_{>m}$ if and only if $\phi_+^m(\mathcal{C}) \subseteq \mathcal{C}$ and $\phi_-^m(\mathcal{C}) \subseteq \mathcal{C}$.*

Proof. $\underline{(\Rightarrow)}$: Assume $\phi_+^m(\mathcal{C}) = \mathcal{C}_{>m}$. Then obviously $\phi_+^m(\mathcal{C}) \subseteq \mathcal{C}$. Since $\lambda_{>m} = \phi_+^m\phi_-^m(\lambda)$ for any $\lambda \in \mathsf{Par}$, (E.1) implies $\phi_+^m\phi_-^m(\mathcal{C}) \subseteq \mathcal{C}_{>m}$ ($= \phi_+^m(\mathcal{C})$), and hence $\phi_-^m(\mathcal{C}) \subseteq \mathcal{C}$ since ϕ_+ is injective. $\underline{(\Leftarrow)}$: Assume $\phi_+^m(\mathcal{C}) \subseteq \mathcal{C}$ and $\phi_-^m(\mathcal{C}) \subseteq \mathcal{C}$. Then obviously $\phi_+^m(\mathcal{C}) \subseteq \mathcal{C}_{>m}$. Since $\phi_-^m(\mathcal{C}) \subseteq \mathcal{C}$ we have $\phi_+^m\phi_-^m(\mathcal{C}_{>m}) \subseteq \phi_+^m(\mathcal{C})$, and $\mathcal{C}_{>m} = \phi_+^m\phi_-^m(\mathcal{C}_{>m})$ since $\phi_+^m\phi_-^m$ is identical on $\mathsf{Par}_{>m}$. Hence $\mathcal{C}_{>m} \subseteq \phi_+^m(\mathcal{C})$. $\square$

Corollary E.2. *A PI having modulus m satisfies* (2.3).

Proof. Since a PI satisfies (E.1) we can apply Proposition E.1. The first condition in (2.3) is obvious from (E.1). □

E.2. **Linked partition ideals.** Recall Definition 2.5.

Definition E.3 ([3, Definition 11])**.** A subset $\mathcal{C}$ of Par is a *linked partition ideal* (*LPI* for short) if there exists $m \in \mathbb{Z}_{>0}$ for which

(L1) $\mathcal{C}$ is a PI having modulus m,
(L2) $|\mathcal{C}_{\leq m}| < \infty$,
(L3) there exist $L\colon \mathcal{C}_{\leq m} \longrightarrow 2^{\mathcal{C}_{\leq m}}$ and $s\colon \mathcal{C}_{\leq m} \longrightarrow \mathbb{Z}_{>0}$ such that $\mathrm{id}^{\bullet}_{\mathcal{C}_{\leq m}}(S) = \mathcal{C}$, where S is the set of $(\lambda^{(i)})_{i\geq 1} \in \mathsf{Seq}(\mathcal{C}_{\leq m}, \mathrm{id}_{\mathcal{C}_{\leq m}})$ with

$$\forall j \geq 1,\ \lambda^{(j+1)} = \cdots = \lambda^{(j+s(\lambda^{(j)})-1)} = \emptyset \text{ and } \lambda^{(j+s(\lambda^{(j)}))} \in L(\lambda^{(j)}).$$

Proposition E.4. *An LPI $\mathcal{C}$ is regularly linked (see Definition 3.8).*

Proof. If $\mathcal{C} = \emptyset$ then we can take m, I, π, X' arbitrarily and $X = I$ in Definition 3.8. Assume $\mathcal{C} \neq \emptyset$ and let m be as in Definition E.3. Since $\mathcal{C}$ is a PI, we have $\emptyset \in \mathcal{C}$ and in particular $\mathcal{C}_{\leq m} \neq \emptyset$. Write $I := \mathcal{C}_{\leq m}$ and $\pi := \mathrm{id}_{\mathcal{C}_{\leq m}}$. Then the set $S \subseteq \mathsf{Seq}(I, \pi)$ in (L3) can be written as

$$\mathsf{avoid}(\mathsf{Seq}(I, \pi), X, \emptyset) = S,$$

where

$$X := \bigcup_{\lambda \in I} \{\lambda\}\Big(I^{s(\lambda)} \setminus \big(\{\underbrace{\emptyset \cdots \emptyset}_{s(\lambda)-1}\}L(\lambda)\big)\Big) \quad (\subseteq I^{+}),$$

which is finite and hence is a regular language over I (recall (3)). □

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