ISOPERIMETRIC INTERPRETATION FOR THE RENORMALIZED VOLUME OF CONVEX CO-COMPACT HYPERBOLIC 3-MANIFOLDS

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Abstract. We reinterpret renormalized volume as the asymptotic difference of the isoperimetric profiles for convex co-compact hyperbolic 3-manifolds. By similar techniques we also prove a sharp Minkowski inequality for horospherically convex sets in $\mathbb{H}^3$. Finally, we include the classification of stable constant mean curvature surfaces in regions bounded by two geodesic planes in $\mathbb{H}^3$ or in cyclic quotients of $\mathbb{H}^3$.

1. Introduction

Renormalized volume is a geometric quantity motivated by the AdS/CFT correspondence and the calculation of gravitational action (see Witten [38]). For a convex co-compact hyperbolic 3-manifold, one can use the duality done by Epstein [13] between conformal metrics at infinity ($\partial_\infty \mathbb{H}^3$) and immersions into $\mathbb{H}^3$ to construct a submanifold $N \subset M$ so that $V_R(M)$ is equal to the volume of N minus half of the integral of the mean curvature of ∂N. In this setup, renormalized volume can be characterized as the antiderivative of a 1-form defined by the Schwarzian derivative of the uniformization maps at infinity. See Section 3 for precise definition of renormalized volume addressed here and further discussion. This present work reinterprets renormalized volume of an acylindrical hyperbolic manifold M as the asymptotic difference between the isoperimetric profile of M and the isoperimetric profile of the representative with totally geodesic convex core in the deformation space of M. An *isoperimetric profile* of a given manifold is a function that for each number $V > 0$ assigns the optimal perimeter of a region of volume V. There are two profiles we consider: $I_M(V)$ and $J_M(V)$, the outermost isoperimetric profile and the isoperimetric profile, respectively. While $J_M(V)$ is taken without restriction, $I_M(V)$ is defined taking optimal perimeters between regions containing all compact minimal surfaces in M. This requires the competitor for $I_M(V)$ to contain a region with minimal boundary, which we call the *outermost region* of M.

Theorem 1.1. *Let M be a convex co-compact hyperbolic 3-manifold that is either acylindrical or quasifuchsian and let Ω_0 be its outermost region. If I_M, I_{TG} denote the outermost isoperimetric profiles of M and M_{TG} (the quasiconformal deformation of M with Fuchsian ends) respectively, then*

$$V_R(M) - |\Omega_0| = \frac{1}{2} \lim_{V \to \infty} \big(I_{TG}(V) - I_M(V) \big).$$

See Section 2 for precise definitions and properties of I_M, J_M, M_{TG} and Ω_0.
It is well known that convex co-compact hyperbolic 3-manifolds have near infinity a foliation with CMC leaves [25]. In Theorem 5.2 we show that such foliation is in fact isoperimetric. Using this

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fact we prove that the renormalized volume is determined by the geometric data of that CMC foliation:

Theorem 1.2. *Let M be a convex co-compact hyperbolic 3-manifold. If J_M is the isoperimetric profile of M, then*

$$V_R(M) + \frac{\pi}{2}\chi(\partial M) = \lim_{V\to\infty}\left(V - \frac{1}{2}J_M(V) + \pi\chi(\partial M)\log\sqrt{\frac{2\,J_M(V)}{\pi|\chi(\partial M)|}}\right).$$

The volume-comparison interpretation of renormalized volume is on a similar spirit to the notions defined for asymptotically hyperbolic manifolds, see for instance the one studied by Brendle and Chodosh [5] (see also [22]). The work [22] also proves a sharp isoperimetric comparison result for AH 3-manifolds with scalar curvature $R \geq -6$. We prove the following comparison for the isoperimetric profile I_{TG} of the convex-co compact hyperbolic 3-manifold with totally geodesic convex core:

Theorem 1.3. *Let M be a convex co-compact hyperbolic 3-manifold that is either acylindrical or quasifuchsian and Ω_0 its outermost region. If $V_R(M) > |\Omega_0|$, then $I_M(V) < I_{TG}(V)$ for every volume $V \geq 0$.*

If M contains only one minimal surface, then $I_M(V) < I_{TG}(V)$ for every volume $V \geq 0$.

The authors in [22] use inverse mean curvature flow to produce a candidate profile that obeys the comparison inequality; the positivity of the Hawking mass plays an important role in their proof. Theorem 1.3 is based on the analytic features of the isoperimetric profile. One fundamental difference is the change of sign of the Euler characteristic of the boundary, and subsequently of the Hawking mass, since this changes the convexity/concavity properties we have at our disposal. On one side, we notice that the Hawking mass, being negative in our setting, allows the difference of profiles to have a positive local maximum. On the other hand, it forbids a negative local minimum. Hence one of our challenges is to successfully use these inverted properties for the Hawking mass in order to conclude a profile camparison. In some perspective Theorem 1.3 reflects a result proved in [6, 37] concerning the infimum of the renormalized volume as a functional in the moduli space of convex co-compact hyperbolic 3-manifolds.

We apply the duality [13], relating horospherically convex sets in $\mathbb{H}^3$ with conformal metrics at infinity, and the renormalized Ricci flow to also prove a sharp Minkowski type inequality that characterizes geodesic balls in $\mathbb{H}^3$.

Theorem 1.4. *If Σ is an horospherically convex surface bounding a compact region $\Omega \subset \mathbb{H}^3$, then*

$$\int_\Sigma H\,d\Sigma - 2|\Omega| \;\geq\; 2\pi\log\left(1 + \frac{1}{2\pi}\int_\Sigma (H+1)d\Sigma\right)$$

with equality if, and only if, Σ is a geodesic sphere.

This inequality is not new. It was proved by J. Natário [30] via an asymptotic analysis at infinity for the normal flow and an application of the isoperimetric inequality. The rigidity statement is not obtained in [30].

Outline. The article is organized as follows. In Section 2 we define an isoperimetric problem for manifolds with outermost minimal surfaces. We present basic properties and then describe the behavior of the Hawking mass function on convex co-compact hyperbolic manifolds that will be needed in Section 6. In Section 3 we introduce renormalized volume, following the correspondence between equidistant foliations and metrics at the conformal infinity. In Sections 4 and 5 we describe how boundaries of isoperimetric regions foliate the ends of convex co-compact hyperbolic manifolds. Section 6 is where we prove one of the main results by relating renormalized volume to the asymptotic behavior of isoperimetric profiles. In Section 7 we apply similar techniques to prove a Minkowski inequality. In the Appendix we describe the isoperimetric profile for the region between two geodesic planes and for cyclic quotients of $\mathbb{H}^3$. Although it is a parallel discussion from the article's main content, this was the starting of the authors collaboration and we include for completeness sake. We show that geodesic spheres and tubes about geodesics are the only stable constant mean curvature surfaces.

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2. Isoperimetric regions in convex co-compact hyperbolic 3-manifolds

A complete hyperbolic 3-manifold M is called *convex co-compact* if there exist a compact convex set U such that the exponential map from ∂U to the conformal infinity ∂M is a diffeomorphism. Each component of ∂M is assumed to be incompressible in M and has negative Euler characteristic. In particular, ∂M is always disconnected. Moreover, we say M is *acylindrical* if any map $(S^1 \times [0,1], S^1 \times \{0,1\}) \to (M, \partial M)$ is homotopic relative to $S^1 \times \{0,1\}$ into ∂M.

An important class of convex co-compact hyperbolic 3-manifold are the quasi-Fuchsian metrics. A Fuchsian 3-manifold is simply $\Sigma \times \mathbb{R}$ with the metric $g = dr^2 + \cosh^2(r) g_\Sigma$ where Σ is a closed orientable hyperbolic surface of genus $g > 1$. One can observe that the surface $\Sigma \times \{0\}$ is totally geodesic and that each boundary at infinity has a conformal structure that can be identified to Σ (after reversing orientation for one of the ends). A quasi-Fuchsian 3-manifold is a convex co-compact manifold of the form $\mathbb{H}^3/G$, where G is a quasi-conformal deformation of a Fuchsian group. This corresponds to having potentially distinct conformal structures at the boundaries, with equality (after reversing the orientation of one component) if and only if the manifold is Fuchsian. If M is a convex co-compact manifold with ∂M incompressible, then the covering associated to each boundary is quasi-Fuchsian.

For M acylindrical hyperbolic 3-manifold, there exists unique hyperbolic structure so that each end is Fuchsian (see for instance [26, Corollary 4.3]). This will be the model hyperbolic structure in M we will use to compare isoperimetric profiles for M acylindrical (we will use Fuchsian structure as models for quasi-Fuchsian structures). To see the existence of the acylindrical hyperbolic structure with Fuchsian ends (following [26]) one uses that the map from the conformal boundary of M to the opposite side of the boundary coverings (also referred as *skinning map*) is a contraction, so one can use a fixed point argument to find the unique hyperbolic structure in M where each end has matching conformal boundaries, hence Fuchsian. We will denote such manifold by M_{TG}. This notation comes from the fact that the convex core (smallest convex set containing all closet geodesics) of M_{TG} has totally geodesic boundary.

Let Ω_0 be the largest volume compact region in M with the property that the boundary $\partial\Omega_0$ is a minimal surface which is homologous and diffeomorphic ∂M. Such region exist since ∂M is incompressible, convex (see [27]). Observe as well that there exists a unique minimal surface configuration when the metric has totally Fuchsian ends. In particular, $\partial\Omega_0$ is connected in each end of M. We call the surface $\partial\Omega_0$ the *outermost minimal surface.* If M contains only one minimal surface, then Ω_0 has zero volume.

We will introduce the related isoperimetric problems which are relevant to the discussion in the next sections. For each $V > 0$ we consider

$$\begin{aligned}\mathcal{R}^1_V &= \{\Omega : \Omega \subset M \text{ is a compact region with } \Omega_0 \subset \Omega \text{ and } vol(\Omega - \Omega_0) = V\}\\ \mathcal{R}^2_V &= \{\Omega : \Omega \subset M \text{ is a compact region with and } vol(\Omega) = V\}\end{aligned}$$

and let

$$I_M(V) = \inf\{area(\partial\Omega) : \ \Omega \in \mathcal{R}^1_V\} \quad \text{and} \quad J_M(V) = \inf\{area(\partial\Omega) : \ \Omega \in \mathcal{R}^2_V\}. \tag{2.1}$$

In order to distiguish both notations, we will refer to $I_M(V)$ as the *outermost isoperimetric profile* of M, while we refer to $J_M(V)$ as the usual *isoperimetric profile.*
For any sequence of points x_i diverging to infinity the injective radius $inj_{x_i}M$ becomes unbounded. In particular, M has bounded geometry. For each volume V there exists a minimizing sequence for the isoperimetric problem (2.1) that does not drift off to infinity. For the regularity near the outermost boundary $\partial\Omega_0$ see [12, Section 4]. In the following lemma we summarize well known existence and regularity results [1, 28, 34].

Lemma 2.1. *For every $V > 0$, there exists an isoperimetric region $\Omega \in \mathcal{R}^i_V$, $i = 1, 2$, with $vol(\Omega) = V$. The surface $\Gamma_\Omega = \partial\Omega$ is a volume preserving stable constant mean curvature surface. Namely,*

$$\int_\Gamma |\nabla f|^2 - (-2 + |A|^2) f^2 \, d\Gamma \geq 0 \quad \textit{whenever} \quad \int_\Gamma f \, d\Gamma = 0.$$

Let us discuss the analytical properties of the isoperimetric profile. Let Ω be an isoperimetric region in M such that $vol(\Omega) = V$. Let I denote both I_M and J_M in what follows. We first note that in our setting I is absolutely continuous and twice differentiable almost everywhere, see [16]. In particular, the function $I(V)$ has left and right derivatives $I'_-(V)$ and $I'_+(V)$ and if H is the mean curvature (average of principal curvatures) of $\Gamma = \partial\Omega$ in the direction of the inward unit vector, then

$$(I)'_+(v) \leq 2H \leq (I)'_-(v). \tag{2.2}$$

The second derivative exists weakly in the sense of comparison functions. More precisely, we say $f'' \leq h$ weakly at x_0 if there exists a smooth function g such that $f \leq g$, $f(x_0) = g(x_0)$, and $g'' \leq h$. In this sense we have

$$I(v)^2 \, I''(v) + \int_\Gamma \left(Ric_g(N,N) + |A|^2\right) d\Gamma \leq \ 0. \tag{2.3}$$

Let us sketch the proof of (2.2) and (2.3):

Let Γ_v be the variation $\Gamma_t = exp_\Gamma(tN)$ of Γ re-parametrized in terms of the enclosed volume $v(t)$ and let $\phi_0(t)$ (resp. $\phi_V(v)$) be the area of Γ_t (resp. Γ_v). Note that $\phi_V(v) \geq I(v)$ and $\phi_V(V) = I(V)$. By the first variation formula for the area and volume we have $\phi'_0(0) = 2\,H\,|\Gamma|$ and $v'(0) = |\Gamma|$ respectively. Since $\phi'_0(t) = \phi'_V(v)v'(t)$, we conclude that $\phi'_V(v(0)) = 2H$ and also that $v'(0)^2\phi''_V(v(0)) = \phi''_0(0) - \phi'_V(v(0))v''(0)$. On the other hand, it follows from the second derivative of area for general variations that

$$\begin{aligned}\phi''_0(0) &= -\int_\Gamma 1\,L\,1\,d\Gamma + 2H\,v''(0)\\ &= -\int_\Gamma \left(Ric_g(N,N) + |A|^2\right) d\Gamma + 2H\,v''(0).\end{aligned}$$

Hence, in the sense of comparison functions, (2.3) follows from:

$$\phi_V(v(0))^2\, \phi_V''(v(0)) + \int_\Gamma \big(Ric_g(N,N) + |A|^2\big)\, d\Gamma = 0. \tag{2.4}$$

One of the main reasons to require that the outermost minimal core is contained in each candidate region is so that the profile I_M is monotone non-decreasing. We quickly justify this in the following lemma.

Lemma 2.2. *The isoperimetric profile $I_M(V)$ is a strictly increasing function.*

Proof. Since I_M satisfies (2.2) it is enough to show $I_M'(V) > 0$ at volumes V for which $I_M'(V)$ exists. Arguing by contradiction, we assume that $I_M'(V) < 0$. It follows from (2.2) that $M - \Omega$ is mean convex. On the other hand, the equidistant surfaces $\Sigma_t = exp_{\partial M}(tN)$ bounds convex regions for sufficiently large t. By Meeks-Simon-Yau [27], there exists a compact minimal surface strictly between $\partial\Omega_0$ and Σ_t which is homologous to ∂M. This contradicts the assumption that Ω_0 is the outermost region. □

Let us now discard spherical and tori components as boundaries of isoperimetric regions for the outermost isoperimetric problem I_M. This will be useful to study the sign and monotonicity for the Hawking mass below.

Lemma 2.3. *If Ω is an isoperimetric region with respect to I_M, then each connected component of $\partial\Omega$ has genus at least two. Moreover, $\chi\big(\partial\Omega\big) \leq \chi(\partial M)$.*

Proof. If $\partial\Omega$ contains a spherical or a torus component, then such component is either a geodesic sphere by Hopf's Theorem or a tube about a geodesic by Ritoré-Ros [32]. In particular, $\partial\Omega$ is disconnected and its mean curvature satisfies $H > 1$, see equation (8.2) in Subsection 6.3. Let us choose two components Γ_1 and Γ_2. We consider the function $f \in C^\infty(\Gamma)$, such that $f = |\Gamma_2|$ on Γ_1 and $f = -|\Gamma_1|$ on Γ_2 and $f = 0$ otherwise. Hence, $\int_\Gamma f d_\Gamma = 0$. By the stability inequality we have

$$\begin{aligned} 0 &\leq -\int_\Gamma fLf\, d_\Gamma = -|\Gamma_2|^2 \int_{\Gamma_1} (-2 + |A|^2) - |\Gamma_1|^2 \int_{\Gamma_2} (-2 + |A|^2) \\ &\leq -2\Big(|\Gamma_1|^2|\Gamma_2| + |\Gamma_2|^2|\Gamma_1|\Big)(-1 + H^2) < 0. \end{aligned}$$

With this contradiction, we conclude that each component of $\partial\Omega$ has genus at least two. If we minimize area in the isotopy class of $\partial\Omega$ with respect to the model metric where the outermost region is totally geodesic following [27], then we obtain the unique connected minimal surface S with multiplicity m such that $g(\partial M) = mg(S) \leq g(\partial\Omega)$. Since each component of $\partial\Omega$ has genus greater than one, we obtain $\chi(\partial M) \geq \chi(\partial\Omega)$. □

In order to study the behavior of the isoperimetric profile, it is convenient to define the Hawking mass function, see [8, 22, 24].

Definition 2.4. The Hawking mass function $m_H : (0, , \infty) \to \mathbb{R}$ is defined in terms of the outermost isoperimetric profile I_M:

$$m_H(V) = \sqrt{I_M(V)}\Big(2\pi\chi(\partial M) + I_M(V) - \frac{1}{4} I_M'^{+}(V)^2\, I_M(V)\Big). \tag{2.5}$$

The quantity $m_H(V)$ is monotone and has a sign as shown below. Theses facts will be useful in the description and comparison for the isoperimetric profile in Section 4.

Lemma 2.5. *The Hawking mass function $m_H(V)$ is a monotone non-decreasing function.*

Proof. Let Ω be an isoperimetric region of volume V and $\Gamma = \partial\Omega$. Recall the Gauss equation

$$Ric(N) + |A|^2 = \frac{R}{2} - K + 3H^2 + \frac{|\mathring{A}|^2}{2}. \tag{2.6}$$

Using (2.6) we can rewrite inequality (2.4) as follows

$$\begin{aligned} I''_M(V) \leq \phi''_V(V) &\leq \frac{1}{\phi_V(V)^2}\int_\Gamma \left(3 + K - 3H^2 - \frac{|\mathring{A}|^2}{2}\right) d_\Gamma \\ &\leq \frac{\sum_{i=1}^{l} 4\pi(1 - g(\Gamma_i))}{I(V)^2} + \frac{3}{I(V)} - \frac{3I'(V)^2}{4I(V)}. \end{aligned} \tag{2.7}$$

The last inequality follows from the Gauss-Bonnet Theorem applied to each component of Γ. Next we compute m'_H in the distribution sense following similar computation in [11, Lemma 3]:

For this we need the quotient difference operator $\Delta_\delta f(V) = \delta^{-1}\big(f(V+\delta) - f(v)\big)$, $\delta \neq 0$. Let $\varphi \in C^1_0(0,\infty)$, then

$$\begin{aligned} -\int \varphi'\, m_H &= -\lim_{\delta\to 0}\int \varphi'\sqrt{I_M}\left(2\pi\chi(\partial M) + I_M - \frac{1}{4}(\Delta_\delta I_M)^2\, I_M\right) \\ -\int \varphi'\, m_H &= \lim_{\delta\to 0}\int \varphi\Delta_{-\delta}\left(\sqrt{I_M}\big(2\pi\chi(\partial M) + I_M - \frac{1}{4}(\Delta_\delta I_M)^2\, I_M\big)\right) \\ -\int \varphi'\, m_H &= \lim_{\delta\to 0}\int \varphi\left(I'_M - \frac{1}{2}I_M\, I'_M\, \Delta_{-\delta}(\Delta_\delta I_M) - \frac{(I'_M)^2}{4}I'_M\right)\sqrt{I_M} \\ &\quad + \varphi\left(\pi\chi(\partial M) + \frac{I_M}{2} - \frac{(I'_M)^2}{8}\, I_M\right)\frac{I'_M}{\sqrt{I_M}} \\ &= \int \varphi\frac{I'_M\, I_M^{\frac{3}{2}}}{2}\left(\frac{(I'_M)^2}{4I_M} + \frac{2\pi\chi(\partial M)}{I^2}\right) - \lim_{\delta\to 0}\int \varphi\frac{I'_M\, I_M^{\frac{3}{2}}}{2}\Delta_{-\delta}(\Delta_\delta I_M). \end{aligned}$$

we used in above formulas that $I'^{+}_M = I'^{-}_M$ except at possibly countable many points. Using that $\limsup_{\delta\to 0}\Delta_{-\delta}\Delta_\delta I_M \leq \phi''_V(V)$ and applying inequality 2.7, we obtain:

$$-\int \varphi'\, m'_H \geq \int \varphi\frac{I'_M(V)\, I_M^{\frac{3}{2}}(V)}{I_M^2(V)}\left(\pi\chi(\partial M) - \sum_{i=1}^{l_V} 2\pi(1 - g(\Gamma_i))\right) \geq 0,$$

by Lemma 2.2 and Lemma 2.3. Hence, $m'_H(V) \geq 0$ in the distribution sense. It follows from (2.2) that at every discontinuity point of $I'_M(V)$, the Hawking mass $m_H(V)$ jumps up. Therefore, $m_H(V)$ is monotone non-decreasing. □

Remark 2.6. If M_e is a connected component of $M - \Omega_0$, then the isoperimetric problem for this end is defined as follows. In the class $\mathcal{R}^0_V = \{\Omega : \Omega \subset M_e$ is a compact region with $\partial M_e \subset \Omega$ and $vol(\Omega) = V\}$, we set the isoperimetric profile I_e as

$$I_e(V) = \inf\{area(\partial\Omega) :\ \Omega \in \mathcal{R}^0_V\}\ -\ area(\partial M_e)$$

The results in this section and their proofs extended naturally to the profile $I_e(V)$. The only relevant change needed is in the definition of the Hawking mass $m_H(V)$.

3. Renormalized volume

The convex co-compact hyperbolic 3-manifolds (M, g) are particular examples of conformally compact manifolds. Namely, there exists on a compact manifold with boundary $\overline{M}$ a defining function x (i.e., $x > 0$ on M, $x = 0$ on ∂M, and $dx \neq 0$ on ∂M) such that $M = int(\overline{M})$ and the conformal metric $\overline{g} = x^2 g$ extends smoothly to the boundary ∂M. The restriction of $\overline{g}$ to ∂M defines a well defined conformal structure $[\partial M]$ on ∂M which we will refer to as the conformal infinity (or conformal boundary) of M.

By works of Epstein [13] and Graham [17], for each metric h on the conformal infinite $[\partial M]$ there exist a unique defining function x defined in a collar neighborhood of ∂M with the special property that the level sets $\{x = r\}$ yield an equidistant foliation of the ends of M by convex sets. Moreover,

$$g = \frac{1}{x^2}\left(dx^2 + h_0 + h_2 x^2 + h_3 x^3 + \dots\right),$$

where h_i are tensors in ∂M. We now look for the quantity $Vol(\{x > \varepsilon\})$ as ε approaches zero. One can check that

$$Vol(\{x > \varepsilon\}) = c_0\, \varepsilon^2 - L \log(\varepsilon) + V + o(1).$$

The constants c_0 e L depend only on the metric $h \in [\partial M]$ uniquely associated to the equidistant foliation $\{x = \varepsilon\}$. The quantity V is the renormalized volume associated to the metric h.

In the context of hyperbolic metrics, the work [23] provided a renormalization procedure for computing V in terms of the geometric data associated to equidistant foliation was proposed. Let M_r, $r \geq 0$, be an equidistant foliation at infinity of M by convex sets. This foliation induces a Riemannian metric at the conformal infinity ∂M as follows:

$$h = \lim_{r\to\infty} e^{-2r} g_r, \tag{3.1}$$

where g_r is the induced Riemannian metric on ∂M_r. Using the unique correspondence between metrics in the conformal class $[\partial M]$ and equidistant foliations as above [17], we define the *W-volume* with respect to h as

$$W(M, h) := \mathrm{vol}(M_r) - \frac{1}{2}\int_{\partial M_r} H\, da + r\pi\mathcal{X}(\partial M). \tag{3.2}$$

One can show that the value on the right hand side is independent of r, see [23]. Among all metrics $h \in [\partial M]$ of fixed area, say $A_0 = Area(\partial M, h)$, the W-volume is maximized by the unique hyperbolic metric h_0 having that area, see Proposition 8.2 in the Appendix. This motivates the following:

Definition 3.1. The Renormalized volume of M is defined as $V_R(M) = W(M, h_{hyp})$ where h_{hyp} is the hyperbolic metric of Gauss curvature -4.

Example 3.2 (Renormalized volume of Fuchsian manifolds)**.** Let (M, g) be a Fuchsian 3-manifold. Recall that $M = \Sigma \times \mathbb{R}$ where Σ is a closed orientable surface of genus g and

$$g = dr^2 + \cosh^2(r) g_\Sigma$$

where g_Σ is the hyperbolic metric (Gauss curvature -1) on Σ. With the warped metric g one can check that the slices Σ_r form a global equidistant foliation of $\Sigma \times \mathbb{R}$ by totally umbilical surfaces. A simple computation gives that the metric at infinity via the limit procedure (3.1) is the hyperbolic metric h_{hyp} of Gauss curvature -4. Because the W-volume $W(\Sigma \times \mathbb{R}, g_{hyp})$ does not depend of the choice of slice Σ_r, we obtain that $V_r(M) = 0$ by choosing $M_r = \Sigma \times [-r, r]$ with r sufficiently closed to zero in (3.2).

The following theorem studies the minimum of V_R along a moduli class of hyperbolic 3-manifolds.

Theorem 3.3 (Bridgeman-Brock-Bromberg [6], Vargas Pallete [36])**.** *Let M be a convex co-compact hyperbolic 3-manifold. Then $V_R(M) \geq \frac{v_3}{2}\|DM\|$, where v_3 is the volume of a regular ideal tetrahedra, $\|.\|$ denotes the Gromov norm of a 3-manifold, and DM is the double of M. Moreover, equality $V_R(M) = \frac{v_3}{2}\|DM\|$ occurs if and only if the convex core of M has totally geodesic boundary, in which case M needs to be either acylindrical with Fuchsian ends or Fuchsian.*

4. Foliation at infinity for hyperbolic ends

As discussed above, for any metric h in the conformal boundary ∂M there exists the unique special defining function x whose level sets form a equidistant foliation of the ends of M and we write

$$g = \frac{1}{x^2}\left(dx^2 + h_0 + h_2x^2 + h_3x^3 + \dots\right),$$

where h_j are tensors on ∂M. It is known that the tensor h_2 is formally undetermined but it satisfies $tr_{h_0}h_2 = K$ where K is the scalar curvature of h_0. The mean curvature of Σ_r satisfies

$$H_r = 1 + \frac{tr_{h_0}h_2}{4}\, r^2 + o(r^2).$$

In particular, if h_0 has negative Gauss curvature, which is always possible since $\chi(\partial M) < 0$ at each component, then H_r is almost constant and $H_r < 1$ when r is sufficiently close to 0.

It is possible to perturb these level sets to to have constant mean curvature. A direct application of the main result of [25] gives:

Theorem 4.1 (Mazzeo-Pacard [25])**.** *Let M be a convex co-compact hyperbolic 3-manifold with $\chi(\partial M) < 0$. There exists a compact subset K such that $M - K$ has an unique monotone increasing foliation by constant mean curvature surfaces.*

Given that the mean curvature is monotone increasing along the foliation, we obtain the following corollary concerning CMC surfaces in the foliated region.

Corollary 4.2. *If Σ is an connected constant mean curvature surface embedded in $M - K$ and homologous to $\partial(M - K)$, then Σ is a leaf of the canonical foliation.*

Proof. Note that Σ is tangent to the outer radius leaf $\partial\Omega_A$ and the inner radius leaf $\partial\Omega_B$ of the canonical foliation with mean curvature vectors having the same orientations. By the Maximum Principle, $H_A \leq H \leq H_B$. Since the foliaton's mean curvature is strictly increasing for large volumes, we have that $\Sigma = \partial\Omega_A = \partial\Omega_B$. □

The next two results strengthen the variational characterization of each leaf of the canonical foliation. Corollary 4.4 will be used in the proof of Theorem 5.2.

Proposition 4.3. *There exist an embedded strongly stable constant mean curvature closed surface Σ with mean curvature H for each $H \in (0, 1)$.*

Proof. We consider sets Ω such that $\partial\Omega = \Gamma$ is homologous to ∂M. In this class, let $\mathcal{F}_H$ be the brane action functional $\mathcal{F}_H(\Omega) = |\Omega| - \frac{1}{2H}|\Gamma|$, see [3]. Recall that the volume element of M satisfies $dM = d\Lambda$ for some $n-1$ form Λ. In particular, $\mathcal{F}_H$ is a functional of Γ and $\mathcal{F}_H(\Gamma) = \int_\Gamma \Lambda - \frac{1}{2H}A(\Gamma)$ where $A(\Gamma)$ is the area of Γ. Note the form of $\mathcal{F}_H$ is to agree with the inequality $V - \frac{1}{2}A > 0$ for large volumes V, see Lemma 5.1 below. We remark that we do not have a sign for $\mathcal{F}_H$. If Ω_r

is such that Γ_r is a connected CMC leaf of the canonical foliation far out at infinity, then Γ_r is a barrier for $\mathcal{F}_H$ in that sense that

$$\delta\mathcal{F}_H(\Omega_r)(fN) = \int_\Gamma (-1 + \frac{H_r}{H}) f \, d\Gamma > 0$$

where H_r is the mean curvature of Γ_r with respect to the inward unit normal vector N of Ω_r and f is a positive function. In other words, the functional $\mathcal{F}_H$ decreases as one approaches ∂M (in fact, $\lim_{r\to\infty} \mathcal{F}_r(\Omega_r) = -\infty$.) We consider the maximization problem $\sup\{\mathcal{F}_H(\Omega)\}$, Ω as described above, inside the compact set Ω_r. By maximization arguments using barriers, see [27], there exist a maximizer Ω_H for the functional $\mathcal{F}_H$ which does not intersect $\partial\Omega_r$. We need to show that Ω is non-trivial, i.e., Ω has non-zero volume: this only happens if $\Omega = S$ where S us a surface homologous to one component of ∂M. In particular, it must be the homological non-trivial surface of least area in M. On the other hand these surfaces are also barrier for maximizing the functional $\mathcal{F}_H$. Indeed,

$$\delta\mathcal{F}_H(\Omega)(fN) = \int_\Gamma (1 - \frac{0}{H}) f \, dS > 0$$

where f is a positive function on S and N is the outward unit normal vector to Ω. Moreover, the first and second variation for Γ_{Ω_H}, see [3], implies it has constant mean curvature H and

$$\delta^2\mathcal{F}(\Omega_H)(f,f) = -\frac{1}{2H}\int_\Gamma |\nabla f|^2 + (2 - |A|^2) f^2 \, d_{\Gamma_H} \le 0, \tag{4.1}$$

for all function $f \in C^\infty(\Gamma)$. In other words, the surface Γ_H is a strongly stable constant mean curvature surface. □

This construction can be made in each end of $M - \Omega_0$, where Ω_0 is the outermost region.

Corollary 4.4. *There are no embedded cmc surface $\Sigma \subset M - \Omega_0$ homologous to ∂M with constant mean curvature $H = 1$.*

Proof. Let Ω be the region bounded by Σ. Following Proposition 4.3, we can use Σ as a barrier for maximizing the brane action functional $\mathcal{F}_{H=1}$ inside Ω among competitors homologous to Σ. Note that the maximizer Γ is non-trivial by the same argument in Proposition 4.3 (in other words, Γ must enclose some volume.) Consequently, we obtain that either Σ is strongly stable (in the sense (4.1)) or we can replace Σ by a strongly stable compact surface Γ with constant mean curvature $H = 1$. Since $H = 1$, the Jacobi operator becomes $L = \Delta + |\mathring{A}|^2$. Therefore, applying the test function $f = 1$ in the stability inequality (4.1) yields a contradiction. □

5. Uniqueness of isoperimetric regions

We start with a simple lemma providing an useful inequality between the enclosed volume and area of equidistant surfaces.

Lemma 5.1. *Let Ω be a strongly convex set in a convex co-compact hyperbolic 3-manifold M such that $\partial\Omega$ is homologous to ∂M. Then $Area(\partial\Omega_r) < 2\,vol(\Omega_r)$, where $\Omega_r = \{x \in M : d(x,\Omega) \le r\}$ and r sufficiently large.*

Proof. Let $\Sigma_r = \partial\Omega_r$. The induced metric on the level set Σ_r is

$$g_r = g_0\big(\cosh(r)I + \sinh(r)A\,,\, \cosh(r)I + \sinh(r)A\big)$$

where I and A is the identity and the second fundamental form of Σ_0. The metric is well define since Σ_0 is strongly convex. Hence,

$$\begin{aligned} |\Sigma_r| &= \int_\Sigma \det\big(\cosh(r)I + \sinh(r)A\big)\, d\Sigma \\ &= \int_\Sigma \Big(\cosh(2r) + \sinh(2r)\, H + K\big(\frac{\cosh(2r)-1}{2}\big)\Big)\, d\Sigma. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\Omega_r| &= |\Omega| + \int_0^r \int_\Sigma \Big(\cosh(2\rho) + \sinh(2\rho)\, H + K\big(\frac{\cosh(2\rho)-1}{2}\big)\Big)\, d\Sigma\, d\rho \\ &= |\Omega| + \frac{1}{2}\int_\Sigma \Big(\sinh(2r) + \big(\cosh(2r)-1\big)\, H + K\,\frac{\sinh(2r)-2r}{2}\Big) d\Sigma \end{aligned}$$

Hence,

$$\begin{aligned} 2|\Omega_r| - |\Sigma_r| = 2|\Omega| \;&+\; \int_\Sigma (\cosh(2r) - \sinh(2r))(H-1)\, d\Sigma - \int_\Sigma H\, d\Sigma \\ &+\; \frac{\pi}{2}\mathcal{X}(\Sigma)(\sinh(2r) - \cosh(2r)) - \pi\mathcal{X}(\Sigma)(2r-1). \end{aligned}$$

Therefore, $\lim_{r\to\infty}(2|\Omega_r| - |\Sigma_r|) = +\infty$ since $\mathcal{X}(\Sigma) < 0$. □

We are now ready to show that leaves of the cmc foliation at infinity are in fact isoperimetric, for leaves sufficiently deep into the end. We will use this result in the next section for the isoperimetric comparison results and characterization of the Renormalized Volume.

Theorem 5.2. *Let M be a convex co-compact hyperbolic 3-manifold and $\{\Sigma_H\}_{H\in\mathbb{R}}$ the cmc foliation at infinity parametrized by constant mean curvature H. If H is sufficiently close to one, then Σ_H is uniquely isoperimetric (with respect to either I_M or J_M) for the volume it encloses in M.*

Proof. Let Ω_{V_i} be an isoperimetric region of volume V_i. Let us study first the case $\partial\Omega_{V_i} \cap K \neq \emptyset$, where $K \subset M$ is some fixed compact set and $\{V_i\}$ is a sequence of volumes satisfying $V_i \to \infty$. By compactness theorem for isoperimetric surfaces, $\partial\Omega_{V_i}$ converges in the graphical sense and with multiplicity one to a non-compact stable constant mean curvature surface Σ_∞ in M. Note that the mean curvature of Σ_∞ satisfies $H \geq 1$. Indeed, by the maximum principle the mean curvature of $\partial\Omega_{V_i}$ is greater than the mean curvature H_{R_i} of Σ_{R_i}, where Σ_{R_i} is a leaf of the cmc foliation that encloses Ω_{V_i} and it is tangent to $\partial\Omega_{V_i}$. On the other hand, $\lim_{i\to\infty} H_{R_i} = 1$. It follows from $H \geq 1$ that the operator $P = \Delta + |\mathring{A}|^2$ satisfies

$$0 \leq -\int_{\Sigma_\infty} fP(f)\, d_{\Sigma_\infty} \quad \text{for every} \quad \int_{\Sigma_\infty} f\, d_{\Sigma_\infty} = 0.$$

By the monotonicity formula, Σ_∞ is either compact or has infinite area. We will deal with the latter case first. In that case, Σ_∞ is also conformally equivalent to a closed Riemann surface with finite points removed by Fisher-Colbrie [15]. Theorem 1.6 in Da Silveira [10] applied to the operator $P = \Delta + |\mathring{A}|^2$ implies that $\mathring{A} \equiv 0$ and Σ_∞ is totally umbilical. As Σ_∞ is a non-compact surface with mean curvature $H \geq 1$, then it has to be an embedded oriented horosphere H_0 in M. Similarly, by the strong compactness properties for sequences of isoperimetric surfaces, any sequence of basepoints in $\partial\Omega_{V_i}$ will locally converge (after a subsequence) to a horosphere in M or in $\mathbb{H}^3$ under the Cheeger-Gromov convergence for manifolds. And since horospheres are strictly convex, we have that $\partial\Omega_{V_i}$ will be locally convex for i large enough.

In general, isoperimetric regions might not be connected. Nevertheless, each connected component is also isoperimetric. By our assumptions, at least one component will have large volume and passing through a compact region, say Ω_{V_i}. Following the prove of [35, Theorem 3.3], we can use the local convexity of $\partial\Omega_{V_i}$ and the normal geodesic flow to see that the covering of M associated to Ω_{V_i} (with the induced hyperbolic metric) is obtained by gluing Ω_{V_i} to $\partial\Omega_{V_i} \times \mathbb{R}_0^+$, where $\partial\Omega_{V_i} \times \mathbb{R}_0^+$ has a metric so that $\partial\Omega_{V_i} \times \{t\}$ is locally convex for any $t \geq 0$. We denote this covering by $\tilde{M}_{V_i}$, and it follows from the normal geodesic flow construction that the surfaces $\partial\Omega_{V_i} \times \{t\}$ are equidistant to one another. From this description of $\tilde{M}_{V_i}$ it follows that Ω_{V_i} is *geodesically convex* in $\tilde{M}_{V_i}$, meaning that any geodesic in $\tilde{M}_{V_i}$ with endpoints in Ω_{V_i} is contained in Ω_{V_i}. Since $\tilde{M}_{V_i}$ is the covering associated to $i_* : \pi_1(\Omega_{V_i}) \to \pi_1(M)$, it follows that any homotopically trivial geodesic segment $\gamma : ([0,1],\{0,1\}) \to (M, \Omega_{V_i})$ (i.e. homotopic into Ω_{V_i} relative to Ω_{V_i}) has image in Ω_{V_i}. We will use this to show in fact that $i_* : \pi_1(\Omega_{V_i}) \to \pi_1(M)$ has trivial image and hence $\tilde{M}_{V_i}$ is $\mathbb{H}^3$.

Assume that $g \in \pi_1(M)$ is a nontrivial element in the image of $i_* : \pi_1(\Omega_{V_i}) \to \pi_1(M)$. Hence we have a homotopically trivial geodesic segment γ of (M, Ω_{V_i}) which, for i sufficiently large, lies close to the orthogeodesic of $\Sigma_\infty = H_0$ associated to g. But then such homotopically trivial geodesic segment γ will not be contained in Ω_{V_i}, which is a contradiction.

Since Ω_{V_i} lifts to $\mathbb{H}^3$, then it must be a hyperbolic geodesic ball. This is impossible for the outermost isoperimetric profile I_M, as Ω_{V_i} should contain Ω_0. For the isoperimetric profile J_M, we saw on Lemma 5.1 we can construct sets Ω_r so that $\lim_{r\to\infty}(2|\Omega_r|-|\partial\Omega_r|) = +\infty$. Such competitors will beat hyperbolic geodesic balls for sufficiently large volumes, so we have a contradiction for the profile J_M as well.

Now we deal with the case when Σ_∞ is compact. In this case $\partial\Omega_V$ is disconnected which implies by the proof of Lemma 2.3 that $H_{V_i} < 1$ for large i. Hence, Σ_∞ has mean curvature $H = 1$. This contradicts Corollary 4.4 and, hence, $\partial\Omega_V$ diverges to infinite as $V \to \infty$.

Let us assume now that the outermost region $\Omega_0 \subset \Omega_{V_i}$ and that $\partial\Omega_{V_i}$ is drifting towards infinity. Since $\partial\Omega_{V_i}$ is homologous to $\partial\Omega_0$, Corollary 4.2 implies at least one component of $\partial\Omega_{V_i}$ in a fixed end is a leaf of the foliation. But this implies Ω_{V_i} is connected since other components would have larger mean curvature. Hence, each component of $\partial\Omega_{V_i}$ is a leaf of the canonical foliation. The argument so far shows that the leafs of the canonical foliation are uniquely isoperimetric with respect to the outermost isoperimetric profile I_M (the characterization also holds at any end of M). Next we assume that $\Omega_0 \subset \Omega_V^c$. Let us show that such configuration is not isoperimetric. Note that all components must be drifting off to infinity. By previous argument, we can deduce that $|\partial\Omega_{V_i}| > |\Sigma_{R_i}| - |\Sigma_{R_j}|$, where the enclosed volume of the leaf Σ_{R_j} is V_j and the enclosed volume of Σ_{R_i} is $V_i + V_j$. By the Fundamental Theorem of Calculus, $|\Sigma_{R_i}| - |\Sigma_{R_j}| = 2H_{s_0}\, V_i$, where $s_0 \in (R_j, R_i)$. In particular, $|\Sigma_{R_i}| - |\Sigma_{R_j}| > 2H_{R_j}\, V_i$. Hence,

$$|\partial\Omega_{V_i}| > |\Sigma_{R_j}|\, \frac{2\, H_{R_j}\, V_i}{|\Sigma_{R_j}|}.$$

Now choose R_j such that $V_j = V_i$. One can check that the profile $I_M(V)$ associated to the foliation Σ_H satisfies $2H_V = I_M'(V) > \frac{I_M(V)}{V}$ for large V. Indeed, this is equivalent showing that $\ln\left(\frac{I_M(V)}{V}\right)$ is an increasing function for large volume V. For this just notice that $\lim_{V\to\infty} \frac{I_M(V)}{V} = 2$, that $|\Sigma_{R_k}| < |\Gamma_r|$, where Γ_r is equidistant to a fixed cmc leaf Γ_0 and encloses the same volume as Σ_{R_k}, and that $|\Gamma_r| < 2\, V_k$ by Lemma 5.1. Therefore, $|\partial\Omega_{V_i}| > |\Sigma_{R_j}|$. □

Remark 5.3. Alternatively, we could have worked with the minimizers Σ_H of $\mathcal{F}_H$, so that if we follow the same steps in the proof above we would obtain that for H sufficiently close to 1 that Σ_H is equal to the cmc leafs of the canonical foliation. Standard comparison implies that minimizers of

$\mathcal{F}$ are isoperimetric with respect to the outermost profile I_M. As a result, the leafs of the canonical foliation are strongly stable in the sense (4.1).

Now we strengthen Lemma 2.5 by showing that the Hawking mass is non-positive.

Lemma 5.4. *The Hawking mass satisfies* $m_H(V) < 0$ *for every* V *unless the ends of* M *are Fuchsian where* $m_H(V) \equiv 0$.

Proof. If V is sufficiently large, then Γ_V is connected on each end by Theorem 5.2 and have the topology of ∂M. Note also that by Theorem 5.2, the isoperimetric profile $I_M(V)$ is differentiable for V sufficiently large. Integrating the Gauss equation and applying the Gauss-Bonnet Theorem on each component of Γ_V gives

$$\begin{aligned} 2\pi\chi(\partial M) &= \int_{\Gamma_V} K_V \, d\Gamma = \int_{\Gamma_V} (-1 + \det(A_V)) \, d_\Gamma = \int_{\Gamma_V} (-1 + H_V^2 - \frac{1}{2}|\mathring{A}_V|^2 \, d\Gamma \\ &\leq \int_{\Gamma_V} (-1 + H_V^2) \, d\Gamma = -I_M(V) + \frac{1}{4} I_M'(V)^2 I_M(V). \end{aligned}$$

Therefore, $m_H(V) \leq 0$. Since $m_H(V)$ is non-decreasing, we conclude that $m_H(V) \leq 0$ for all volumes. If $m_H(V) = 0$, then Γ_V is totally umbilical and each end of M is Fuchsian. ☐

6. Isoperimetric profile comparison

Since our goal is to relate V_R to the isoperimetic profile, we start with an expression of V_R in terms of volume and area only. We achieve this by using the fact that for an equidistant foliation M_r, the mean curvature H approaches 1 exponentially on r. We recall that the equidistant foliation M_r corresponds to the hyperbolic metric at infinity of Gauss curvature -4.

Proposition 6.1.

$$\lim_{r\to\infty} Vol(M_r) - \frac{1}{2}Area(\partial M_r) + \pi\chi(\partial M) \log \sqrt{\frac{2Area(\partial M_r)}{\pi|\chi(\partial M)|}} = V_R(M) + \frac{\pi}{2}\chi(\partial M) \tag{6.1}$$

Proof. Let us first prove that

$$\lim_{r\to\infty} \int_{\partial M_r} (H-1)\, da = \pi\chi(\partial M). \tag{6.2}$$

Fix s and denote the metric at ∂M_s by g, shape operator A and its principal curvatures by $k_{1,2}$. Then the metric and principal curvatures at ∂M_r are given by (see [35])

$$g_r(u,v) = g(\cosh(r-s)u + \sinh(r-s)Au, \cosh(r-s)v + \sinh(r-s)Av)$$

$$k_{1,2}^r = \frac{\sinh(r-s) + \cosh(r-s)k_{1,2}}{\cosh(r-s) + \sinh(r-s)k_{1,2}}$$

In particular, the volume element at ∂M_r is given by

$$da_r = (\cosh(r-s) + \sinh(r-s)k_1)(\cosh(r-s) + \sinh(r-s)k_2)da_s$$

So then

$$\int_{\partial M_r} (H_r - 1)da_r = \frac{1}{4}\int_{\Sigma_s} (-1+k_1)(1+k_2) + (-1+k_2)(1+k_1)da_s + O(e^{s-r})$$

Applying the Gauss equation and Gauss-Bonnet Theorem:

$$\lim_{r\to\infty}\int_{\partial M_r}(H_r-1)da_r=\frac{1}{2}\int_{\partial M_s}(-1+k_1k_2)da_s=\pi\chi(\partial M_s)$$

The equidistant foliation is parametrized such that $\lim_{r\to\infty} e^{-2r}g_r = h_0$, where h_0 is the hyperbolic metric at infinity of Gauss curvature -4. In particular,

$$\lim_{r\to\infty} e^{-2r}Area(\partial M_r)=\frac{\pi}{2}|\chi(\partial M)|$$

This is equivalent to

$$\lim_{r\to\infty}\left(r-\log\sqrt{\frac{2Area(\partial M_r)}{\pi\chi(\partial M)}}\right)=0 \tag{6.3}$$

If we add and subtract both terms $\frac{1}{2}$ Area(∂M_r) and $\log\sqrt{\frac{2Area(\partial M_r)}{\pi\chi(\partial M)}}$ in the renormalized volume formula (3.2), the proposition will follow from an application of (6.2) and (6.3). □

6.1. **Isoperimetric comparison.** Recall that Ω_0 denotes the outermost region of the convex co-compact hyperbolic manifold M. In what follows we will show an isoperimetric comparison for the outermost isoperimetric profile I_M

$$I_M(V)=\inf\{\text{area}(\partial\Omega)\,:\,\Omega_0\subset\Omega\quad\text{and}\quad\text{vol}(\Omega-\Omega_0)=V\}$$

with that of the hyperbolic metric with totally geodesic convex core. Notice that I_M takes into consideration all ends of M. The standard isoperimetric profile of M, i.e. infimum of boundary area among all regions of a given volume, is denoted by J_M. .

Theorem 6.2. *Let M be a hyperbolic 3-manifold that is either acylindrical or quasi Fuchsian. Let $I_M(V)$ be the isoperimetric profile of M with respect to its outermost region Ω_0 and I_{TG} (resp. J_{TG}) the outermost isoperimetric profile (resp. standard isoperimetric profile) of the hyperbolic metric in the deformation space of M that has totally geodesic convex core Ω_{TG}. Then*

$$\lim_{V\to\infty}\Big(J_{TG}(V)-J_M(V)\Big)>0\quad\textit{and}\quad I_M(V)<I_{TG}(V+|\Omega_0|-|\Omega_{TG}|).$$

In particular, if M contains only one closed minimal surface, then $I_M(V) < I_{TG}(V)$ for every volume $V > 0$.

One should observe by the special case in the main theorem in [2] that $|\Omega_0| - |\Omega_{TG}| \geq 0$.

Proof. Let U_r be the equidistant foliation of M at infinity by convex sets inducing the hyperbolic metric h_0 of Gauss curvature -4 at the conformal infinity ∂M via

$$h_0=\lim_{r\to\infty}e^{-2r}g_r,$$

where g_r is the induced Riemannian metric on ∂U_r.

Proposition 6.1 and an isoperimetric comparison yields

$$\begin{aligned} V_R(M) &= \lim_{r\to\infty}\left(\text{vol}(U_r)-\frac{1}{2}\text{area}(\partial U_r)+\pi\chi(\partial M)\log\sqrt{\frac{2\,Area(\partial U_r)}{\pi|\chi(\partial M)|}}\right)-\frac{\pi}{2}\chi(\partial M)\\ &\leq \lim_{V\to\infty}\left(V-\frac{1}{2}J_M(V)+\pi\chi(\partial M)\log\sqrt{\frac{2\,J_M(V)}{\pi|\chi(\partial M)|}}\right)-\frac{\pi}{2}\chi(\partial M) \qquad (6.4)\end{aligned}$$

where we are using that the function $x-\pi\chi(\partial M)\log\sqrt{x}$ is increasing for large values of x.

By Theorem 3.3 and Proposition 6.1, the Renormalized volume $V_R(M)$ as a functional in the moduli space of convex co-compact 3-manifolds attains its global minimum at the geodesic class of the conformal infinity ∂M. Using this comparison and

$$V_R(g_{TG}) = \lim_{V\to\infty}\left(V - \frac{1}{2}J_{TG}(V) + \pi\chi(\partial M)\log\sqrt{\frac{2\,J_{TG}(V)}{\pi\chi(\partial M)}}\right) - \frac{\pi}{2}\chi(\partial M)$$

we obtain that

$$\lim_{V\to\infty}\left((J_{TG}(V) - \pi\chi(\partial M)\log\sqrt{J_{TG}(V)}) - (J_M(V) - \pi\chi(\partial M)\log\sqrt{J_M(V)})\right) \geq V_R(M) - V_R(M_{TG}).$$

Since $x - \pi\chi(\partial M)\log\sqrt{x}$ is increasing for large x, we have that

$$\lim_{V\to\infty}\left((J_{TG}(V) - J_M(V)\right) > 0.$$

Using for large volume V that $I_M(V) = J_M(V + |\Omega_0|)$ and $I_{TG}(V) = J_{TG}(V + |\Omega_{TG}|)$, we conclude after a relabeling that the isoperimetric profile of M with respect to the outermost region Ω_0 satisfies

$$\lim_{V\to\infty}\left(I_{TG}(V + |\Omega_0| - |\Omega_{TG}|) - I_M(V)\right) > 0.$$

As observed earlier, we have as a special case of the main theorem in [2] that $|\Omega_0| - |\Omega_{TG}| \geq 0$. In particular, we obtain

$$I_M(0) \leq I_{TG}(0) < I_{TG}(0 - |\Omega_0| - |\Omega_{TG}|).$$

The first inequality follows from the Gauss-Bonnet Theorem and the second from the monotonicity of the isoperimetric profile. On the other hand, we have by definition of the Hawking mass function the following equation for I_M and I_{TG}

$$-2\pi\mathcal{X}(M) - \frac{I_M'^{+}(V)^2 I_M(V)}{4} + I_M(V) = \frac{m_H(V)}{\sqrt{I_M(V)}}.$$

$$2\pi\mathcal{X}(M) + \frac{I_{TG}'(V + |\Omega_0| - |\Omega_{TG}|)^2 I_{TG}(V + |\Omega_0| - |\Omega_{TG}|)}{4} - I_{TG}(V + |\Omega_0| - |\Omega_{TG}|) = 0.$$

By adding these two equations, we obtain

$$I_M(V) - I_{TG}(V + |\Omega_0| - |\Omega_{TG}|) - \frac{(I_M'^{+})^2 I_M(V) - (I_{TG}'^{+})^2 I_{TG}(V + |\Omega_0)| - |\Omega_{TG}|)}{4} = \frac{m_H(V)}{\sqrt{I_M(V)}}.$$

Therefore, the function $f(V) = I_M(V) - I_{TG}(V + |\Omega_0| - |\Omega_{TG}|)$ does not have a positive local maximum point since that would imply $m_H(V) > 0$, contradicting Lemma 5.4. While the isoperimetric profile I_{TG} is smooth, the graph of the isoperimetric profile I_M can have corners. To deal with this possibility at the local maximum point V_0, we replace I_M locally near V_0 by the area profile function $f(V)$ associated to the equidistant deformation of the isoperimetric surface Γ_{V_0}. □

Theorem 6.2 brings a connection between V_R and $J_M(V)$. The following proposition shows that V_R is in fact determined by the asymptotic of the isoperimetric profile $J_M(V)$.

Theorem 6.3 (Theorem 1.2)**.** *Let M be a convex co-compact hyperbolic 3-manifold. Then*

$$V_R(M) + \frac{\pi}{2}\chi(\partial M) = \lim_{V\to\infty}\left(V - \frac{1}{2}J_M(V) + \pi\chi(\partial M)\log\sqrt{\frac{2\,J_M(V)}{\pi|\chi(\partial M)|}}\right).$$
$$= \lim_{V\to\infty}\left(V + |\Omega_0| - \frac{1}{2}I_M(V) + \pi\chi(\partial M)\log\sqrt{\frac{2\,I_M(V)}{\pi|\chi(\partial M)|}}\right).$$

Proof. Let Ω_V be the isoperimetric region of volume V and Ω_r its equidistant enlargement region at distance r. Following Lemma 5.1, we have that

$$\lim_{r\to\infty} e^{-2r}|\partial\Omega_r| = \frac{1}{2}|\partial\Omega_V| + \frac{1}{2}\int_{\partial\Omega_V} H\,d\Sigma + \frac{\pi}{2}\mathcal{X}(\partial M) = \beta \tag{6.5}$$

By the computations in Lemma 5.1 and the variational characterization of the renormalized volume in the space of metrics conformal to ∂M having area β, we obtain

$$|\Omega_V| - \frac{1}{2}\int_{\partial\Omega_V} H\,d\Sigma + \frac{\pi}{2}\mathcal{X}(\partial M) = \lim_{r\to\infty}\Big(|\Omega_r| - \frac{1}{2}|\partial\Omega_r| + r\pi\mathcal{X}(\partial M)\Big) \le V_R(M,\beta) + \frac{\pi}{2}\chi(\partial M).$$

The notation $V_R(M,\beta)$ reflects the constraint on the area of the conformal metric at infinity. On the other hand, one has

$$V_R(M,\beta) = V_R\big(M, \frac{\pi}{2}|\chi(\partial M)|\big) - \frac{\pi}{2}\chi(\partial M)\log\left(\frac{2\beta}{\pi|\chi(\partial M)|}\right).$$

Therefore,

$$|\Omega_V| - \frac{1}{2}|\partial\Omega_V| - \frac{1}{2}\int_{\partial\Omega_V}(H-1)\,d\Sigma + \frac{\pi}{2}\mathcal{X}(\partial M) + \frac{\pi}{2}\chi(\partial M)\log\left(\frac{2\beta}{\pi|\chi(\partial M)|}\right) \le V_R\big(M, \frac{\pi}{2}|\chi(\partial M)|\big) + \frac{\pi}{2}\chi(\partial M). \tag{6.6}$$

Substituting equation (6.5) into (6.6), we obtain that

$$\begin{aligned} V - \frac{1}{2}J_M(V) \quad &+ \quad \frac{\pi}{2}\mathcal{X}(\partial M)\log\left(\frac{2\,J_M(V)}{\pi|\chi(\partial M)|} + \frac{1}{|\pi\chi(\partial M)|}\int_{\partial\Omega_V}(H-1)\,da - 1\right) \\ &\le \quad V_R\big(M, \frac{\pi}{2}|\chi(\partial M)|\big) + \frac{1}{2}\int_{\partial\Omega_V}(H-1)\,da - \frac{\pi}{2}\chi(\partial M) + \frac{\pi}{2}\chi(\partial M). \end{aligned} \tag{6.7}$$

Since the Hawking mass $m_H(V)$ is monotone increasing by Lemma 2.5 and bounded by Lemma 5.4, we have

$$\lim_{V\to\infty}\int_{\Sigma_V}\big(H-1\big)d\Sigma = \pi\mathcal{X}(\partial M).$$

By Taking the limit as $V\to\infty$ in both sides of (6.7) and applying this identity, we obtain

$$\lim_{V\to\infty}\left(V - \frac{1}{2}J_M(V) + \pi\chi(\partial M)\log\sqrt{\frac{2\,J_M(V)}{\pi|\chi(\partial M)|}}\right) \le V_R(M) + \frac{\pi}{2}\chi(\partial M).$$

The reverse inequality, obtained in (6.4), is a straightforward isoperimetric comparison for the terms in the expression of $V_R(M, h_0)$. □

Remark 6.4. The following isoperimetric inequality for strongly convex regions $\Omega \subset M$ such that $\partial\Omega \in [\partial M]$ follows from the definition of the renormalized isoperimetric constant $V_R(M)$ and Proposition 6.3:

$$|\Omega| - \frac{1}{2}\int_{\partial\Omega} H\, da \leq V_R(M) + \frac{\pi|\chi(\partial M)|}{2}\log\left(\frac{1}{\pi|\chi(\partial M)|}\int_{\partial\Omega} H\, da + \frac{|\partial\Omega| - \pi|\chi(\partial M)|}{\pi|\chi(\partial M)|}\right).$$

Corollary 6.5 (Theorem 1.1)**.** *Let M be a convex co-compact 3-manifold that is either acylindrical or quasifuchsian, and let Ω_0 be its outermost region. If I_M, I_{TG} denote the outermost isoperimetric profiles of M and M_{TG} (the quasiconformal deformation of M with Fuchsian ends) respectively, then*

$$\frac{1}{2}\lim_{V\to\infty}\big(I_{TG}(V) - I_M(V)\big) = V_R(M) - |\Omega_0|.$$

Equivalently,

$$V_R(M) = \frac{1}{2}\lim_{V\to\infty}\big(J_{TG}(V) - J_M(V)\big).$$

Proof. Using Theorem 6.3 for M with $V = V' + |\Omega_0|$ and for M_{TG} with $V = V' + |\Omega_{TG}|$ and then relabeling V' by V we obtain

$$V_R(M) - V_R(M_{TG}) = \lim_{V\to\infty}\left(|\Omega_0| - |\Omega_{TG}| + \frac{1}{2}(I_{TG}(V) - I_M(V)) + \pi\chi(\partial M)\log\sqrt{\frac{I_{TG}(V)}{I_M(V)}}\right) \tag{6.8}$$

Since $V_R(M_{TG}) = |\Omega_{TG}|$ then this reduces to

$$V_R(M) = |\Omega_0| + \lim_{V\to\infty}\left(\frac{1}{2}(I_{TG}(V) - I_M(V)) + \pi\chi(\partial M)(\log\sqrt{I_{TG}(V)} - \log\sqrt{I_M(V)})\right)$$

This in particular implies that $\limsup_{V\to\infty}|I_{TG}(V) - I_M(V)| < +\infty$, since otherwise the left-side of (6.8) would not converge. We also have $\lim_{V\to\infty} I_{TG}(V) = \lim_{V\to\infty} I_M(V) = +\infty$, then $\lim_{V\to\infty}\log(\sqrt{I_{TG}(V)}) - \log(\sqrt{I_M(V)}) = 0$. Hence we have

$$V_R(M) = |\Omega_0| + \lim_{V\to\infty}\frac{1}{2}(I_{TG}(V) - I_M(V)) \tag{6.9}$$

which finishes the proof. □

Remark 6.6. Observe that in Corollary 6.5 M_{TG} is not uniquely defined if M is quasifuchsian, but since I_{TG} is independent from the Fuchsian model considered, we proceed as normal. This remark remains valid for later statements.

Corollary 6.7 (Theorem 1.3)**.** *Let M be a convex co-compact hyperbolic 3-manifold that is either acylindrical or quasifuchsian and Ω_0 its outermost region. If $V_R(M) > |\Omega_0|$, then $I_M(V) < I_{TG}(V)$ for every volume $V \geq 0$.*

Proof. If $V_R(M) > |\Omega_0|$, then $\lim_{V\to\infty} I_M(V) - I_{TG}(V) > 0$. We also have $I_{TG}(0) - I_M(0) > 0$ by an application of the Gauss-Bonnet Theorem. The proof of Theorem 6.2 applies verbatim when $I_{TG}(V + |\Omega_0|)$ is replaced by $I_{TG}(V)$ to show that $I_{TG} - I_M$ cannot have a non-negative local maximum. Therefore, $I_M(V) < I_{TG}(V)$ for every V. □

Remark 6.8. In order to see that Corollary 6.7 is not an empty statement, observe that Theorem 3.3 ([6],[36]) implies $V_R(M) > 0$ when M is quasi Fuchsian (but not Fuchsian) and contains a unique compact minimal surface. This last condition (unique minimal surface) is non-empty, since it contains in its interior the set of *almost-Fuchsian* manifolds (studied by Uhlenbeck [35]) , which are manifolds that contain a minimal surface with principal curvatures $|k_{1,2}| < 1$.

Question 6.9. *It is natural to ask what is the reach of Corollary 6.7, particularly since the comparison between $V_R(M)$ and Ω_0 answers if I_M ever surpasses I_{TG} or not. For which M convex co-compact hyperbolic 3-manifold we have that $V_R(M) > |\Omega_0|$?*

Remark 6.10. Results up until this Section hold for relatively acylindrical hyperbolic manifolds $(M^3, S \subseteq \partial M)$ when one considers the definitions of isoperimetric profile and renormalized volume for the set of ends $S \subseteq \partial M$ (for example one can consider the case of a Bers slice). Similarly, one can write the results for convex co-compact manifolds with incompressible boundary (but not necessarily acylindrical of quasifuchsian), with the caviat that the isoperimetric model I_{TG} is replaced by a disjoint union of Fuchsian ends. If the boundary is not incompressible, the existence of an outermost minimal core needs to be assumed for the results to work. We focused in the acylindrical/quasifuchsian case to illustrate the isoperimetric features of renormalized volume while avoiding the technical setup required to establish a more general statement.

7. Minkowski inequality for Horospherically convex sets

In this section we are interested in studying the geometric objects in $\mathbb{H}^3$ that, according to Epstein's description, correspond to conformal metrics in $\partial_\infty \mathbb{H}^3 = \mathbb{S}^2$.

Definition 7.1 ([14])**.** A hypersurface Σ in $\mathbb{H}^3$ is said to be *horospherically convex* (h-convex) if at every point Σ lies locally on one side of its tangential horosphere.

An oriented surface $\Sigma \subset \mathbb{H}^3$ is horospherically convex at $p \in \Sigma$ if, and only if, all the principal curvatures of Σ at p verify simultaneously $\kappa_i(p) < -1$ or $\kappa_i(p) > -1$, see [14]. Here we assuming that the orientation of Σ coincide with the outward orientation of the horosphere tangent to Σ at p. This definition is more general than geodesic convexity. If Σ is closed and lies in the concave side of the tangential horosphere at each point for example, then the principal curvatures satisfy $\kappa_i > -1$.

If Σ is horospherically convex bounding a compact region Ω, then the outward exponential map $\psi : \Sigma \times [0, \infty) \to \mathbb{H}^3 - \Omega$, given by $\psi(p, r) = exp(p, rN)$, is a diffeomorphism. The family $\Sigma_r = \psi(\Sigma, r)$ are called the normal flow of Σ . The foliation $\{\Sigma_r\}$ induces a metric h at the conformal boundary $\partial_\infty \mathbb{H}^3 = \mathbb{S}^2$ by

$$h = \lim_{r \to \infty} e^{-2r} g_r$$

where g_r is the first fundamental form of the parallel surface Σ_r. More importantly, for each metric h in the conformal class at infinity there exist an unique equidistant foliation such that the associated metric at infinity is h. This well known result has its root in the work of C. Epstein [13] through the envelopes of horospheres construction which we briefly describe:

Consider the Poincare ball model for $\mathbb{H}^3$. For any $x \in \mathbb{H}^3$ define its *visual metric* v_x as a metric in the conformal $\partial_\infty \mathbb{H}^3 = S^2$ by

(1) v_0 is the canonical round metric in S^2.
(2) If γ is an isometry of $\mathbb{H}^3$ so that $\gamma(x) = 0$, then $v_x = \gamma^*(v_0)$

Observe that v_x is well defined because isometries of $\mathbb{H}^3$ fixing the origin 0 are the isometries of the round metric in S^2. It is not a hard exercise to see that for $x \in \mathbb{H}^3, b \in S^2$ the set $H(b, v_x(b)) := \{y \in \mathbb{H}^3 \,|\, v_y(b) = v_x(b)\}$ is the horosphere tangent at b passing through x.

Epstein [13] shows that given a C^1 conformal metric ρ in a open set $U \subseteq S^2$, there exists a unique continuous map $Y_\rho : U \to T^1\mathbb{H}^3$ so that $Y(b)$ is a unit normal vector to $H(b, \rho(b))$ oriented towards b. This map satisfies that for t constant, $Y_{e^t\rho}$ is equal to Y_ρ after translating t units by the geodesic flow. Moreover, if ρ is smooth and we fix some compact $K \subset U$, we have that $Y_{e^t}|_K$ is an embedding for sufficiently large t. Hence if we take a convex co-compact manifold M and a metric h in the conformal boundary $\partial_\infty M$, the maps $Y_{e^th}(\partial M)$ are well defined, and define a equidistant foliation of the ends of M when t is sufficiently large.

In the rest of this section we will see that the W-volume is maximized among metrics of fixed area by constant curvature metrics. This result can also be established from the work of Osgood, Phillips and Sarnak [31], where they show that constant curvature metrics in S^2 maximize $\log(\mathrm{Det}(\Delta))$, the logarithm of the determinant of the Laplace-Beltrami operator. The result for W-volume follows because its first variation formula is a constant multiple (3π in fact) of the first variation formula of $\log(\mathrm{Det}(\Delta))$. Our proof is based on the renormalized Ricci flow for surfaces.

7.1. **Minkowski-type inequality.** The following Minkowski-type inequality is obtained by comparing the quantity $|\Omega| - \frac{1}{2}\int_{\partial\Omega} H\, da$ for a given compact region with horospherically convex boundary with that of a geodesic ball via the Renormalized Ricci flow for conformal metrics in $\partial_\infty \mathbb{H}^3 = S^2$ (as done in [31, Section 3, Theorem 2.A], [37, Section 4] for $\chi(\Sigma) \leq 0$).

Theorem 7.2. *If Σ is an horospherically convex surface bounding a compact region $\Omega \subset \mathbb{H}^3$, then*

$$\int_\Sigma H\, d\Sigma - 2|\Omega| \;\geq\; 2\pi \log\left(1 + \frac{1}{2\pi}\int_\Sigma (H+1) d\Sigma\right)$$

with equality if, and only if, Σ is a geodesic sphere.

The result without the rigidity statement was also obtained by J. Natário [30].

Proof. Given a horospherically convex domain $\Omega \subset \mathbb{H}^3$, we consider the W-volume functional

$$W(\Omega) = |\Omega| - \frac{1}{2}\int_\Sigma H\, d\Sigma.$$

By mean of the correspondence between equidistant foliation and metrics at infinity, one can prove that the function $W(\Omega)$ depends only on the metric $h \in [\partial\mathbb{H}^3]$. The first variation for W for conformal deformations of h was computed in [23]:

$$\delta W(\hat{h}) = \frac{1}{4}\int_{\mathbb{S}^2} \delta K(\hat{h})\, dvol_h,$$

where K is the Gauss curvature of $(\mathbb{S}^2, h)$. In other words, the Ricci flow is the gradient-like flow for the functional W. In particular, if $\partial_t h_t = (\frac{8\pi}{area(h)} - 2K)h_t$ is the Renormalized Ricci Flow that keeps the area of h constant, then

$$\delta W(h) = \frac{1}{4}\int_{\mathbb{S}^2}\left(\Delta K + K\left(2K - \frac{8\pi}{area(h)}\right)\right) dvol_h = \frac{1}{2}\int_{\mathbb{S}^2} K^2\, dvol_h - \frac{1}{2area(h)}\left(\int_{\mathbb{S}^2} K\, dvol_h\right)^2.$$

It follows from the Cauchy-Schwarz inequality that $\delta W(h) \geq 0$. By the strong convergence results for the Renormalized Ricci flow [21], we conclude that the W-functional has a global maximum among conformal metrics of a fixed area in the conformal boundary $[\partial M]$ at constant curvature metrics. These round metrics correspond to the normal flow of geodesic spheres in $\mathbb{H}^3$.

If h is the conformal metric at infinity for the equidistant foliation associated to Ω, then Lemma 5.1 implies that

$$area(h) = \frac{1}{2}|\Sigma| + \frac{1}{2}\int_\Sigma H\, d\Sigma + \pi.$$

Therefore, if B_r is a geodesic ball such that $|\partial B_r| + \int_{\partial B_r} H_r = |\Sigma| + \int_\Sigma H\, d\Sigma = 4\pi\lambda$, then

$$\int_\Sigma H\, d\Sigma - 2|\Omega| \geq \int_{\partial B_r} H_r\, dS_r - 2|B_r|.$$

Note that $H_r = \frac{\cosh(r)}{\sinh(r)}$, $|\partial B_r| = 4\pi \sinh^2(r)$, and $|B_r| = \pi\sinh(2r) - 2\pi r$. Using these formulas we obtain

$$\int_{\partial B_r} H_r\, dS_r - 2|B_r| = 4\pi r$$

and $\lambda = \cosh(r)\sinh(r) + \sinh^2(r)$. From this last equality we deduce that $r = \sinh^{-1}\left(\frac{\lambda}{\sqrt{1+2\lambda}}\right)$. Since $\sinh^{-1}(x) = \log\left(\sqrt{1+x^2}+x\right)$, we conclude that $r = \frac{1}{2}\log(1+2\lambda)$. Therefore,

$$\int_\Sigma H\, d\Sigma - 2|\Omega| \geq 2\pi \log\left(1 + \frac{1}{2\pi}\int_\Sigma \left(H+1\right) d\Sigma\right),$$

with equality if, and only if, Σ is up to a rigid motion the geodesic sphere ∂B_r. □

Remark 7.3. For an horospherically convex surface surface Σ bounding a compact region $\Omega \subset \mathbb{H}^3$, there is another sharp Minkowski inequality [18]:

$$\int_\Sigma H\, d\Sigma \;\geq\; 2\pi\sqrt{\frac{|\Sigma|^2}{4\pi^2} + \frac{|\Sigma|}{\pi}} \tag{7.1}$$

Combining (7.1) with the inequality in Theorem 7.2, we obtain

$$\int_\Sigma H\, d\Sigma \;\geq\; 2|\Omega| + 2\pi\log\left(1 + \frac{|\Sigma|}{2\pi} + \sqrt{\frac{|\Sigma|^2}{4\pi^2} + \frac{|\Sigma|}{\pi}}\right), \tag{7.2}$$

with equality if, and only if, Σ is a geodesic sphere.

In contrast with Theorem 7.2, whose poof relies on 2-dimensional features of the renormalized volume, the proof of the Minkowski inequality (7.1) involves a mean curvature type flow and generalizes to higher dimensions [18, Theorem 6.1].

7.2. **Polyakov-type formula.** As stated in [19] we have the following Polyakov type formula for conformal metrics in the sphere

$$W(e^{2\omega}h_0) - W(h_0) = -\frac{1}{4}\int_{S^2} |\nabla\omega|^2_{h_0} + \mathrm{Scal}_{h_0}\omega \mathrm{dvol}_{h_0}. \tag{7.3}$$

This formula follows, as in the geometrically finite case, by applying the Fundamental Theorem of Calculus and the first variation formula for the W-volume to the 1-parameter family of metrics $h_t = e^{2t\omega}h_0$, see Proposition 8.2 in the Appendix.

As done in [33, Proposition 3.11] for the convex co-compact case with $\chi(\partial M) < 0$, we have the following monotonicity for the W-volume. In this case W-volume is monotone decreasing, contrary to [33, Proposition 3.11], which boils down to the signature of the Euler characteristic of the boundary.

Proposition 7.4. *Let h_0, h_1 be non-negatively curved conformal metrics on $\partial_\infty \mathbb{H}^3 = S^2$ so that $h_0 \leq h_1$ pointwise. Then*

$$W(h_0) \geq W(h_1).$$

Moreover, equality occurs if and only if $h_0 = h_1$ pointwise.

Proof. Define $\omega : S^2 \to \mathbb{R}$ so that $h_1 = e^{2\omega} h_0$. Since $h_0 \leq h_1$, then $\omega \geq 0$. Hence, we have $\mathrm{Scal}_{h_0}\omega \geq 0$ pointwise. By the Polyakov-type formula (7.3)

$$W(h_1) - W(h_0) = -\frac{1}{4}\int_{S^2} |\nabla\omega|^2_{h_0} + \mathrm{Scal}_{h_0}\omega \,\mathrm{dvol}_{h_0} \leq 0, \tag{7.4}$$

so the inequality follows.

For the equality, note that $\int_{S^2} |\nabla\omega|^2_{h_0}\mathrm{dvol}_{h_0} = \int_{S^2} \mathrm{Scal}_{h_0}\omega\mathrm{dvol}_{h_0} = 0$. Hence ω is a constant function, which by the Gauss-Bonnet yields $\int_{S^2} \mathrm{Scal}_{h_0}\omega\mathrm{dvol}_{h_0} = 8\pi\omega$. Therefore, $\omega = 0$ and $h_0 = h_1$ pointwise. □

Proposition 7.4 can be written in terms of horospherically convex spheres in $\mathbb{H}^3$.

Corollary 7.5. *Let Σ_0, Σ_1 be horospherically convex spheres in $\mathbb{H}^3$ bounding regions Ω_0, Ω_1 so that $\Omega_0 \subset \Omega_1$. If $Scal_{\Sigma_0} \geq 0$, then*

$$\int_{\Sigma_0} H d\Sigma_0 - 2|\Omega_0| \leq \int_{\Sigma_1} H d\Sigma_1 - 2|\Omega_1|,$$

where equality occurs if and only if Σ_0, Σ_1 are the same surface.

Proof. Let h_0, h_1 be the conformal metrics in $\partial_\infty\mathbb{H}^3 = S^2$ corresponding to Σ_0, Σ_1, respectively. Since $\Sigma_0 \subset \Omega_1$ then $h_0 \leq h_1$, because any outer-tangent horosphere to Σ_1 will not intersect Σ_0.

If $k_{1,2}(p)$ are the principal curvatures of Σ_0, then the scalar curvature at p^+ (point at infinity whose outer-tangent horosphere to Σ_0 is tangent at p) is given by $\frac{-1+k_1(p)k_2(p)}{(1+k_1(p))(1+k_2(p))}$. Hence $\mathrm{Scal}_{h_0}(p^+) \geq 0$ if and only if $(-1 + k_1(p)k_2(p)) \geq 0$, which by Gauss equation is equivalent to $\mathrm{Scal}_{\Sigma_0}(p) \geq 0$.

Hence we have met the conditions of Proposition 7.4, from which the result follows. □

Finally, let's observe how the Polyakov formula (7.3) relates to Theorem 7.2. Assume that h_0 is a conformal metric in $\partial_\infty\mathbb{H}^3 = S^2$ with constant scalar curvature $K > 0$ and take $\omega : S^2 \to \mathbb{R}$ so that $\int_{S^2}(e^{2\omega} - 1)\mathrm{dvol}_{h_0} = 0$. In other words, the conformal metric $h_1 = e^{2\omega}h_0$ and the constant curvature metric h_0 have the same area. As detailed in [9, Lemma7], this assumption implies that $\int_{S^2} \omega\mathrm{dvol}_{h_0} \leq 0$. The proof of Theorem 7.2 show for $h_1 = e^{2\omega}h_0$ that $W(h_1) \leq W(h_0)$ with equality if, and only if, h_1 has constant scalar curvature K. Note that unlike the convex co-compact case, this fact does not follow from (7.3). Consequently, the Polyakov formula (7.3) yields:

Corollary 7.6. *Let h_0 be the round metric in $\partial_\infty\mathbb{H}^3 = \mathbb{S}^2$ with constant Gauss curvature 1, and take $\omega : S^2 \to \mathbb{R}$ so that $\int_{S^2}(e^{2\omega} - 1)\,dvol_{h_0} = 0$. Then*

$$\left|\int_{\mathbb{S}^2} 2\omega \, dvol_{h_0}\right| \leq \int_{\mathbb{S}^2} |\nabla\omega|^2_{h_0} \, dvol_{h_0}$$

Moreover, equality occurs if and only if $h_1 = e^{2\omega}h_0$ has constant Gauss curvature 1, or equivalently, $e^{2\omega}$ is given by the derivative of a Möbius transformation.

Remark 7.7. A complete proof of this result with the line of reasoning mentioned above is done in [31, Section 2.3] using the $\log(\mathrm{Det}(\Delta))$ functional. As mentioned earlier, the first variation formula of $\log(\mathrm{Det}(\Delta))$ is a constant multiple of the first variation formula of the W-volume.

8. Appendix

8.1. **Polyakov formula for the W-volume.** In this section we prove the Polyakov formula from the first variation formula for the W-volume as a functional on the conformal boundary class at infinity.

Proposition 8.1. *The W-volume of a convex set Ω inside a convex co-compact hyperbolic 3-manifold M depends only on the metric at infinity h associated to Ω. Moreover, the first variation among conformal deformations is*

$$\delta W(\hat{h}) = \frac{1}{4}\int_{\partial M} \delta K(\hat{h})\, dh.$$

Proof. See Krasnov-Schlenker [23, Section 7]. □

Proposition 8.2. *The W-volume satisfies the Polyakov-type formula:*

$$W(e^{2\omega}h_0) - W(h_0) = -\frac{1}{4}\int_{\partial M} |\nabla\omega|^2_{h_0} + Scal_{h_0}\omega\, dvol_{h_0}.$$

Proof. Let $h_t = e^{2t\omega}h_0$ and $t \in [0,1]$. Applying the Fundamental Theorem of Calculus and Proposition 8.1, we obtain

$$\begin{aligned} W(e^{2\omega}h_0) - W(h_0) &= \int_0^1 \delta W(h_t)(2\omega\, h_t)\, dt = \frac{1}{4}\int_0^1\int_{\partial M} \delta K_{h_t}(2\omega\, h_t) dh_t\, dt \\ &= \frac{1}{4}\int_0^1\int_{\partial M} \big(- 2\omega e^{-2t\omega} K + 2\omega e^{-2t\omega}\Delta(t\omega) - e^{-2t\omega}\Delta\omega\big) dh_t\, dt \\ &= \frac{1}{4}\int_0^1\int_{\partial M} \big(- 2\omega\, K + 2t\,\omega\Delta\omega\big) dh_0\, dt \\ &= \frac{1}{4}\int_{\partial M} \big(- 2\omega\, K - |\nabla\omega|^2\big)\, dh_0. \end{aligned}$$

□

8.2. **Free boundary stability for cmc surfaces between two parallel geodesic planes.** In this section we study the isoperimetric problem for regions bounded by two geodesic planes in $\mathbb{H}^3$. In contrast with previous works for slabs, the Alexandrov reflection principle is not available in this setting. Instead, we exploit the reflections across the geodesic boundary and a result of Hsiang [21] to reduce the problem to rotationally invariant surfaces. With this simplification, we are able to extend a characterization result of Ritoré-Ros [32] to the free boundary case discussed here. As the classical result for slabs in $\mathbb{R}^3$, we will show that geodesic spheres and tubes about geodesics are the only solutions. The isoperimetric problem in cyclic quotients of $\mathbb{H}^3$ is also treated.
We start describing the basic cyclic actions in $\mathbb{H}^3$ by isometries and the main model for the slabs between geodesic planes. Let G_a and G_λ be cyclic subgroups of $Iso(\mathbb{H}^3)$ generated respectively by the isometries

$$\gamma_1(x,y,z) = (x + a_1, y + a_2, z) \quad \text{and} \quad \gamma_2(x,y,z) = (\lambda(x - x_0), \lambda(y - y_0), \lambda z).$$

Let M denote either the slab bounded by two vertical planes with Euclidean distance a or the slab bounded by two concentric hemispheres perpendicular to $\partial\mathbb{R}^3_+$ with Euclidean distance λ. Without loss of generality, we can assume that M is the fundamental domain of the cyclic group G_a or G_λ, respectively.

Theorem 8.3. *Let Σ be a free boundary stable constant mean curvature surface in M. If the distance between the geodesic planes of ∂M is positive, then Σ is either a geodesic hemisphere or a tube about a free boundary geodesic connecting the two planes. If the distance is zero, then Σ is a geodesic hemisphere.*

Proof. If $\partial\Sigma$ has only one component, then Σ is a geodesic hemisphere by Alexandrov's Theorem. Since ∂M is totally geodesic, we can apply a hyperbolic reflection across one boundary component to obtain another free boundary surface between hyperbolic planes. This compact extended surface can now be iterated infinity many times, using the isometry $\gamma_i \circ \gamma_i$, $i = 1, 2$, to obtain a complete cmc surface $\hat{\Sigma}$ properly embedded in $\mathbb{H}^3$ that is cylindrically bounded. By Hsiang [21], $\hat{\Sigma}$ is a rotationally invariant surface and so does Σ. The result will follow from Ritoré-Ros [32] as sketched below:

The Hopf holomorphic quadractic differential applied to (Σ, ds^2) implies that the metric $ds_0^2 = b|A^{2,0}|$, $b^2 = 4(-1 + H^2)$, is flat and conformal do the metric ds^2. Note by the maximum principle comparison with a flat tube that $H > 1$. If G is the cyclic subgroup generated by $\gamma_i \circ \gamma_i$, then $\hat{\Sigma}/G$ has genus one in $\mathbb{H}^3/G$ and, hence, is without umbilical points. In particular, ds_0^2 is a smooth metric. If we write $ds^2 = \frac{e^{2\omega}}{b^2} ds_0^2$, then it is well known that ω satisfies the Sinh-Gordon equation

$$\Delta_0 \omega + \sinh(\omega)\cosh(\omega) = 0$$

It is noted in [32] that ω and the Gauss curvature K of Σ share the same sign. Arguing by contradiction, let us assume that Ω_1 and Ω_2 are the sign components of ω. Ritoré-Ros [32] considered the test function $f = a_1 \sinh(\omega)$ in Ω_1 and $f = a_2 \sinh(\omega)$ in Ω_2; a_1 and a_2 are chosen so that f has mean zero on Σ. Hence,

$$0 \le I(f,f) = -\int_\Sigma f\Delta f + (-2 + |A|^2) f^2 dA + \int_{\partial\Sigma} f \frac{\partial f}{\partial \nu}\, d\sigma - \int_{\partial\Sigma} \Pi_{\partial M}(N,N) f^2\, d\sigma \tag{8.1}$$

The last integral in (8.1) is zero since ∂M is totally geodesic. Since Σ can be reflected across ∂M and ω depends only at the geometric data of Σ, the second integral is also zero. For the first integral, we follow computation in [32] to obtain

$$\begin{aligned} 0 \le I(f,f) &= -\int_\Sigma f\Delta_0 f + (\cosh^2(\omega) + \sinh^2(\omega)) f^2 dA_0 \\ &= -\sum_{i=1}^{2} \int_{\Omega_i} a_i^2 \sinh^2(\omega)\big(\sinh^2(\omega) + |\nabla_0 \omega|^2\big)\, dA_0 < 0. \end{aligned}$$

Hence, the Gauss curvature of Σ has a sign and by the Gauss-Bonnet Theorem we conclude that Σ is flat. □

Corollary 8.4. *If Σ is a compact embedded stable cmc surface in $\mathbb{H}^3/G_\lambda$, then Σ is either a geodesic sphere or a tube about a closed geodesic. If Σ is a compact embedded stable cmc surface in $\mathbb{H}^3/G_a$, then Σ is a geodesic sphere.*

Remark 8.5. The isoperimetric problem between horospheres is studied in [7]. Building on the symmetries of this setting and the Alexandrov reflection method, the authors show that isoperimetric surfaces are rotationally invariant and classified those that meet the boundary orthogonally. The stability of these surfaces is not investigated and the possibility of onduloids be isoperimetric for certain volumes is left open.

8.3. **Stability of tubes.** Let $\phi(r,\theta) = (e^r\cos(\theta), e^r\sin(\theta), a\,e^r)$ be the parametrization of a tube T of radius $R = \log\big(\frac{1+\sqrt{a^2+1}}{a}\big)$ about the z-axis in $\mathbb{H}^3$. The metric and the second fundamental form of T are

$$g = \begin{pmatrix} \frac{1+a^2}{a^2} & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \sqrt{a^2+1} & 0 \\ 0 & \frac{1}{\sqrt{a^2+1}} \end{pmatrix} \tag{8.2}$$

Note that the mean curvature of T satisfies $H > 1$. The Jacobi operator in this coordinate system is

$$L = \frac{a^2}{1+a^2}\partial_{rr} + a^2\partial_{\theta\theta} + \big(a^2 + \frac{1}{a^2+1} - 1\big).$$

Let's look at the free boundary stability of T between the hyperbolic planes $z = \sqrt{1 - x^2 + y^2}$ and $z = \sqrt{e^{2\lambda} - x^2 - y^2}$. For this we look at the eigenvalue problem:

$$\left(\frac{a^2}{1+a^2}\partial_{rr} + a^2\partial_{\theta\theta}\right)\varphi + \big(a^2 + \frac{1}{a^2+1} - 1\big)\varphi = -\lambda\varphi \quad \text{and} \quad \frac{\partial\varphi}{\partial r}(0,\theta) = \frac{\partial\varphi}{\partial r}(\lambda,\theta) = 0$$

The constant functions are the first eigenfunctions with eigenvalue $\lambda_1 = 1 - a^2 - \frac{a^2}{a^2+1} < 0$. The stability of T is then equivalent to have the second eigenvalue $\lambda_2 \geq 0$. Eigenfunctions of the form $\varphi(r,\theta) = \varphi(\theta)$ contribute positive eigenvalues for the Jacobi operator. An analysis of the eigenfunctions $\varphi(r,\theta) = \varphi(r)$ shows that $\lambda_2 \geq 0$ if, and only if, $a \leq \frac{\pi}{\lambda}$. Note that the distance between those two planes is $d_{\mathbb{H}} = \lambda$. Therefore, the tube T is stable if, and only if,

$$R \geq \log\left(\frac{d_{\mathbb{H}}}{\pi} + \sqrt{\frac{d_{\mathbb{H}}^2}{\pi^2} + 1}\right). \tag{8.3}$$

Let's look now at the volume preserving stability of T in the cyclic quotient $\mathbb{H}^3/G_{e^\lambda}$. For this we look at the eigenvalue problem:

$$\left(\frac{a^2}{1+a^2}\partial_{rr} + a^2\partial_{\theta\theta}\right)\varphi + \big(a^2 + \frac{1}{a^2+1} - 1\big)\varphi = -\lambda\varphi \quad \text{and} \quad \varphi(0,\theta) = \varphi(k\lambda,\theta) \quad \forall k \in \mathbb{Z}$$

Following the discussion above, we have that $\lambda_1 = 1 - a^2 - \frac{a^2}{a^2+1} < 0$ corresponding to the constant functions. The stability of T in $\mathbb{H}^3/G_{e^\lambda}$ is then equivalent to $\lambda_2 \geq 0$. As before, it is enough to consider eigenfunctions of the form $\varphi(r,\theta) = \varphi(r)$ which under the constraint above implies that $a \leq \frac{2\pi}{\lambda}$.

8.4. **Isoperimetric regions between two parallel geodesic planes.** Up to a hyperbolic reflection, any region bounded by two parallel geodesic planes of positive distance apart is congruent to the slab M bounded by the hyperbolic planes $z = \sqrt{1 - x^2 + y^2}$ and $z = \sqrt{e^{2\lambda} - x^2 - y^2}$ for some λ.

Let us now discuss the existence of isoperimetric regions in M. We follow Morgan [29]. Let Ω_α be a minimizing sequence for a fixed volume V. First we take a partition of $\mathbb{H}^3$ into congruent polyhedron Q_j and consider only those such that $Q_j \cap M \neq \emptyset$. Moreover, we choose Q_j large enough so that Ω_α does not contain any Q_j. Through hyperbolic reflections across the boundary of M, we can extend $\partial\Omega_\alpha$ such that its boundary does not intersect Q_j. The number of reflections is independent of Ω_α. The key observation is that Q_j satisfies a relative isoperimetric inequality for some constant γ. As detailed in Morgan [29], this isoperimetric inequality implies the existence of a number $\delta = \delta(\gamma, I(V))$ (I is the isoperimetric profile of M) such that $vol(\Omega_\alpha \cap Q_i) > \delta V$ for some of the Q_i. Choose a rigid motion in $\mathbb{H}^3$ (not necessarily preserving ∂M) that brings the center of

Q_i to a fixed point $O \in \mathbb{H}^3$. Standard compactness and regularity applied to the sequence $\partial\Omega_\alpha$ passing through O imply it will converge to a constant mean curvature surface $\partial\Omega$ enclosing volume $V_0 \geq \delta V$. By the monotonicity formula, $\partial\Omega$ must be bounded in $\mathbb{H}^3$ since it has finite area. It is enough to repeat the process finitely many times to recover the volume V; this also folows from the monotonicity formula since the value of the constant mean curvature at each repetition does not change. Therefore, the minimizing sequence Ω_α can be replaced by another which does not drift off to infinity.

The situation is different when the hyperbolic distance between the planes is zero. For the region bounded by two vertical planes in the half-space model of $\mathbb{H}^3$ for example, there are no isoperimetric regions. If such set existed, it would be either a half geodesic ball or a tube about a geodesic. The latter is ruled out since there is no free boundary geodesic connecting the two planes. Moreover, half geodesic spheres of a given radius can always be constructed so that it is centered at one of the planes and tangent to the other. Such configuration is not a critical point of the area functional under volume constraints. In particular, there are no isoperimetric regions $\mathbb{H}^3/G_a$ (G_a composed of horizontal translations). This argument also shows that half geodesic balls cannot be isoperimetric for all volumes in the slab bounded by two geodesic planes of positive distance apart. Therefore, there exists a critical volume V_0 for which tubes about geodesics are the isoperimetric surfaces when the enclosing volume satisfies $V \geq V_0$.

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