Lelong numbers of currents of full mass intersection

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In memory of Nessim Sibony

Abstract

We study Lelong numbers of currents of full mass intersection on a compact Kähler manifold in a mixed setting. Our main theorems cover some recent results due to Darvas-Di Nezza-Lu. The key ingredient in our approach is a new notion of products of pseudoeffective $(1,1)$-classes which captures some "pluripolar part" of the "total intersection" of given pseudoeffective $(1,1)$-classes.

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1 Introduction

Let X be a compact Kähler manifold of dimension n. For every closed positive current S on X, we denote by $\{S\}$ its cohomology class. For cohomology (q,q)-classes α and β on X, we write $\alpha \le \beta$ if $\beta - \alpha$ can be represented by a closed positive (q,q)-current.

Let $\alpha_1, \dots, \alpha_m$ be pseudoeffective $(1,1)$-classes, where $1 \le m \le n$. Let T_j and T'_j be closed positive $(1,1)$-currents in α_j for $1 \le j \le m$ such that T_j is more singular than T'_j, *i.e,* potentials of T_j is smaller than those of T'_j modulo an additive constant. By monotonicity of non-pluripolar products (see [27, Theorem 1.1] and also [7, 12, 30]), there holds

$$\{\langle T_1 \wedge \cdots \wedge T_m \rangle\} \le \{\langle T'_1 \wedge \cdots \wedge T'_m \rangle\}. \tag{1.1}$$

We refer to the beginning of Section 2 for a brief recap of non-pluripolar products.

We are interested in comparing the singularity types of T_j and T'_j when the equality in (1.1) occurs. Given the generality of the problem, it is desirable to formulate it in a more concrete way. In what follows, we focus on the important setting where $T_1, \dots, T_m$ are of full mass intersection (*i.e,* T'_j's have minimal singularities in their cohomology classes).

Let us recall that $T_1, \dots, T_m$ are said to be of *full mass intersection* if the equality in (1.1) occurs for T'_j to be a current with minimal singularities $T_{j,\min}$ in α_j for $1 \le j \le m$.

This is independent of the choice of $T_{j,\min}$. The last notion has played an important role in complex geometry, for example, see [2, 7, 11, 14, 19, 23, 28, 29]. We also notice that a connection of the notion of full mass intersection with the theory of density currents (see [20]) was established in [26], see also [22].

One of the most basic objects to measure the singularity of a current is the notion of Lelong numbers. We refer to [15] for its basic properties. Hence, the purpose of this paper is to compare the Lelong numbers of T_j and $T_{j,\min}$ when $T_1, \ldots, T_m$ are of full mass intersection. To go into details, we need some notions.

Let S be a closed positive current on X and x be a point in X. Denote by $\nu(S, x)$ the Lelong number of S at x. One can compute $\nu(S, x)$ as follows. We write $S = dd^c\psi$ for some psh function ψ defined on an open neighborhood U of x such that U is a local chart of X which we identify with an open subset in $\mathbb{C}^n$ and the point x corresponds to the origin in $\mathbb{C}^n$. Then we have

$$\nu(S, x) = \max\{\gamma \in \mathbb{R}_{\geq 0} : \psi(z) \leq \gamma \log |z| + O(1) \text{ near } 0\},$$

see [15, Chapter III]. Let V be an irreducible analytic subset of X. By Siu's analytic semi-continuity of Lelong numbers ([15, 24]), for every $x \in V$ outside some proper analytic subset of V, we have

$$\nu(S, x) = \min_{x' \in V} \nu(S, x').$$

The last number is called *the generic Lelong number of S along V* and is denoted by $\nu(S, V)$.

Let α be a pseudoeffective $(1, 1)$-class on X. Following [16], we recall that α is said to be *big* if there is a Kähler current in α, *i.e,* there is a closed positive current T in α such that $T \geq \omega$ for some Kähler form ω on X. Let $T_{\alpha,\min}$ be a current with minimal singularities in α (see [16, page 41-42] for definition). We denote by $\nu(\alpha, V)$ the generic Lelong number of $T_{\alpha,\min}$ along V. This number is independent of the choice of $T_{\alpha,\min}$. It is clear that for every current $S \in \alpha$, we have $\nu(S, V) \geq \nu(\alpha, V)$. Here is our first main result.

Theorem 1.1. *Let $1 \leq m \leq n$ be an integer. Let $\alpha_1, \ldots, \alpha_m$ be big cohomology classes in X and let T_j be a closed positive $(1, 1)$-currents in α_j for $1 \leq j \leq m$. Let V be a proper irreducible analytic subset of X of dimension $\geq n - m$. Assume that $T_1, \ldots, T_m$ are of full mass intersection. Then there exists an index $1 \leq j \leq m$ such that*

$$\nu(T_j, V) = \nu(\alpha_j, V). \tag{1.2}$$

We note that when $\alpha_1, \ldots, \alpha_m$ are Kähler, Theorem 1.1 was proved in [27, Theorem 1.2]; see also the discussion after Corollary 1.4 below. The proof presented there is not applicable in the setting of Theorem 1.1.

When $\dim V = n - m$, the above result is optimal because in general, it might happen that there is only one index j satisfying (1.2); see Example 3.5. However, motivated from the Kähler case, we wonder whether it is true that the number of $1 \leq j \leq m$ such that $\nu(T_j, V) = \nu(\alpha_j, V)$ is at least $\dim V - (n - m) + 1$ (recall $V \subsetneq X$).

In the case where $m = n$, our above result can be improved quantitatively as follows.

Theorem 1.2. *Let $\mathscr{B}_0$ be a closed cone in $H^{1,1}(X,\mathbb{R})$ which is contained in the cone of big $(1,1)$-classes of X. Then, there exists a constant $C>0$ such that for every $x_0 \in X$, every $\alpha_j \in \mathscr{B}_0$ and every closed positive $(1,1)$-current $T_j \in \alpha_j$ for $1 \le j \le n$, we have*

$$\int_X \left(\langle \wedge_{j=1}^n \alpha_j \rangle - \{\langle \wedge_{j=1}^n T_j \rangle\}\right) \ge C\big(\nu(T_1,x_0)-\nu(\alpha_1,x_0)\big)\cdots\big(\nu(T_n,x_0)-\nu(\alpha_n,x_0)\big). \quad (1.3)$$

The dependence of C on $\mathscr{B}_0$ is necessary, see Example 3.4. We have the following direct consequences of Theorem 1.1.

Corollary 1.3. *Let $1 \le m \le n$ be an integer. Let α be a big class and let $T \in \alpha$ be a closed positive $(1,1)$-current so that*

$$\{\langle T^m \rangle\} = \langle \alpha^m \rangle.$$

Let V be an irreducible analytic subset of X of dimension at least $n-m$. Then there holds

$$\nu(T,V) = \nu(\alpha_j, V).$$

In particular, if α is big and nef, then T has zero Lelong number at a generic point in V.

Recall that $\langle \alpha^m \rangle$ is defined to be the cohomology class of $\langle T^m_{\alpha,\min} \rangle$, where $T_{\alpha,\min}$ is a current with minimal singularities in α, see Section 2 below for details. Combining Corollary 1.3 with results in [4, 8], we recover the following known result.

Corollary 1.4. *Let θ be a smooth closed $(1,1)$-form in a big cohomology class α. Let φ be a θ-psh function of full Monge-Ampère mass,* i.e,

$$\{\langle (dd^c\varphi + \theta)^n \rangle\} = \langle \alpha^n \rangle.$$

Let $\varphi_{\alpha,\min}$ be a θ-psh function with minimal singularities. Then, we have

$$\mathcal{I}(t\varphi) = \mathcal{I}(t\varphi_{\alpha,\min}) \quad (1.4)$$

for every $t>0$, where for every quasi-psh function ψ on X, we denote by $\mathcal{I}(\psi)$ the multiplier ideal sheaf associated to ψ.

Corollary 1.4 was proved in [10, 12, 13] (hence answering a question posed in [18]); see also [21] for the case where θ is Kähler. In fact, [12] gives a stronger fact which we describe below. For every closed positive $(1,1)$-current T' with $\int_X \langle T'^n \rangle > 0$, Theorem 1.3 in [12] gives a characterization (in terms of certain plurisubharmonic rooftop envelopes) of potentials of every closed positive $(1,1)$-current T cohomologous to T' such that T is less singular than T' and

$$\int_X \langle T^n \rangle = \int_X \langle T'^n \rangle.$$

Consequently, the multiplier ideal sheafs associated to the potentials of T and T' are the same by arguments from the proof of [13, Theorem 1.1]. Nevertheless, in the present setting of our main results, it is unclear how to formulate such a characterization because either $T_1,\dots,T_m$ can be different or $m \le n$ (even if one takes $T_1 = \cdots = T_m$). In fact, a direct analogue of the envelope characterization given [12] is not true in our setting when $m \leq n$; see the comment after Theorem 1.1 and [12, Remark 3.3].

Let us now have a few comments on our approach. Due to the above discussions, we present here a *completely new strategy* to the study of singularity of currents of full mass intersection. We stress that although our main results only involve the usual non-pluripolar products, the notion of relative non-pluripolar products introduced in [27] will play an essential role in our proof. The reason, which will be more clear later, is that relative non-pluripolar products allow us to better control the loss of masses.

The key ingredient in our proof of Theorem 1.1 is a new notion of products of pseudoeffective classes which was briefly mentioned in [27, Remark 4.5]. This new product of pseudoeffective classes is bounded from below by the positive product introduced in [5, 7]. The feature is that this new product also captures some pluripolar part of "total intersection" of classes. This explains why we have a better control on masses.

Theorem 1.2 is a direct consequence of the proof of Theorem 1.1. We underline that our arguments in the proof of Theorem 1.1 are not quantifiable as soon as $\dim V \geq 2$. This is due to the fact that we need to use the blowup along V and the desingularization of V (in case V is singular). Despite of this, we think that it is still reasonable to expect an estimate similar to Theorem 1.2 in the case where V is of higher dimension.

Finally, in view of the above discussion of results in [12], one can wonder what should be expected for the equality case of (1.1) when T_j's are not necessarily of minimal singularities. It is not unrealistic to hope that our approach can be extended to this setting. But there are non-trivial obstructions. To single out one: the condition that T_j's have minimal singularities are needed in our proof of Theorem 1.1 because we will use the fact that there are Kähler currents with analytic singularities which are more singular than T_j for every j.

The paper is organized as follows. In Section 2, we present basic properties of relative non-pluripolar products and introduce the above-mentioned notion of products of pseudoeffective classes. Theorems 1.1 and 1.2 are proved in Section 3.

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2 Relative non-pluripolar products

We first recall some basic facts about relative non-pluripolar products. This notion was introduced in [27] as a generalization of the usual non-pluripolar products given in [3, 7, 21]. To simplify the presentation, we only consider the compact setting.

Let X be a compact Kähler manifold of dimension n. Let $T_1, \ldots, T_m$ be closed positive $(1,1)$-currents on X. Let T be closed positive current of bi-degree (p,p) on X. By [27], we can define *the T-relative non-pluripolar product* $\langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle$ in a way similar to that of the usual non-pluripolar product. For readers' convenience, we recall how to do it.

Write $T_j = dd^c u_j + \theta_j$, where θ_j is a smooth form and u_j is a θ_j-psh function. Put

$$R_k := \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \wedge_{j=1}^m (dd^c \max\{u_j, -k\} + \theta_j) \wedge T$$

for $k \in \mathbb{N}$. By the strong quasi-continuity of bounded psh functions ([27, Theorems 2.4 and 2.9]), we have

$$R_k = \mathbf{1}_{\cap_{j=1}^m \{u_j > -k\}} \wedge_{j=1}^m (dd^c \max\{u_j, -l\} + \theta_j) \wedge T$$

for every $l \geq k \geq 1$. A similar equality also holds if we use local potentials of T_j instead of global ones. We can show that R_k is positive (see [27, Lemma 3.2]).

As in [7], since X is Kähler, one can check that R_k is of mass bounded uniformly in k and $(R_k)_k$ admits a limit current which is closed as $k \to \infty$. The last limit is denoted by $\langle \wedge_{j=1}^m T_j \dot{\wedge} T \rangle$. The last product is, hence, a well-defined closed positive current of bi-degree $(m+p, m+p)$; and it is symmetric with respect to $T_1, \ldots, T_m$ and homogeneous. We refer to [27, Proposition 3.5] for more properties of relative non-pluripolar products. When $T \equiv 1$, the T-relative non-pluripolar product $\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle$ is exactly the non-pluripolar product $\langle T_1 \wedge \cdots \wedge T_m \rangle$ of $T_1, \ldots, T_m$ defined in [7].

Let $\alpha_1, \ldots, \alpha_m$ be pseudoeffective $(1,1)$-classes on X. Recall that by using a monotonicity of relative non-pluripolar products ([27, Theorem 1.1]), we can define the cohomology class $\{\langle \alpha_1 \wedge \ldots \wedge \alpha_m \dot{\wedge} T \rangle\}$ which is the one of the current $\langle \wedge_{j=1}^m T_{j,\min} \dot{\wedge} T \rangle$, where $T_{j,\min}$ is a current with minimal singularities in α_j for $1 \leq j \leq m$. When T is the current of integration along X, we write $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ for $\{\langle \alpha_1 \wedge \cdots \wedge \alpha_m \dot{\wedge} T \rangle\}$. By [27, Proposition 4.6], the class $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ is equal to the positive product of $\alpha_1, \ldots, \alpha_m$ defined in [7, Definition 1.17] provided that $\alpha_1, \ldots, \alpha_m$ are big.

In the next paragraph, we are going to introduce a related notion of products of $(1,1)$-classes. This idea was already suggested in [27]. This new notion will play a crucial role in our proof of Theorem 1.1. We are interested in the case where T is of bi-degree $(1,1)$. We recall the following key monotonicity property.

Theorem 2.1. *([27, Remark 4.5]) Let X be a compact Kähler manifold and let $T_1, \ldots, T_m, T$ be closed positive $(1,1)$-currents on X. Let T'_j and T' be closed positive $(1,1)$-currents in the cohomology class of T_j and T respectively such that T'_j is less singular than T_j for $1 \leq j \leq m$ and T' is less singular than T. Then we have*

$$\{\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \dot{\wedge} T' \rangle\}.$$

Recall that for closed positive $(1,1)$-currents P and P' on X, we say that P' is less singular than P if for every global potential u of P and u' of P', then $u \leq u' + O(1)$.

Proof. Since this result is crucial for us, we will present its proof below. Write $T_j = dd^c u_j + \theta_j$, $T'_j = dd^c u'_j + \theta_j$, where θ_j is a smooth form and u'_j, u_j are negative θ_j-psh functions, for every $1 \leq j \leq m$. Similarly, we have $T = dd^c \varphi + \eta$, $T' = dd^c \varphi' + \eta'$.

Step 1. Assume for the moment that T_j, T'_j are of the same singularity type for every $1 \leq j \leq m$ and T, T' are also of the same singularity type. We will check that

$$\{\langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T \rangle\} = \{\langle T'_1 \wedge \cdots \wedge T'_m \dot{\wedge} T' \rangle\}. \tag{2.1}$$

Since T_j, T'_j are of the same singularity type, we have $\{u_j = -\infty\} = \{u'_j = -\infty\}$ and $w_j := u_j - u'_j$ is bounded. We have similar properties for φ, φ'. Let $A := \cup_{j=1}^m \{u_j = -\infty\}$ which is a complete pluripolar set. Put $u_{jk} := \max\{u_j, -k\}$, $u'_{jk} := \max\{u'_j, -k\}$ and

$$\psi_k := k^{-1} \max\{\sum_{j=1}^n (u_j + u'_j), -k\} + 1 \tag{2.2}$$

which is quasi-psh and $0 \le \psi_k \le 1$, $\psi_k(x)$ increases to 1 for $x \notin A$. We have $\psi_k(x) = 0$ if $u_j(x) \le -k$ or $u'_j(x) \le -k$ for some j. Put $w_{jk} := u_{jk} - u'_{jk}$. Since w_j is bounded, we have

$$|w_{jk}| \lesssim 1 \tag{2.3}$$

on X. Let $J, J' \subset \{1, \dots, m\}$ with $J \cap J' = \varnothing$. Put

$$R_{JJ'k} := \wedge_{j\in J}(dd^c u_{jk} + \theta_j) \wedge \wedge_{j'\in J'}(dd^c u'_{j'k} + \theta_{j'}) \wedge T$$

and

$$R_{JJ'} := \big\langle \wedge_{j\in J} (dd^c u_j + \theta_j) \wedge \wedge_{j'\in J'}(dd^c u'_{j'} + \theta_{j'}) \dot\wedge T \big\rangle.$$

Let

$$B_k := \cap_{j\in J}\{u_j > -k\} \cap \cap_{j'\in J'}\{u'_{j'} > -k\}.$$

Observe

$$0 \le \mathbf{1}_{B_k} R_{JJ'} = \mathbf{1}_{B_k} R_{JJ'k}$$

for every J, J', k. Put $\tilde R_{JJ'} := \mathbf{1}_{X\backslash A} R_{JJ'}$. The last current is closed positive. Using the fact that $\{\psi_k \ne 0\} \subset B_k \backslash A$, we get

$$\psi_k \tilde R_{JJ'} = \psi_k R_{JJ'} = \psi_k R_{JJ'k}. \tag{2.4}$$

Put $p' := n - |J| - |J'| - p - 1$. By Claim in the proof of [27, Proposition 4.2], for every $j'' \in \{1, \dots, m\} \backslash (J \cup J')$ and every closed smooth form Φ of bi-degree (p', p') on X, we have

$$\lim_{k\to\infty} \int_X \psi_k dd^c w_{j''k} \wedge R_{JJ'k} \wedge \Phi = 0. \tag{2.5}$$

Let

$$S_0 := \langle T_1 \wedge \dots \wedge T_n \dot\wedge T\rangle - \langle T'_1 \wedge \dots \wedge T'_n \dot\wedge T\rangle$$

and

$$S_1 := \langle T_1 \wedge \dots \wedge T_n \dot\wedge T\rangle - \langle T'_1 \wedge \dots \wedge T'_n \dot\wedge T\rangle, \quad S_2 := \langle T'_1 \wedge \dots \wedge T'_n \dot\wedge (T - T')\rangle.$$

We have $S_0 = S_1 + S_2$. Using $T_{jk} = T'_{jk} + dd^c w_{jk}$, one can check that

$$\int_X \psi_k S_1 \wedge \Phi = \sum_{s=1}^m \int_X \psi_k \wedge_{j=1}^{s-1} T'_{jk} \wedge dd^c w_{sk} \wedge \wedge_{j=s+1}^m T_{jk} \wedge T \wedge \Phi$$

for every closed smooth Φ. This together with (2.5) yields

$$\langle S_1, \Phi\rangle = \lim_{k\to\infty} \langle \psi_k S_1, \Phi\rangle = 0. \tag{2.6}$$

Let $\varphi_l := \max\{\varphi, -l\}$ and $\varphi'_l := \max\{\varphi', -l\}$ for $l \in \mathbb{N}$. By [27, Theorem 2.2], observe

$$\int_X \psi_k S_2 \wedge \Phi = \lim_{l\to\infty} \int_X \psi_k dd^c(\varphi_l - \varphi'_l) \wedge T'_{1k} \wedge \cdots T'_{mk} \wedge \Phi. \tag{2.7}$$

Since $\varphi_l - \varphi'_l$ is bounded uniformly in $l \in \mathbb{N}$, reasoning as in the proof of (2.5), we see that the term under limit in the right-hand side of (2.7) converges to 0 as $k \to \infty$ uniformly in l. Hence

$$\int_X \psi_k S_2 \wedge \Phi \to 0$$

as $k \to \infty$. Consequently, we get $\int_X \psi_k S \wedge \Phi \to 0$ as $k \to \infty$. In other words, (2.1) follows. This finishes Step 1.

Step 2. Consider now the general case, *i.e,* T'_j and T' are less singular than T_j and T respectively. Without loss of generality, we can assume that $u'_j \geq u_j$ and $\varphi' \geq \varphi$. For $l \in \mathbb{N}$, put $u^l_j := \max\{u_j, u'_j - l\}$ which is of the same singularity type as u'_j. Notice that $dd^c u^l_j + \theta_j \geq 0$. Similarly, put $\varphi^l := \max\{\varphi, \varphi' - l\}$ and $T^l := dd^c\varphi^l + \eta \geq 0$.

Since X is Kähler, the family of currents $\langle \wedge_{j=1}^m (dd^c u^l_j + \theta_j) \dot\wedge T^l\rangle$ parameterized by l is of uniformly bounded mass. Let S be a limit current of the last family as $l \to \infty$. Since u^l_j, u'_j are of the same singularity type for every j and φ^l, φ' are so, using Step 1, we see that

$$\{S\} = \{\langle \wedge_{j=1}^m T'_j \dot\wedge T'\rangle\}. \tag{2.8}$$

On the other hand, since u^l_j, φ^l decrease to u_j, φ as $l \to \infty$ respectively, we can apply [27, Lemma 4.1] (and [27, Theorem 2.2]) to get

$$S \geq \langle \wedge_{j=1}^m T_j \dot\wedge T\rangle.$$

This combined with (2.8) gives the desired assertion. The proof is finished. □

We note here the following remark which could be useful for other works.

Remark 2.2. *Let P and P' be closed positive $(1,1)$-currents and Q a closed positive currents such that P' is less singular than P and potentials of P are integrable with respect to the trace measure of Q. Put $T := P \wedge Q$ and $T' := P' \wedge Q$. Then Theorem 2.1 still holds for these T', T with the same proof. The only minor modification is that the potentials φ, φ' of T, T' in the last proof are replaced by those of P, P'.*

For a $(1,1)$-current P, recall that *the polar locus* I_P of P is the set of $x \in X$ so that the potentials of P are equal to $-\infty$ at x. By abuse of language, we say that a closed positive current T has no mass on a Borel set $A \subset X$, if the trace measure of T has no mass on A.

For every pseudoeffective $(1,1)$-class β in X, we define its *polar locus* I_β to be that of a current with minimal singularities in β. This is independent of the choice of a current with minimal singularities. We have the following.

Lemma 2.3. *Assume that T is of bi-degree $(1,1)$. Then we have*

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T\rangle = \langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} (\mathbf{1}_{X\setminus I_T} T)\rangle, \tag{2.9}$$

In particular, T has no mass on I_T, then

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T\rangle = \langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T\rangle.$$

Proof. By [27, Proposition 3.6], we get

$$\langle T_1 \wedge \cdots \wedge T_m \wedge T\rangle = \mathbf{1}_{X\setminus I_T} \langle T_1 \wedge \cdots \wedge T_m \dot{\wedge} T\rangle. \tag{2.10}$$

Now using (2.10) and [27, Proposition 3.5] (vii) gives (2.9). This finishes the proof. □

Let $1 \le l \le m$. Let $\alpha_l, \ldots, \alpha_m, \beta$ be pseudoeffective $(1,1)$-classes of X. Let $T_{j,\min}, T_{\min}$ be currents with minimal singularities in the classes α_j, β respectively, where $l \le j \le m$. By Theorem 2.1, the class

$$\big\{\langle T_1 \wedge \cdots T_{l-1} \wedge T_{l,\min} \wedge \cdots \wedge T_{m,\min} \dot{\wedge} T_{\min}\rangle\big\}$$

is a well-defined pseudoeffective class which is independent of the choice of $T_{\min}$ and $T_{j,\min}$ for $l \le j \le m$. We denote the last class by

$$\big\{\langle T_1 \wedge \cdots \wedge T_{l-1} \wedge \alpha_l \wedge \cdots \wedge \alpha_m \dot{\wedge} \beta\rangle\big\}.$$

For simplicity, when $l = 1$, we remove the bracket $\{\quad\}$ from the last notation.

The following result holds for the class $\big\{\langle T_1 \wedge \cdots \wedge T_{l-1} \wedge \alpha_l \wedge \cdots \wedge \alpha_m \dot{\wedge} \beta\rangle\big\}$ but to avoid cumbersome notations (while keeping the essence of the statements), we only write it for $l = 1$.

Proposition 2.4. (i) *The product $\langle \wedge_{j=1}^m \alpha_j \dot{\wedge} \beta\rangle$ is symmetric and homogeneous in $\alpha_1, \ldots, \alpha_m$.*

(ii) *If β' is a pseudo-effective $(1,1)$-class, then*

$$\langle \wedge_{j=1}^m \alpha_j \dot{\wedge} \beta\rangle + \langle \wedge_{j=1}^m \alpha_j \dot{\wedge} \beta'\rangle \le \langle \wedge_{j=1}^m \alpha_j \dot{\wedge} (\beta + \beta')\rangle.$$

(iii) *Let $1 \le l \le m$ be an integer. Let $\alpha''_1, \ldots, \alpha''_l$ be a pseudoeffective $(1,1)$-class such that $\alpha''_j \ge \alpha_j$ for $1 \le j \le l$. Assume that there is a current with minimal singularities in β having no mass on $I_{\alpha''_j - \alpha_j}$ for every $1 \le j \le l$. Then, we have*

$$\langle \wedge_{j=1}^l \alpha''_j \wedge \wedge_{j=l+1}^m \alpha_j \dot{\wedge} \beta\rangle \ge \langle \wedge_{j=1}^m \alpha_j \dot{\wedge} \beta\rangle.$$

(iv) *If there is a current with minimal singularities in β having no mass on proper analytic subsets on X, then the product $\{\langle \wedge_{j=1}^m \alpha_j \dot{\wedge} \beta\rangle\}$ is continuous on the set of $(\alpha_1, \ldots, \alpha_m)$ such that $\alpha_1, \ldots, \alpha_m$ are big.*

(v) *We have*

$$\langle \wedge_{j=1}^m \alpha_j \wedge \beta\rangle \le \langle \wedge_{j=1}^m \alpha_j \dot{\wedge} \beta\rangle$$

and the equality occurs if there is a current with minimal singularities P in β such that $P = 0$ on I_P.

Proof. We see that (v) is a direct consequence of Lemma 2.3 and the definition of the product $\langle \wedge_{j=1}^m \alpha_j \dot{\wedge} \beta \rangle$. The other desired statements can be proved by using arguments similar to those in the proof of [27, Proposition 4.6]; see also [9] for related materials. This finishes the proof. $\square$

The following result will be useful later.

Lemma 2.5. *Let α be a big class and let $T_{\alpha,\min}$ be a current with minimal singularities in α. Let T be a current in α. Then, the current $T_\alpha := \mathbf{1}_{I_{T_{\alpha,\min}}} T_{\alpha,\min}$ is a linear combination of currents of integration along irreducible hypersurfaces of X, and we have*

$$T_\alpha \leq \mathbf{1}_{I_T} T. \tag{2.11}$$

In particular, for every pluripolar set A, if T has no mass on A, then neither does $T_{\alpha,\min}$.

Proof. Recall that $I_\alpha = I_{T_{\alpha,\min}}$. By Demailly's analytic approximation of $(1,1)$-currents ([16]), there exists a Kähler current with analytic singularities P in α. It follows that I_α is contained in a proper analytic subset V of X. This together the fact that $\mathrm{Supp} T_\alpha$ is contained in the closure of I_α implies that T_α is supported on V.

Since T_α is of bi-dimension $(n-1, n-1)$, using the first support theorem [15, Page 141], we see that T_α is supported on the union of hypersurfaces of X contained in V. Now the second support theorem [15, Page 142-143] impplies that T_α must be a linear combination of currents of integration along hypersurfaces. Hence the first desired assertion follows.

We prove (2.11). It is enough to consider the case where $\mathbf{1}_{I_{T_{\alpha,\min}}} T_{\alpha,\min}$ is nonzero. Let W be the support of the last current. By the above observation, W is a hypersurface. Since T is less singular than $T_{\alpha,\min}$, we get

$$\nu(T, x) \geq \nu(T_{\alpha,\min}, x)$$

for every x. In particular, the generic Lelong number of T along every irreducible component W' of W is greater than or equal to that of $T_{\alpha,\min}$ along W'. We deduce that $T \geq \mathbf{1}_{I_{T_{\alpha,\min}}} T_{\alpha,\min}$. Hence, (2.11) follows.

Let A be a pluripolar set in X. Let $\varphi_{\min}$ be a potential of $T_{\alpha,\min}$. We have

$$T_{\alpha,\min} = \mathbf{1}_{\{\varphi_{\min} > -\infty\}} T_{\alpha,\min} + \mathbf{1}_{\{\varphi_{\min} = -\infty\}} T_{\alpha,\min}.$$

Denote by I_1, I_2 the first and second term in the right-hand side of the last equality respectively. By (2.11) and the hypothesis, we see that I_2 has no mass on A. We now show that I_1 satisfies the same property.

If $\{\varphi_{\min} > -\infty\}$ is open, then it is clear that I_1 has no mass on A because $\varphi_{\min}$ is locally bounded on the open set $\{\varphi_{\min} > -\infty\}$. However in general, when $\{\varphi_{\min} > -\infty\}$ is not necessarily open, some more arguments are needed. Recall that I_1 is actually equal to the non-pluripolar product $\langle T_{\alpha,\min} \rangle$ of $T_{\alpha,\min}$ itself (e.g, by applying [27, Proposition 3.6 (i)] to $T \equiv 1$ and $m = 1$). Since the current $\langle T_{\alpha,\min} \rangle$ has no mass on pluripolar sets, we see that I_1 has no mass on A. Hence, $T_{\alpha,\min}$ has no mass on A. This finishes the proof. $\square$

We note that (2.11) actually holds in a much more general setting; see [1, Lemma 4.1].

3 Proof of Theorems 1.1 and 1.2

We will sometimes use the notations $\gtrsim, \lesssim$ to denote the inequalities $\geq, \leq$ modulo some strictly positive multiplicative constant independent of parameters in consideration. For every analytic set W in a complex manifold Y, we denote by $[W]$ the current of integration along W.

Let X be a compact Kähler manifold. Let $\alpha_1, \ldots, \alpha_m$ be big classes in X. Let $T_{j,\min}$ be a current with minimal singularities in α_j and

$$T_{\alpha_j} := \mathbf{1}_{I_{\alpha_j}} T_{j,\min}$$

(recall here that I_{α_j} is the set of $x \in X$ such that potentials of $T_{j,\min}$ are equal to $-\infty$ at x). By Lemma 2.5, the current T_{α_j} is a linear combination of currents of integration along irreducible hypersurfaces of X. In view of proving Theorem 1.1, we first explain how to reduce the problem to the case where T_{α_j}'s are zero.

Lemma 3.1. *For every j, the class $\alpha_j - \{T_{\alpha_j}\}$ is big and there holds*

$$\langle \wedge_{j=1}^m \alpha_j \rangle = \langle \wedge_{j=1}^m (\alpha_j - \{T_{\alpha_j}\}) \rangle. \tag{3.1}$$

Proof. Let ω be a Kähler form on X. Fix an index $1 \leq j \leq m$. Let W_j be the support of T_{α_j}. Consider a Kähler current $P_j \in \alpha_j$. By Lemma 2.5, the set W_j is a hypersurface (or empty), and $P_j - T_{\alpha_j}$ is a closed positive current. Note that

$$P_j - T_{\alpha_j} = P_j \gtrsim \omega$$

on $X \backslash W_j$. Since ω is smooth, we get $P_j - T_{\alpha_j} \gtrsim \omega$ on X. In other words, $P_j - T_{\alpha_j}$ is a Kähler current. Hence, $\alpha_j - \{T_{\alpha_j}\}$ is big.

It remains to prove (3.1). The inequality direction " $\geq$ " is clear because $\alpha_j \geq \alpha_j - \{T_{\alpha_j}\}$. To get the converse inequality, one only needs to notice that

$$\langle \wedge_{j=1}^m T_{j,\min} \rangle = \langle \wedge_{j=1}^m (T_{j,\min} - T_{\alpha_j}) \rangle$$

which is true because both sides are currents which have no mass on

$$W := \cup_{j=1}^m W_j$$

(which is a closed pluripolar set) and are equal on $X \backslash W$ (which is an open subset of X). The proof is finished. □

Let $T_j \in \alpha_j$ be a closed positive current as in Theorem 1.1. By Lemma 2.5, we have $\mathbf{1}_{I_{T_j}} T_j \geq T_{\alpha_j}$. It follows that $T_j - T_{\alpha_j}$ is positive. Using the fact that T_{α_j} is supported on proper analytic subsets on X gives

$$\langle \wedge_{j=1}^m T_j \rangle = \langle \wedge_{j=1}^m (T_j - T_{\alpha_j}) \rangle.$$

This combined with Lemma 3.1 yields that $(T_1 - T_{\alpha_1}), \ldots, (T_m - T_{\alpha_m})$ are of full mass intersection. Hence, by considering $T_j - T_{\alpha_j}$, $\alpha_j - \{T_{\alpha_j}\}$ instead of T_j, α_j, *we can assume, from now on, that T_{α_j} is zero as desired.*

Assume for the moment that V is a smooth submanifold of X of dimension $\leq n-1$. Let $\sigma : \widehat{X} \to X$ be the blowup of X along V. Denote by $\widehat{V}$ the exceptional hypersurface. Let ω be a Kähler form on X. Let ω_h be a closed smooth form cohomologous to $-[\widehat{V}]$ so that the restriction of ω_h to each fiber of the natural projection from $\widehat{V}$ to V is strictly positive (the existence of such a form is classical, see [25, Lemma 3.25]). Thus, there exists a strictly positive constants c_V satisfying that

$$\widehat{\omega} := c_V \sigma^* \omega + \omega_h > 0 \tag{3.2}$$

We note that when $\dim V = n-1$, by convention, we put $\widehat{X} := X$, $\sigma := \mathrm{id}$, $\widehat{V} := V$, $c_V := 1$ and $\omega_h := 0$.

For every closed positive current S on X, let λ_S be the generic Lelong number of S along V. By a well-known result on Lelong numbers under blowups (see [5, Corollary 1.1.8]), the generic Lelong number of $\sigma^* S$ along $\widehat{V}$ is equal to λ_S. Hence, we can decompose

$$\sigma^* T_j = \lambda_{T_j} [\widehat{V}] + \eta_j, \quad \sigma^* T_{j,\min} = \lambda_{T_{j,\min}} [\widehat{V}] + \eta_{j,\min},$$

where η_j and $\eta_{j,\min}$ are currents whose generic Lelong numbers along $\widehat{V}$ are zero. Since $T_{j,\min}$ is less singular than T_j, we have $\lambda_{T_j} \geq \lambda_{T_{j,\min}}$.

Let

$$\gamma_j := \{\eta_j\}, \quad \gamma_{j,\min} := \{\eta_{j,\min}\}, \quad \beta := \{[\widehat{V}]\}.$$

These classes are important in the sequel. By [6, 17], the class $\gamma_{j,\min}$ is big. For every closed smooth $(n-m, n-m)$-form Φ, using the fact that $T_{j,\min}$ has minimal singularities and the monotonicity of non-pluripolar products gives

$$\int_X \langle \wedge_{j=1}^m T_{j,\min} \rangle \wedge \Phi = \int_{\widehat{X}} \langle \wedge_{j=1}^m \eta_{j,\min} \rangle \wedge \sigma^* \Phi = \int_{\widehat{X}} \langle \wedge_{j=1}^m \gamma_{j,\min} \rangle \wedge \sigma^* \Phi. \tag{3.3}$$

Lemma 3.2. *We have*

$$\langle \wedge_{j=1}^m \eta_j \rangle \leq \langle \wedge_{j=1}^{m-1} \eta_j \dot{\wedge} \eta_m \rangle, \quad \langle \wedge_{j=1}^m \eta_{j,\min} \rangle = \langle \wedge_{j=1}^{m-1} \eta_{j,\min} \dot{\wedge} \eta_{m,\min} \rangle, \tag{3.4}$$

and

$$\langle \wedge_{j=1}^m \gamma_{j,\min} \rangle = \langle \wedge_{j=1}^{m-1} \gamma_{j,\min} \dot{\wedge} \gamma_{m,\min} \rangle \tag{3.5}$$

Proof. The first desired inequality in (3.4) is clear by Proposition 2.4. Observe that $\mathbf{1}_{I_{\eta_{m,\min}}} \eta_{m,\min}$ has no mass on $\widehat{V}$ because the generic Lelong number of $\eta_{m,\min}$ along $\widehat{V}$ is equal to zero. We deduce that

$$\mathbf{1}_{I_{\eta_{m,\min}}} \eta_{m,\min} = \mathbf{1}_{I_{\eta_{m,\min}} \backslash \widehat{V}} \eta_{m,\min} \leq \sigma^* \big(\mathbf{1}_{\sigma(I_{\eta_{m,\min}})} T_{m,\min} \big) \leq \sigma^* (\mathbf{1}_{I_{T_{m,\min}}} T_{m,\min}) = 0.$$

Hence, $\eta_{m,\min}$ has no mass on $I_{\eta_{m,\min}}$. Combining this with Lemma 2.3 yields (3.4).

We now prove (3.5). Let Q_m be a current with minimal singularities in $\gamma_{m,\min}$. By Lemma 2.5 and the fact that $\gamma_{m,\min}$ is big, we see that

$$\mathbf{1}_{I_{Q_m}} Q_m \leq \mathbf{1}_{\eta_{m,\min}} \eta_{m,\min} = 0.$$

Hence, Q_m has no mass on I_{Q_m}. Using this and Lemma 2.3 gives the desired equality and finishes the proof. □

Fix a norm $\|\cdot\|$ on $H^{1,1}(X,\mathbb{R})$. For $1 \le j \le m$, let P_j be a Kähler current with analytic singularities in α_j. Let $\epsilon > 0$ be a constant small enough so that $P_j \ge \epsilon\omega$ for every $1 \le j \le m$.

Lemma 3.3. *For every constant $\delta \in (0,1)$, there exist a constant $c_\delta > 0$ and a Kähler current with analytic singularities $Q_j \in \gamma_{j,\min} - c_\delta \beta$ for $1 \le j \le m$ such that I_{Q_j} does not contain $\widehat{V}$, and $Q_j \ge \frac{\delta\epsilon}{2c_V}\widehat{\omega}$, and*

$$\frac{\delta\epsilon}{2c_V} \le c_\delta \le \big(c\|\alpha_j\| + \frac{\epsilon}{2c_V}\big)\delta, \tag{3.6}$$

for some constant $c > 0$ independent of δ, β and α_j. In particular, the currents with minimal singularities in $\gamma_{j,\min} - c_\delta\beta$ has no mass on $\widehat{V}$, and the current $[\widehat{V}]$ has no mass on the polar locus of the class $\gamma_{j,\min} - c_\delta\beta - \frac{\delta\epsilon}{2c_V}\{\widehat{\omega}\}$.

Proof. Using Demailly's analytic approximation of currents ([16]) applied to the Kähler current $(1-\delta)T_{j,\min} + \delta P_j$ for $\delta \in (0,1)$, we obtain that for every $\delta \in (0,1)$, there exits a Kähler current $P_{j,\delta}$ with analytic singularities in the class α_j such that $P_{j,\delta}$ is less singular than $(1-\delta)T_{j,\min} + \delta P_j$ and

$$P_{j,\delta} \ge \delta\epsilon\omega/2. \tag{3.7}$$

We deduce that

$$\lambda_{T_{j,\min}} \le \lambda_{P_{j,\delta}} \le \lambda_{T_{j,\min}} + a_j\delta, \tag{3.8}$$

where $a_j := \lambda_{P_j} - \lambda_{T_{j,\min}} \ge 0$. Write

$$\sigma^* P_{j,\delta} = \lambda_{P_{j,\delta}}[\widehat{V}] + \eta_{j,\delta}.$$

Since $P_{j,\delta}$ has analytic singularities, so does $\eta_{j,\delta}$ and the polar locus of $\eta_{j,\delta}$ is an analytic subset of X which doesn't contain $\widehat{V}$. Hence, $[\widehat{V}]$ has no mass on the polar locus of $\eta_{j,\delta}$.

Recall that by the choice of ω_h, we have $\omega_h \in -\beta$. By (3.7) and (3.2), we also get

$$Q_j := \eta_{j,\delta} + \frac{\delta\epsilon}{2c_V}\omega_h \ge \frac{\delta\epsilon}{2c_V}\widehat{\omega}.$$

The last current is in the class

$$\gamma_{j,\min} - c_\delta\beta,$$

where

$$c_\delta := \lambda_{P_{j,\delta}} - \lambda_{T_{j,\min}} + (\delta\epsilon)/(2c_V) \le \big(\lambda_{P_j} - \lambda_{T_{j,\min}} + \epsilon/(2c_V)\big)\delta$$

by (3.8). Since P_j is a current in α_j, we get $\lambda_{P_j} \le c\|\alpha_j\|$ for some positive constant c independent of α_j. Hence, (3.6) follows.

We have proved that there is a Kähler current with analytic singularities Q_j in $\gamma_{j,\min} - c_\delta\beta$ such that $\widehat{V} \not\subset I_{Q_j}$. It follows that Q_j has no mass on $\widehat{V}$. Using this and Lemma 2.5 yields that the currents with minimal singularities in $\gamma_{j,\min} - c_\delta\beta$ has no mass on $\widehat{V}$. The last desired assertion is also immediate because the polar locus of $Q_j - \frac{\delta\epsilon}{2c_V}\widehat{\omega}$ does not contain $\widehat{V}$. This finishes the proof. □

End of the proof of Theorem 1.1. Let

$$b_j := \lambda_{T_j} - \lambda_{T_{j,\min}} \geq 0.$$

Note that $\gamma_j = \gamma_{j,\min} - b_j\beta$. Suppose on contrary that $b_j > 0$ for every j. Recall that we are assuming that V is smooth. The case where V is singular is dealt with later.

Let c_δ be the constant associated to a number $\delta \in (0,1)$ as in Lemma 3.3. Let c be the constant appearing in (3.6). Put

$$\delta_j := \big(c\|\alpha_j\| + \frac{\epsilon}{2c_V}\big)^{-1} b_j$$

for $1 \leq j \leq m$. Note that since $b_j \lesssim \|\alpha_j\|$, we can increase c in order to have $\delta_j \in (0,1)$. By (3.6), we get $c_{\delta_j} \leq b_j$ for every j. Let $\gamma'_{j,\min} := \gamma_{j,\min} - c_{\delta_j}\beta$. By Lemma 3.3 and the fact that

$$I_{\gamma'_{j,\min}-\gamma_j} = I_{(b_j - c_{\delta_j})\beta} \subset \widehat{V},$$

we obtain that the currents with minimal singularities in $\gamma'_{m,\min}$ has no mass on $I_{\gamma'_{j,\min}-\gamma_j}$. This combined with Proposition 2.4 (iii) gives

$$\{\langle \wedge_{j=1}^m \eta_j\rangle\} \leq \langle \wedge_{j=1}^{m-1}\gamma_j \dot\wedge \gamma_m\rangle \leq \langle \wedge_{j=1}^{m-1}\gamma_j \dot\wedge \gamma'_{m,\min}\rangle \leq \langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge \gamma'_{m,\min}\rangle.$$

Using the supper-additivity of products of classes (Proposition 2.4 (ii)), we get

$$\langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge \gamma'_{m,\min}\rangle \leq \langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge \gamma_{m,\min}\rangle - c_{\delta_m}\langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge \beta\rangle$$

Let I be the first term in the right-hand side in the last inequality. Recall that the currents with minimal singularities in $\gamma_{m,\min}$ has no mass on $\widehat{V}$. The last set contains I_β. Hence, using Proposition 2.4 (iii) implies

$$I \leq \langle \wedge_{j=1}^{m-1}\gamma_{j,\min} \dot\wedge \gamma_{m,\min}\rangle.$$

Consequently,

$$\begin{aligned}\langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge \gamma'_{m,\min}\rangle &\leq \langle \wedge_{j=1}^{m-1}\gamma_{j,\min} \dot\wedge \gamma_{m,\min}\rangle - c_{\delta_m}\langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge \beta\rangle \\ &\leq \langle \wedge_{j=1}^{m}\gamma_{j,\min}\rangle - c_{\delta_m}\langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge [\widehat{V}]\rangle\end{aligned}$$

by Lemma 3.2. Now let Φ be a closed smooth positive $(n-m, n-m)$-form on X. Put $M_j := \frac{\delta_j \epsilon}{2c_V}$. Note that by (3.6), we get $M_j \leq c_{\delta_j}$ for every j. Taking into account Lemma 3.3 and Proposition 2.4 (iii), we see that

$$\int_{\widehat{X}} \langle \wedge_{j=1}^{m-1}\gamma'_{j,\min} \dot\wedge [\widehat{V}]\rangle \wedge \sigma^*\Phi \geq M_1 \cdots M_{m-1} \int_{\widehat{V}} \widehat{\omega}^{m-1} \wedge \sigma^*\Phi.$$

Consequently, we obtain

$$\begin{aligned}\int_X \langle \wedge_{j=1}^m T_j\rangle \wedge \Phi &= \int_{\widehat{X}} \langle \wedge_{j=1}^m \eta_j\rangle \wedge \sigma^*\Phi \\ &\leq \int_{\widehat{X}} \langle \wedge_{j=1}^m \gamma_{j,\min}\rangle \wedge \sigma^*\Phi - M_1 \cdots M_m \langle [\widehat{V}] \wedge \sigma^*\Phi, \widehat{\omega}^{m-1}\rangle\end{aligned} \tag{3.9}$$

which is, by (3.3), equal to

$$\int_X \langle \wedge_{j=1}^m T_{j,\min} \rangle \wedge \Phi - M_1 \cdots M_m \langle [\widehat{V}] \wedge \sigma^* \Phi, \widehat{\omega}^{m-1} \rangle.$$

Using this and the hypothesis that

$$\int_X \langle \wedge_{j=1}^m T_j \rangle \wedge \Phi = \int_X \langle \wedge_{j=1}^m T_{j,\min} \rangle \wedge \Phi, \tag{3.10}$$

we infer that $[\widehat{V}] \wedge \sigma^* \Phi = 0$ for every closed smooth $(n-m, n-m)$-form Φ. The last property means that $[V] \wedge \Phi = 0$ for every closed smooth $(n-m, n-m)$-form Φ. By choosing $\Phi := \omega^{n-m}$, we obtain a contradiction because $\dim V \geq n - m$. This finishes Step 1 of the proof. We observe that we didn't fully use the assumption that $\{\langle \wedge_{j=1}^m T_j \rangle\} = \{\langle \wedge_{j=1}^m T_{j,\min} \rangle\}$. We only needed that there is a closed positive smooth $(n-m, n-m)$-form Φ on X such that (3.10) holds and $[V] \wedge \Phi \neq 0$. We will use this remark in the next paragraph.

We now explain how to treat the case where V is not necessarily smooth. By Hironaka's desingularization, there is $\sigma' : X' \to X$ which is a composition of consecutive blowups along smooth centers starting from X so that the centers don't intersect the regular part of V and the strict transform V' of V by σ' is smooth. Note that V' is of the same dimension as V.

Let $T'_j := \sigma'^* T_j$ and $\alpha'_j := \sigma'^* \alpha_j$. One should note that $T'_1, \ldots, T'_m$ might not be of full mass intersection, however, we still have

$$\int_X \langle \wedge_{j=1}^m T'_j \rangle \wedge \sigma'^* \Phi = \int_X \langle \wedge_{j=1}^m \alpha_j \rangle \wedge \Phi = \int_{X'} \langle \wedge_{j=1}^m \alpha'_j \rangle \wedge \sigma'^* \Phi, \tag{3.11}$$

for every closed smooth $(n-m, n-m)$-form Φ on X. We will use $\Phi := \omega^{n-m}$. Observe that

$$[V'] \wedge \sigma'^* \Phi \neq 0$$

because σ' is a biholomorphism on an open Zariski set containing the regular part of V and $[V] \wedge \Phi \neq 0$ (here we use $\dim V \geq n - m$). This together with (3.11) and the observation at the end of Step 1 allows us to apply Step 1 to X', α'_j and T'_j to obtain that there exist an index j_0 such that

$$\nu(T'_{j_0}, V') = \nu(\alpha'_{j_0}, V').$$

On the other hand, by construction of σ', we get $\nu(T'_j, V') = \nu(T_j, V)$ for every j, a similar property also holds for $T_{j,\min}$. It follows that

$$\nu(T_{j_0}, V) = \nu(\alpha'_{j_0}, V') \leq \nu(T'_{j_0,\min}, V') = \nu(T_{j_0,\min}, V) \leq \nu(T_{j_0}, V).$$

Hence, we get $\nu(T_{j_0,\min}, V) = \nu(T_{j_0}, V)$. This finishes the proof. □

We now present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let ω be a fixed Kähler form on X. Observe that by homogeneity, in order to prove the desired inequality, it suffices to consider $\alpha_j/\|\alpha_j\|$ in place of α_j. Hence, from now on, without loss of generality, we can assume that $\alpha_j \in \mathscr{B}_0 \cap \mathscr{S}$, where $\mathscr{S}$ is the unit sphere in $H^{1,1}(X,\mathbb{R})$. Since $\mathscr{B}_0$ is closed and contained in the big cone, we deduce that $\mathscr{B}_0 \cap \mathscr{S}$ is compact in the big cone. It follows that there exist a constant $\epsilon > 0$ such that for every $\alpha \in \mathscr{B}_0 \cap \mathscr{S}$, there exists a current with analytic singularities $P \in \alpha$ such that $P \geq \epsilon\omega$. In particular, we obtain currents with analytic singularities $P_j \in \alpha_j$ such that $P_j \geq \epsilon\omega$ for $1 \leq j \leq m$.

Now, we follow the arguments in the proof of Theorem 1.1. One only needs to review carefully the constants involving in estimates used there. Our submanifold V is now the point set $\{x_0\}$. Let the notations be as in the proof of Theorem 1.1. By the construction of $\widehat{X}$, the constant $c_V > 0$ in (3.2) can be chosen to be independent of x_0. As in the proof of Theorem 1.1, put

$$b_j := \nu(T_j, x_0) - \nu(\alpha_j, x_0), \quad \delta_j := \big(c\|\alpha_j\| + \frac{\epsilon}{2c_V}\big)^{-1} b_j, \quad M_j := \frac{\delta_j \epsilon}{2c_V}$$

for $1 \leq j \leq n$, where c is a constant big enough depending only on X (and a fixed Kähler form ω on X and a fixed norm on $H^{1,1}(X,\mathbb{R})$). Since $\alpha_1, \ldots, \alpha_n \in \mathscr{B}_0 \cap \mathscr{S}$, we get

$$\delta_j \gtrsim b_j,$$

and the constant ϵ can be chosen independent of $\alpha_1, \ldots, \alpha_n$. Using (3.9) for Φ to be the constant function equal to 1 gives

$$\int_X \big(\langle \wedge_{j=1}^n \alpha_j \rangle - \{\langle \wedge_{j=1}^n T_j \rangle\}\big) \geq M_1 \cdots M_n = \frac{\delta_1 \epsilon}{2c_V} \cdots \frac{\delta_n \epsilon}{2c_V} \gtrsim b_1 \cdots b_n.$$

The proof is finished. □

Example 3.4. *Let Y be a compact Kähler manifold and θ be a semi-positive $(1,1)$-form in Y such that there is a current P in $\{\theta\}$ with $\nu(P, x_0) > 0$ for some $x_0 \in Y$ (one can take, for example, Y to be the complex projective space and θ to be its Fubini-Study form). Let $X := Y^2$ and $\alpha := \pi_1^*\{\theta\}$ which is a semi-positive class, where $\pi_1 : Y^2 \to Y$ is the projection to the first component. We have $\int_X \alpha^{2\dim Y} = 0$. Hence, α is not big. Let ω be a Kähler form on X. Let $\alpha_\epsilon := \alpha + \epsilon\{\omega\}$. We have*

$$\int_X \alpha_\epsilon^{2\dim Y} \to \int_X \alpha^{2\dim Y} = 0.$$

Hence, if the constant C in Theorem 1.2 were independent of $\mathscr{B}_0$, then (1.3) for x_0 would hold for $\alpha_j := \alpha_\epsilon$ and $T_j = \pi_1^* P$ for every j for some constant C independent of ϵ. Letting $\epsilon \to 0$ gives a contradiction because the left-hand side converges to 0, whereas the right-hand side converges to a positive constant.

Proof of Corollary 1.4. We explain how to obtain Corollary 1.4 from Corollary 1.3. Let $\rho : X' \to X$ be a smooth modification of X and E an irreducible hypersurface in X'. Let

$\varphi' := \varphi \circ \rho$, $\varphi'_{\alpha,\min} := \varphi_{\alpha,\min} \circ \rho$, $\theta' := \rho^*\theta$ and $\alpha' := \rho^*\alpha$. Since non-pluripolar products have no mass on pluripolar sets, we have

$$\langle (dd^c\varphi' + \theta')^n \rangle = \langle \alpha'^n \rangle = \langle \alpha^n \rangle > 0,$$

and a similar equality also holds if φ' is replaced by $\varphi'_{\alpha,\min}$ (note that we don't know if the latter is a quasi-psh function with minimal singularities in α'; anyway we will only need that $\varphi'_{\alpha,\min}$ is of full Monge-Ampère mass in α'). By a well-known result in [6], the class α' is big.

Applying Corollary 1.3 to $dd^c\varphi' + \theta'$ and $V := E$, we obtain that the generic Lelong number of φ' along E is equal to $\nu(\alpha', E)$. We also get an analogous property for $\varphi'_{\alpha,\min}$ by applying Corollary 1.3 to $dd^c\varphi'_{\alpha,\min} + \theta'$. It follows that the generic Lelong numbers of φ' and $\varphi'_{\alpha,\min}$ along E are equal. Now using this property and [4, Corollary 10.18] (or [8, Theorem A]) gives the desired assertion. The proof is finished. □

We end the paper with the example mentioned in Introduction.

Example 3.5. *Let $X := \mathbb{P}^n$ and $[x_0 : x_1 : \cdots : x_n]$ the homogeneous coordinates. Let ω be the Fubini-Study form on $\mathbb{P}^n$. Let $2 \le m \le n$ be an integer. Consider*

$$V := \big\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n : x_j = 0, \quad 0 \le j \le m-1 \big\},$$

and

$$T_j := dd^c(|x_0|^2 + \cdots + |x_{m-1}|^2) = dd^c \frac{|x_0|^2 + \cdots + |x_{m-1}|^2}{|x_0|^2 + \cdots + |x_n|^2} + \omega$$

for $1 \le j \le m-1$. We have $\dim V = n - m$. Put $T_m := \omega$. Observe that the currents $T := T_1 \wedge \cdots \wedge T_m$ and $T' := T_1 \wedge \cdots \wedge T_{m-1}$ are well-defined (classically) by [15, Corollary 4.11, Page 156]. Moreover since V is of dimension $n-m$, and T' is of bi-dimension $(n-m+1, n-m+1)$, we see that the trace measure of T' has no mass on V by [15, Page 141]. This combined with the fact that T_m is smooth yields that the trace measure of T also has no mass on V. Using this and the fact that T_j is smooth outside V, we obtain

$$T = \langle T_1 \wedge \cdots \wedge T_m \rangle$$

(both sides have no mass on V). It follows that $T_1, \ldots, T_m$ are of full mass intersection, but $\nu(T_j, V) > 0$ for $1 \le j \le m-1$, and $\nu(T_m, V) = 0$.

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