The spectrum of an operator associated with G_2-instantons with $1-$dimensional singularities and Hermitian Yang-Mills connections with isolated singularities

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Abstract

This is the first step in an attempt at a deformation theory for G_2-instantons with $1-$dimensional conic singularities. Under a set of model data, the linearization yields a Dirac operator P on a certain bundle over $\mathbb{S}^5$, called the *link operator*. As a dimension reduction, the link operator also arises from Hermitian Yang-Mills connections with isolated conic singularities on a Calabi-Yau 3-fold.

Using the quaternion structure in the Sasakian geometry of $\mathbb{S}^5$, we describe the set of all eigenvalues of P , denoted by $SpecP$. We show that $SpecP$ consists of finitely many integers induced by certain sheaf cohomologies on $\mathbb{P}^2$, and infinitely many real numbers induced by the spectrum of the rough Laplacian on the pullback endomorphism bundle over $\mathbb{S}^5$. The multiplicities and the form of an eigensection can be described fairly explicitly.

In particular, there is a relation between the spectrum on $\mathbb{S}^5$ to certain sheaf cohomologies on $\mathbb{P}^2$.

Moreover, on a Calabi-Yau $3-$fold, the index of the linearized operator for admissible singular Hermitian Yang-Mills connections is also calculated, in terms of these sheaf cohomologies.

Using the representation theory of $SU(3)$ and the subgroup $S[U(1)\times U(2)]$, we show an example in which $SpecP$ and the multiplicities can be completely determined.

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1 Introduction

1.1 Overview

G_2-instantons and projective G_2-instantons are the analogue of both flat connections in dimension 3, and anti self-dual connections in dimension 4. Understanding their singularities is important for the programs proposed by Donaldson-Thomas [12] and Donaldson-Segal [10]. In conjunction with Jacob-Walpuski [20], to construct (projective) G_2-instantons with $1-$dimensional singularities on twisted connected-sum G_2-manifolds via gluing, an important step is a Fredholm theory for the linearized operator. In [42], it is shown that a Fredholm theory and consequently a deformation theory always exist for instantons with isolated singularities. However, instantons with $1-$dimensional singularities are expected to be different. The linearized operator yields a self-adjoint elliptic operator P on the domain bundle over $\mathbb{S}^5$. It is also referred to as *the link operator*. The set of all eigenvalues of P, denoted by $SpecP$, is crucial in the construction of a deformation theory. It determines the indicial roots.

In this paper we describe $SpecP$ and the multiplicities. We relate the eigenvalues and eigenspaces of P on $\mathbb{S}^5$ to certain sheaf cohomologies on $\mathbb{P}^2$.

1.2 Context and motivations

1.2.1 Gauge theoretic moduli spaces and gluing constructions

The Donaldson-Thomas invariant [12] on a Calabi-Yau $3-$folds "counts" the Gieseker stable sheaves in a certain sense. The definition by Thomas [37] employs the virtual moduli cycle theory constructed by Li-Tian [24]. Please also see Behrend-Fantechi [3] theory via the intrinsic normal cone. In the $7-$dimensional G_2-setting, Donaldson-Segal [10] propose to "count" G_2-instantons, which can be viewed as a perspective generalization of the Casson invariant for $3-$manifolds and Donaldson-Thomas invariant. Also see the work of Walpuski [41]. The lack of general algebraic geometry in this setting means that we have to rely to a large extent on differential geometry. However, the moduli of smooth G_2-instantons is not expected to be compact. According to the compactification by Tian [40], besides the bubbling along an associative rectifiable "submanifolds", essential singularities of dimension 1 and 0 are expected to appear. Tian conjectured that their Hausdorff co-dimension is ≥ 6. This possible phenomenon should be based on the removal of singularities by Tao-Tian [39]. Please also see the work of Smith-Uhlenbeck [34] on Yang-Mills Higgs equations. A step in understanding the 'boundary" of the compactified moduli is a Fredholm theory for G_2-instantons with $1-$dim conic singularities. However, no example is known except product ones. A Fredholm theory is also inevitable toward a gluing construction of such examples. Please see the discussion by Jacob-Walpuski [20].

The difficulty is that the linearization is expected to have infinite dimensional co-kernel caused by non-regularizable indicial roots. Please see the work of Chen [8]. On singular G_2-instantons, we expect a relation between the essential obstruction and the eigenspace of the eigenvalue -1 of the link operator. For classical literature, please see the work conducted by Lockhart-McOwen [25], Melrose-Mendoza [28], Mazzeo [26], and Mazzeo-Vertman [27]. In order to construct a "remedial" deformation, it is necessary to understand the eigenvalues of the link operator P and the eigenspaces. This is our purpose here.

1.2.2 Spectral theory and Sasakian geometry

Given a Dirac operator i.e. the square is a generalized Laplacian (see [30]), it is tempting to know the precise values for some of the eigen-values, and also to describe some of the eigen-sections. For example, we know explicitly the eigenvalues and eigen-functions of the fundamental Dirac operator $i\frac{\partial}{\partial\theta}$ on the circle differentiating complex valued functions. More generally, under appropriate homogeneous conditions, the spectrum can be satisfactorily understood, thanks to the Peter-Weyl formulation. The scheme uses representation theory and Casimir operators of certain Lie groups. For Hodge Laplacians on spheres and complex projective spaces, pleasee see for example the work of Ikeda-Taniguchi [18] and Gallot-Meyer [14]. Beyond the homogenous setting, in general, it is challenging.

There are more recent investigations that are closer related to our theme here. Moroianu–Semmelmann [29] calculated the deformation space of homogeneous nearly Kähler $6-$folds. In their work, the nearly Kähler deformations are identified with co-closed forms in an eigenspace of the Hermitian Laplace operator. The spectrum and eigenspaces of this operator are calculated for homogeneous examples. Via spinorial point of view on nearly Kähler $6-$folds, Charbonneau-Harland [7] identified the deformation space of nearly Kähler instantons with a subspace of the kernel of an index 0 Dirac operator. They showed that abelian instantons are rigid. Via Peter-Weyl formalism, they characterized deformations of canonical connections on homogeneous nearly Kähler $6-$folds and also other eigenvalues and eigen-spaces.

There are numerical algorithms for the spectrum of Laplace-Beltrami operators. For example, please see the work of Braun-Brelidze-Douglas-Ovrut [5] on certain Calabi-Yau $3-$folds. This is related to the goal in computing all the observable quantities of particle physics in certain physics settings.

Our link operator lives on a domain bundle $Dom_{\mathbb{S}^5}$ over $\mathbb{S}^5$, induced by a holomorphic Hermitian vector bundle $E \to \mathbb{P}^2$. The quaternion structure in the Sasakian geometry

$$U(1) \to \mathbb{S}^5 \to \mathbb{P}^2$$

yields a *fairly explicit characterization of all the eigen-values and eigen-sections of P, without assuming homogeneity of E.* We wonder whether *the relation we established, between the spectrum of P on $\mathbb{S}^5$ and sheaf cohomologies on $\mathbb{P}^2$*, can be generalized to more operators on regular or even quasi regular Sasakian manifolds in general dimensions. The idea underpin the above spectral reduction is called the *Sasakian Fourier-Series.* It is the orbit-wise Fourier series under the Reeb-action on $\mathbb{S}^5$, even if the $U(1)-$ bundle is non-trivial. The "Fourier-co-efficients" are no longer functions, but are sections of the twisted bundles over $\mathbb{P}^2$. Particularly, a smooth complex-valued function on $\mathbb{S}^5$ can be represented by such a series. Though we do not address it here, we are not surprised if it works in general dimensions.

If $E \to \mathbb{P}^2$ is a twisted tangent bundle, we can determine *all the eigen-values and eigen-sections* of P on $\mathbb{S}^5$. This can be understood as of "co-homogeneity 1". The calculation requires both representation theory and the spectral reduction.

1.2.3 Cohomologies and Ext groups

The link operator also appears in the linearization of singular Hermitian Yang-Mills connections on a Calabi-Yau $3-$fold. Our index calculation (Theorem B) suggests that the difference between $Ext^1_{O_{CY^3}}(\mathcal{F},\mathcal{F})$ and $H^1[CY^3, End\mathcal{F}]$ might be related to the cohomologies of the twisted endomorphism bundles on $\mathbb{P}^2$ for the model of $\mathcal{F}$ near each singular point. Please see the work of Jacob-Sá Earp-Walpuski [19]. Moreover, it is tempting to ask whether $Ext^1_{O_{CY^3}}(\mathcal{F},\mathcal{F})$ and $H^1[CY^3, End\mathcal{F}]$ are related to kernel of the linearized operator $\square_{CY^3}$ in certain weighted spaces. These questions can be generalized to reflexive sheaves with more complicated singularities. Please see the work of Chen-Sun [9].

1.3 Main result

Given a holomorphic Hermitian vector bundle E on $\mathbb{P}^2$ with the Chern connection A, let $\nabla^\star\nabla|_{\mathbb{S}^5}$ denote the rough Laplacian on the pullback adjoint bundle on $\mathbb{S}^5$ associated to the standard round metric and pullback connection. We define the following subsets of $\mathbb{R}$.

- $S_{\nabla^\star\nabla} \triangleq \{\mu|(\mu^2+2\mu-3)\in Spec(\nabla^\star\nabla|_{\mathbb{S}^5})\}\cup\{\mu|\mu^2+4\mu\in Spec(\nabla^\star\nabla|_{\mathbb{S}^5})\}$.
- $S_{coh} \triangleq \{l|l$ is an integer and $H^1[\mathbb{P}^2,(EndE)(l)]\neq 0\}$.

The set $S_{\nabla^\star\nabla}$ is infinite and given by the bundle rough Laplacian on $\mathbb{S}^5$. The set S_{coh} is finite and given by the sheaf cohomologies on $\mathbb{P}^2$. Intuitively speaking, under natural assumptions, our main theorem says $SpecP$ is their union.

Theorem A. *Let $(E,A)\to\mathbb{P}^2$ be a non-projectively flat Hermitian Yang-Mills bundle of rank $r\geq 2$. Let P be the link operator of the linearization of G_2-instantons with circle singularities, defined in Lemma 4.3 below under the model setting.*

Spectrum

- *If $(E,\overline{\partial}_A)$ is stable, then*
$$SpecP = S_{\nabla^\star\nabla}\cup S_{coh}. \tag{1}$$
- *If $(E,\overline{\partial}_A)$ is poly-stable but not stable, then*
$$SpecP = [S_{\nabla^\star\nabla}\setminus\{0,-3\}]\cup S_{coh}. \tag{2}$$

Consequently, in both cases,

- *the set $(SpecP)\cap(-3,0)$ contains and only contains the two numbers $-1,\ -2$.*
- *Let l_0 be the smallest positive integer such that $H^0[\mathbb{P}^2,(EndE)(l_0)]\neq\{0\}$. For any integer $l\geq l_0$, the integers*
$$l+1,\ -l-3,\ l,\ -l-4$$
are eigenvalues of P.

Multiplicities

There is an isometry I on the domain bundle of P that induces an almost complex structure on an arbitrary eigen-space. A real number μ is an eigenvalue of P if and only if $-3-\mu$ is, and their eigen-spaces are conjugate complex isomorphic. Suppose $\mu\in SpecP$.

- *If μ is not an integer, then $Eigen_\mu P$ is isomorphic to*
$$[Eigen_{\mu^2+2\mu-3}(\nabla^\star\nabla|_{\mathbb{S}^5})]^{\oplus 2}\oplus[Eigen_{\mu^2+4\mu}(\nabla^\star\nabla|_{\mathbb{S}^5})]^2.$$
- *If 1 and -4 are eigenvalues of P, then*
$$Mult_1P = Mult_{-4}P = 2dimKer(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2Mult_5(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2c_2(EndE) - 6(r^2-1).$$

- *If* $m = 0,\ -1,\ -2,$ *or* -3*, then the eigenspace* $Eigen_m P$ *is complex isomorphic to*
$$H^1[\mathbb{P}^2, (EndE)(m)].$$
- *Suppose* μ *is an integer and* $\mu \neq 1,\ 0,\ -1,\ -2,\ -3$ *or* -4.
If $\mu \in S_{coh}$ *but* $\mu \notin S_{\nabla^\star\nabla}$*, then* $Eigen_\mu P$ *is complex isomorphic to*
$$H^1[\mathbb{P}^2, (EndE)(\mu)].$$
If $\mu \in S_{\nabla^\star\nabla}$ *but* $\mu \notin S_{coh}$*, then*
$$\begin{aligned} Mult_\mu P &= 2Mult_{\mu^2+2\mu-3}(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2Mult_{\mu^2+4\mu}(\nabla^\star\nabla|_{\mathbb{S}^5}) \\ &\quad -2h^0[\mathbb{P}^2, (End_0E)(\mu)] - 2h^0[\mathbb{P}^2, (End_0E)(-\mu-3)]. \end{aligned}$$
If $\mu \in S_{\nabla^\star\nabla} \cap S_{coh}$*, then*
$$\begin{aligned} Mult_\mu P &= 2Mult_{\mu^2+2\mu-3}(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2Mult_{\mu^2+4\mu}(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2c_2(EndE) \\ &\quad -(r^2-1)(\mu+1)(\mu+2). \end{aligned}$$

Notation Convention 1.1. We abbreviate the multiplicity to $Mult_a(operator)$. This is the dimension of the eigenspace $Eigen_a(operator) \triangleq Ker(operator - a \cdot Id)$. The real number a does not have to be an eigenvalue but it is if and only if the eigenspace is non-trivial i.e. $Mult_a \neq 0$. Most of the time we abbreviate $Eigen_a(operator)$ even more compactly to $\mathbb{E}_a(operator)$.

The link operator has infinitely many positive and negative integer eigenvalues. In the description of multiplicities, the binomial
$$-c_2(EndE) + \frac{(r^2-1)(\mu+1)(\mu+2)}{2}$$
is the Hilbert polynomial of the twisted traceless endomorphism bundle. The Hermitian Yang-Mills condition implies that the holomorphic bundle structure $(E, \overline{\partial}_A)$ must be (slope) poly-stable. Some of our intermediate results also hold for projectively flat connections or Hermitian Yang-Mills connections on line bundles. These are regarded as trivial (Section 16).

The splittings (1) and (2) are due to that the domain Hilbert space of the link operator is the direct sum of a finite dimensional subspace given by the cohomologies and $Ker\nabla^\star\nabla_{\mathbb{S}^5}$, and an infinite-dimensional orthogonal complement (Definition 9.1). The binomials $\mu^2 + 2\mu - 3$ and $\mu^2 + 4\mu$ are given by the Bochner formulas for $P^2 + 2P - 3$ and $P^2 + 4P$, respectively (Lemma 5.4). They "correspond" to each other via the Serre duality
$$\mu \leftrightarrow -\mu - 3: \qquad H^1[\mathbb{P}^2, (EndE)(\mu)] = H^1[\mathbb{P}^2, (EndE)(-\mu-3)].$$
The Hermitian Yang-Mills condition is also required for these formulas. Non-projective flatness is required for the Chern number inequality (84) that implies -1 and -2 must be eigenvalues. Via the two binomials, any eigenvalue $\lambda \in Spec(\nabla^\star\nabla|_{\mathbb{S}^5})$ generates the following 4 numbers
$$\mu_{\lambda,+} \triangleq -1 + \sqrt{4+\lambda},\ \mu_{\lambda,-} \triangleq -1 - \sqrt{4+\lambda},\ \underline{\mu}_{\lambda,+} = -2 + \sqrt{4+\lambda},\ \underline{\mu}_{\lambda,-} = -2 - \sqrt{4+\lambda}. \quad (3)$$
Our theorem means that when $\lambda \neq 0$, all of them are eigenvalues of P. However, if 0 is an eigenvalue of $\nabla^\star\nabla|_{\mathbb{S}^5}$ i.e. if E is strictly poly-stable, it only generates eigenvalues 1 and -4 among $\{1, -4, 0, -3\}$. The other two numbers $0,\ -3$ can only be generated by co-homologies $H^1[\mathbb{P}^2, EndE]$ and $H^1[\mathbb{P}^2, (EndE)(-3)]$ if non-trivial.

Remark 1.2. The eigensections of P admit an explicit form (92) in terms of the eigensections of $\nabla^\star\nabla|_{\mathbb{S}^5}$ (Remark 10.4 and Lemma 10.2).

1.4 The index

On a stable reflexive sheaf over a Calabi-Yau 3−fold, using weighted analysis, we can formulate a deformation problem for admissible Hermitian Yang-Mills connections. It is supposed to preserve the singular points and tangent cones (Section 15). While we do not establish the full package here, the index can be calculated with the help of Theorem A. The existence of admissible Hermitian Yang-Mills connections is proved by Bando-Siu [2], generalizing Donaldson-Uhlenbeck-Yau theorem [11, 38] on stable vector bundles. Jacob-Sá Earp-Walpuski [19] and Chen-Sun [9] characterized the tangent cone connections when the singularities are isolated and admissible.

Theorem B. *On a compact Calabi-Yau* 3*−fold, let $\mathcal{F}$ be a non-locally free reflexive sheaf with finitely-many non-trivial and admissible isolated singularities. Then at an admissible Hermitian connection, the index of the linearized operator* (144) *equals*

$$-2\Sigma_j h^1[\mathbb{P}^2, EndE_j(-1)] - 2\Sigma_j h^1[\mathbb{P}^2, EndE_j]$$

where the summation is over the singular points of the sheaf. Particularly, the index is always negative.

The appearance of the dimension of deformation space $h^1[\mathbb{P}^2, EndE_j]$ is consistent with that our setup (144) is supposed to preserve the tangent connections at the singularities.

This negative index and the discussion in [17] on vanishing of c_3 suggest that, at least for rank 2, we can ask whether these reflexive sheaves appear generically in the gauge theoretic compactification of the moduli space.

1.5 The example

As another application, if E is a twisted holomorphic tangent bundle of $\mathbb{P}^2$, we can completely determine $SpecP$.

Theorem C. *Let (E, A) be a twisted holomorphic tangent bundle $T^{1,0}\mathbb{P}^2(k)$ with the twisted Fubini-Study connection and metric. Then*

$$S_{coh} = \{-1\} \cup \{-2\}.$$

Consequently, $SpecP = S_{\nabla^\star\nabla} \cup \{-1\} \cup \{-2\}$. The first row of the following table contains all the eigenvalues of P in the closed interval $[-4, 1]$. The second row addresses the multiplicity of each.

eigenvalue of P	-4	$-2\sqrt{2}-1$	*-2*	*-1*	$2\sqrt{2}-2$	*1*
multiplicity	*12*	*16*	*6*	*6*	*16*	*12*

(4)

The other eigenvalues and their multiplicities are also determined by Theorem A and D.

Theorem D below determines the spectrum of the rough Laplacian. Though we do not know a direct homogeneous formulation on $\mathbb{S}^5$, we have a homogeneous formulation on $\mathbb{P}^2$ thanks to the spectral reduction in Lemma 8.3 below. Moreover, both Fubini-Study connection and the Chern connection of the standard metric on $O(l) \to \mathbb{P}^2$ are $SU(3)-$homogeneous and given by the same horizontal distribution in the Lie algebra $su(3)$ (Lemma 12.1,12.2, and Proposition 12.3). Then Peter-Weyl theory applies. A little bit of algebraic geometry (Lemma 14.2) determines S_{coh}.

1.6 Sketch of the proof for Theorem A

- We establish a fine formula for the link operator (Lemma 4.3) according to the Sasaki-quaternion structure on $\mathbb{S}^5$. This is a special case of a Sasaki-Einstein $SU(2)-$structure mentioned in [13, 4.1]. We need more explicit information.
- Then we establish two Bochner formulas (Lemma 5.4) for the link operator using the Hermitian Yang-Mills condition. The observation is that the first and second rows of $P^2 + 2P$ are "autonomous" i.e. they are independent of the unknowns corresponding to the other rows. The same is true for the third and fourth rows of $P^2 + 4P$.
- We found two finite dimensional invariant subspaces of P. One is identified with direct sum of sheaf cohomologies and the other one is generated by kernel of the rough Laplacian on $adE \to \mathbb{S}^5$ i.e. bundle of trace-less skew-Hermitian endomorphisms of E. It suffices to calculate $SpecP$ on the orthogonal complement. Then a "spectral separation" identifies it with the $S^0_{\nabla^\star\nabla}$ below.
- For multiplicities, we construct a projection map (Definition 10.1) from a direct sum of two eigen-spaces of the bundle rough Laplacian to a given eigen-space of the restricted P. This map admits an explicit formula (92) as well, leading to its surjectivity and that the kernel is isomorphic to a direct sum of H^0 of certain twisted endomorphism bundles.

The paper is organized as follows. In Sections 2—4, we fully employ the Sasaki-Quaternion structure of $(\mathbb{C}^3 \setminus O) \to \mathbb{S}^5$ to prove the fine formula (Lemma 4.3) for the operator P. We establish the Bochner formulas in Section 5 regarding the two binomials. The Sasakian-Fourier series is defined in Section 6. We identify the finite dimensional $P-$invariant subspace to a directly sum of cohomologies in Section 7. In Section 8 we prove the spectral reduction for the bundle rough Laplacian on $\mathbb{S}^5$. We prove Theorem A (spectrum) in Section 9, and determine the multiplicities in Section 10. In Sections 11—14, we employ the spectral reduction and representation theory for $SU(3)$ to determine $SpecP$ assuming E is a twisted tangent bundle of $\mathbb{P}^2$, and prove Theorem C, D. Section 15 is devoted to the Fredholm index on a Calabi-Yau $3-$fold. The Appendix collects some results obtained by routine calculations.

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2 The Sasakian geometry of $\mathbb{S}^5$

The purpose of this section, aiming at our fine formula for the link operator, is to collect some pedestrian facts on the Sasakian geometry of $\mathbb{S}^5$ in an elementary way. For more sophisticated monographs, please see [36] and [4].

2.1 General conventions

We have a chain of natural fibration maps

$$(\mathbb{C}^3 \setminus O) \times \mathbb{R} \text{ or } (\mathbb{C}^3 \setminus O) \times \mathbb{S}^1 \to \mathbb{C}^3 \setminus O \to \mathbb{S}^5 \to \mathbb{P}^2. \tag{5}$$

Except the Hopf fibration represented by the last arrow, the others are all topologically trivial fibrations. To avoid heavy notations, we adopt the following.

Convention 2.1. Unless otherwise specified, *all pullback bundles, connections, (bundle-valued) forms are denoted the same.*

For example, the contact form

$$\eta \triangleq d^c \log r \triangleq \sqrt{-1}(\bar{\partial} - \partial) \log r \tag{6}$$

originally defined on $\mathbb{C}^3 \setminus O$ also means the one on $\mathbb{S}^5$. Throughout, we consider the Fubini-Study form $\frac{d\eta}{2}$ on $\mathbb{P}^2$, the standard round metric on $\mathbb{S}^5$, and the Euclidean metrics on $\mathbb{C}^3 \times \mathbb{S}^1$ and $\mathbb{C}^3$.

Similar abusing of notations also applies to differential operators. For example, in (23) below, the exterior derivative on $\mathbb{P}^2$, denoted by $d_{\mathbb{P}^2}$, also means the local pullback operator on $\mathbb{S}^5$.

Definition 2.2. Let $m,\ n$ be two integers among $7, 6, 5, 4$ and $m > n$. Let $\pi_{m,n}$ denote the right-bound fibration map in (5) from the m dimensional manifold and the $n-$dimensional therein, where m and n mean the real dimensions. For example, the map $\pi_{5,4}$ is the Hopf-fibration map $\mathbb{S}^5 \to \mathbb{P}^2$.

Given a Riemannian manifold (M, g), for any tangent vector $v_x \in T_x M$, let $v_x^{\sharp_M}$ denote the metric dual form in $T_x^\star M$. Conversely, for any $1-$form $\theta_x \in T_x^\star M$, let $\theta_{x,\sharp_M}$ denote the metric dual vector in $T_x M$. Given a $1-$form $h \in T_x^\star M$ and a $p-$form $\Omega \in \wedge^p T_x^\star M$, $p \geq 1$, we define the metric contraction by $h \lrcorner_g \Omega \triangleq h_{\sharp_M} \lrcorner \Omega$.

The superscript $\mathbb{C}$ on a (real) vector-bundle (vector space) means the complexification. Associated with the Riemannian metric, in the below, the tensor operators $\lrcorner$ (contraction), $^\sharp$ (pulling up), $_\sharp$ (pushing down), $^{\|_X}$ (projection onto a real vector field X), $P_{X^\perp}$ (projection onto the orthogonal complement of a real vector field X), and the star operators $\star_0$, $\star_{\mathbb{P}^2}$ etc are all extended $\mathbb{C}-$linearly to the complexified tangent and co-tangent bundle.

2.2 Sasakian coordinate system

The purpose of this section is to define the Sasakian coordinate.

We denote the contact distribution $Ker\eta$ by D, and call $D^\star \triangleq \eta^\perp$ the contact co-distribution. Let ξ denote the standard Reeb vector field on $\mathbb{S}^5$ that is tangential to orbit of the $U(1)-$multiplications.

The Fubini-Study form $\frac{d\eta}{2} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)$ induces the Fubini-Study metric on $\mathbb{P}^2$, and also the pullback metric on the contact co-distribution $D^\star$. Henceforth, *we unanimously call the metrics on $\mathbb{P}^2$, D, $D^\star$ the Fubini-Study metric.*

The contact co-distribution $D^\star$ is equal to the pullback of the (real) co-tangent bundle $T^\star\mathbb{P}^2$. On the other hand, because $\pi_{5,4}$ is a Riemannian submersion, the tangent map $\pi_{5,4,\star}$ is an isometry $D \to T\mathbb{P}^2$. We split orthogonally the tangent bundle of $\mathbb{C}^3 \setminus O$ as

$$T^{\mathbb{C}}(\mathbb{C}^3 \setminus O) = span(\frac{\partial}{\partial r}, \xi) \oplus \pi_{6,4}^\star D^{\mathbb{C}}. \tag{7}$$

Similarly, the tangent bundle of the $7-$dimensional $(\mathbb{C}^3 \setminus O) \times \mathbb{S}^1$ splits orthogonally as

$$T^{\mathbb{C}}[(\mathbb{C}^3 \setminus O) \times \mathbb{S}^1] = span(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}, \xi) \oplus \pi_{7,4}^\star D^{\mathbb{C}}. \tag{8}$$

The coordinate neighborhoods

For any $\beta = 0, 1$, or 2, let $U_{\beta,\mathbb{P}^2},\ U_{\beta,\mathbb{S}^5},\ U_{\beta,\mathbb{C}^3}$ be the subset of $\mathbb{P}^2,\ \mathbb{S}^5,\ \mathbb{C}^3$ respectively defined by $Z_\beta \neq 0$. We recall the complex coordinate functions:

Coordinate neighborhoods	Part of the coordinate functions
$U_{0,\mathbb{P}^2},\ U_{0,\mathbb{S}^5},\ U_{0,\mathbb{C}^3}$	$u_1 = \frac{Z_1}{Z_0},\ u_2 = \frac{Z_2}{Z_0}$
$U_{1,\mathbb{P}^2},\ U_{1,\mathbb{S}^5},\ U_{1,\mathbb{C}^3}$	$v_0 = \frac{Z_0}{Z_1},\ v_2 = \frac{Z_2}{Z_1}$
$U_{2,\mathbb{P}^2},\ U_{2,\mathbb{S}^5},\ U_{2,\mathbb{C}^3}$	$w_0 = \frac{Z_0}{Z_2},\ w_1 = \frac{Z_1}{Z_2}$

(9)

In $U_{0,\mathbb{S}^5}$, the complexification $D^{\star,\mathbb{C}}$ is spanned by $du_1,\ du_2,\ d\bar{u}_1,\ d\bar{u}_2$ everywhere. Define the real coordinates $(x_i,\ i = 1, 2, 3, 4)$ by $u_1 = x_1 + \sqrt{-1}x_2,\ u_2 = x_3 + \sqrt{-1}x_4$. Both the real vector bundle $D^\star$ and the complex bundle $D^{\star,\mathbb{C}}$ are spanned by $dx_1,\ dx_2,\ dx_3,\ dx_4$. The same holds in $U_{1,\mathbb{S}^5}$ and $U_{2,\mathbb{S}^5}$.

The coordinate maps

Based on the above table, on $\mathbb{C}^3 \setminus O$, we define the Sasakian coordinate.

Definition 2.3. Let $Z_\beta = |Z_\beta|e^{\sqrt{-1}\theta_\beta}$, $\theta_\beta \in \mathbb{S}^1 \triangleq \mathbb{R}/2\pi\mathbb{Z}$. On $U_{\beta,\mathbb{C}^3}$, the *Sasakian coordinate* is defined to be the functions $(r, \theta_\beta, u_j, u_k)$ which is a homeomorphism from $U_{\beta,\mathbb{C}^3}$ to $\mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{C}^2$. Similarly, we also call the homeomorphism (θ_β, u_j, u_k) from $U_{\beta,\mathbb{S}^5}$ to the trivial circle bundle $\mathbb{S}^1 \times \mathbb{C}^2$ the *Sasakian coordinate*.

The Reeb vector field can be described satisfactorily.

Fact 2.4. Let $\beta = 0, 1$, or 2. The Reeb vector field ξ equals the coordinate vector field $\frac{\partial}{\partial\theta_\beta}$ in $U_{\beta,\mathbb{S}^5}$ under the Sasakian coordinate.

Proof of Fact 2.4: For any point $Z = (r, \theta_\beta, u_j, u_k) \in U_{\beta,\mathbb{C}^3}$, the scalar multiplication $e^{\sqrt{-1}t}Z$ is the translation in the angular variable: $e^{\sqrt{-1}t} \cdot (r, \theta_\beta, u_j, u_k) = (r, \theta_\beta + t, u_j, u_k)$. $\square$

2.3 The Sasaki-quaternion structure

Preliminary

We introduce a formula for the contact form under a Sasakian coordinate chart. We shall mainly work in $U_{0,\mathbb{S}^5}$, because $U_{0,\mathbb{S}^5}$ is dense and open in $\mathbb{C}^3 \setminus O$, so it suffices to prove the desired identities therein.

For any $\beta = 0,\ 1$, or 2, let the function

$$\phi_\beta \triangleq \frac{r^2}{|Z_\beta|^2} = \frac{|Z_0|^2 + |Z_1|^2 + |Z_2|^2}{|Z_\beta|^2}$$

be the Kähler potential of the Fubini-Study form $\frac{d\eta}{2}$ on $U_{\beta,\mathbb{P}^2}$. The complex coordinate function Z_β satisfies the following.

$$|Z_\beta| = \frac{r}{\sqrt{\phi_\beta}}, \quad Z_\beta = \frac{r}{\sqrt{\phi_\beta}}e^{\sqrt{-1}\theta_\beta}. \tag{10}$$

Formula 2.5. For any $\beta = 0, 1$, or 2, $\eta = d\theta_\beta + \frac{d^c \log \phi_\beta}{2}$ in $U_{\beta,\mathbb{S}^5}$.

The proof is deferred to Appendix 17.1.

The semi-basic forms G and H

We call a form θ semi-basic if $\xi \lrcorner \theta = 0$. A section of $D^\star$ is a semi-basic $1-$form. On the cone $\mathbb{C}^3 \setminus O$, the vector field $\frac{1}{2r^3}(r\frac{\partial}{\partial r} - \sqrt{-1}\xi)$ is $(1,0)$. Contracting it with the standard $(3,0)-$form on $\mathbb{C}^3$, we obtain a $(2,0)$ semi-basic form.

Lemma 2.6. *There are smooth real semi-basic $2-$forms H and G with the following properties. Let $\Omega_{\mathbb{C}^3} \triangleq dZ_0dZ_1dZ_2$ be the standard holomorphic volume form on $\mathbb{C}^3$, we have*

$$\Omega_{\mathbb{C}^3} = (r^2dr + \sqrt{-1}r^3\eta) \wedge H + (r^3\eta - \sqrt{-1}r^2dr) \wedge G, \tag{11}$$

$$Re\Omega_{\mathbb{C}^3} = r^2dr \wedge H + r^3\eta \wedge G,\ Im\Omega_{\mathbb{C}^3} = r^3\eta \wedge H - r^2dr \wedge G,\ \text{and} \tag{12}$$

$$[\frac{1}{2r^3}(r\frac{\partial}{\partial r} - \sqrt{-1}\xi)] \lrcorner \Omega_{\mathbb{C}^3} \triangleq \Theta = H - \sqrt{-1}G. \tag{13}$$

Θ is $(2,0)$ semi-basic, and $\overline{\Theta} = H + \sqrt{-1}G$ is $(0,2)$ semi basic. Under the Sasakian coordinate in $U_{0,\mathbb{S}^5}$, $U_{1,\mathbb{S}^5}$, $U_{2,\mathbb{S}^5}$ respectively, the following holds for G, H, and Θ.

In $U_{0,\mathbb{S}^5}$	$G = -\frac{1}{2\sqrt{-1}}(Z_0^3du_1du_2 - \bar{Z}_0^3d\bar{u}_1d\bar{u}_2),\ H = \frac{1}{2}(Z_0^3du_1du_2 + \bar{Z}_0^3d\bar{u}_1d\bar{u}_2),$ $\Theta = Z_0^3du_1du_2.$	(14)
In $U_{1,\mathbb{S}^5}$	$G = \frac{1}{2\sqrt{-1}}(Z_1^3dv_0dv_2 - \bar{Z}_1^3d\bar{v}_0d\bar{v}_2),\ H = -\frac{1}{2}(Z_1^3dv_0dv_2 + \bar{Z}_1^3d\bar{v}_0d\bar{v}_2),$ $\Theta = -Z_1^3dv_0dv_2.$	
In $U_{2,\mathbb{S}^5}$	$G = -\frac{1}{2\sqrt{-1}}(Z_2^3dw_0dw_1 - \bar{Z}_2^3d\bar{w}_0d\bar{w}_1),\ H = \frac{1}{2}(Z_2^3dw_0dw_1 + \bar{Z}_2^3d\bar{w}_0d\bar{w}_1),$ $\Theta = Z_2^3dw_0dw_1.$	

The proof is routine and is also deferred to Appendix 17.1. Schematically speaking, the role of Θ is analogous to the holomorphic volume form on a $K3-$surface.

We search for more properties of the forms G and H. Let $\star_0$ denote the Hodge star operator of the Fubini-Study metric on the exterior algebra of $D^\star$. Because H and G are both in $(\wedge^{(2,0)} \oplus \wedge^{(0,2)})D^{\mathbb{C},\star}$, they are $\star_0-$self-dual i.e.

$$\star_0 G = G,\ \star_0 H = H. \tag{15}$$

At the point $(1,0,0) \in \mathbb{S}^5$,

$$G = -Im(du_1du_2),\ H = Re(du_1du_2). \tag{16}$$

For any point $[Z] \in \mathbb{P}^2$, let $\chi \in SU(3)$ map $[Z]$ to $[1,0,0]$. The pullback coordinate $z_1 = \chi^\star u_1,\ z_2 = \chi^\star u_2$ is called a *Sasaki-quaternion coordinate* of the Reeb orbit $\pi_{5,4}^{-1}[Z]$. Because the Fubini-Study metric, the $(1,0)$ vector field $\frac{1}{2r^3}(r\frac{\partial}{\partial r} - \sqrt{-1}\xi)$, and the standard $(3,0)-$form $dZ_0dZ_1dZ_2$ on $\mathbb{C}^3$ are $SU(3)-$invariant, so are G and H. As forms on $\mathbb{S}^5$, they must equal the standard form

$$G = -Im(dz_1dz_2),\ H = Re(dz_1dz_2)$$

everywhere on the orbit $\pi_{5,4}^{-1}[Z]$. This coordinate also yields transverse geodesic coordinate in the sense of Lemma 17.1 below.

The Quaternion structure

Before proceeding, we stipulate the following.

Convention 2.7. Let $\lrcorner$ denote the contraction between two forms on $\mathbb{S}^5$ under the standard round metric. The same notation might also denote the usual contraction between a tangent vector and a form (without involving the Riemannian metric). Our rationale is that if a tensor operation or operator has no subscript for the domain, then it is on $\mathbb{S}^5$.

We now get into the crucial properties of G and H. At an arbitrary point on $\mathbb{S}^5$, it is straight forward to verify the following (for example, under the Sasakian-quaternion coordinate so that G and H is of the canonical form (16)).

$$(a\lrcorner G)\lrcorner G = -a,\ (a\lrcorner H)\lrcorner H = -a. \tag{17}$$

Consequently, both $\lrcorner G$ and $\lrcorner H$ are almost complex structures on $D^\star$. Henceforth, let $J_G,\ J_H$ denote $\lrcorner G,\ \lrcorner H$. Let J_0 denote $\lrcorner\frac{d\eta}{2}$. They are all isometries. The complex structures on D are defined by the metric pulling up and down i.e.

$$J_0(X) \triangleq [J_0(X^{\sharp_0})]_{\sharp_0},\ J_H(X) \triangleq [J_H(X^{\sharp_0})]_{\sharp_0},\ J_G(X) \triangleq [J_G(X^{\sharp_0})]_{\sharp_0}. \tag{18}$$

Hence the Sasaki-quaternion structure applies to the contact distribution D as well.

Based on the above, we routinely verify our main Lemma in the underlying section.

Lemma 2.8. *(The Sasaki-quaternion structure) On the contact co-distribution $D^\star \to \mathbb{S}^5$ and its pullbacks (complexification), the following holds.*

$$J_G J_H = J_0,\ J_H J_0 = J_G,\ J_0 J_G = J_H,\ J_0^2 = J_H^2 = J_G^2 = -Id. \tag{19}$$

Moreover, the Fubini-Study metric on D and $D^\star$ is preserved by each of $J_0,\ J_H,\ J_G$.

In the general setting of the first sentence in Lemma 8.3, the identities in (19) *hold for an endomorphism-valued semi-basic* $1-$*form.*

2.4 The Reeb Lie derivative and the transverse exterior derivative

To describe certain eigensections of the operator P, we need the two first order differential operators in Formula 2.9 and Definition 2.11.

Formula 2.9. Let L_ξ denote the Lie derivative in the direction of the Reeb vector field. Then the followings is true.

$$L_\xi H = 3G,\quad L_\xi G = -3H,\quad L_\xi(\frac{d\eta}{2}) = 0. \tag{20}$$

Proof of Formula 2.9: Differentiating (14) with respect to θ_0, the equalities hold everywhere in $U_{0,\mathbb{S}^5}$, therefore everywhere in $\mathbb{C}^3 \setminus O$ by continuity. $\square$

In the general setting of the first sentence in Lemma 8.3, let a_0 be a pullback $EndE-$valued semi-basic $1-$form on $\mathbb{S}^5$. The following holds by the formula for G and H (Lemma 14) and the local formula for the Reeb vector field (Fact 2.4).

$$J_G L_\xi(a_0) = L_\xi J_G(a_0) + 3J_H(a_0),\ J_H L_\xi(a_0) = L_\xi J_H(a_0) - 3J_G(a_0) \tag{21}$$

Notation Convention 2.10. Most of the time, to avoid heavy notation, for an differential operator on the bundle, we shall suppress the subscript for the connection.

We now turn to the definition of a derivative operator with respect to the contact co-distribution $D^\star$.

Definition 2.11. In the general setting of the first sentence in Lemma 8.3, on the pullback bundle $\wedge^p D^\star \otimes EndE \to \mathbb{S}^5$ of endomophism-valued semi-basic $p-$forms, the transverse exterior derivative operator d_0 is defined by

$$d_0 \triangleq d - \eta \wedge L_\xi. \tag{22}$$

It turns out that if θ is semi-basic, so is $d_0\theta$. The Lie derivative L_ξ is well defined because the endomorphism bundle is pulled back from $\mathbb{P}^2$.

The operator d_0 admits a local splitting in the following sense.

For any $\beta = 0,1,2$, let "b" be a section (form) of $\wedge^p D^{\star,\mathbb{C}} \otimes EndE \to U_{\beta,\mathbb{S}^5}$. For any $\theta_\beta \in [0,2\pi)$, the restriction $b(\cdot,\theta_\beta)$ onto the $\theta_\beta-$slice is a form on $U_{\beta,\mathbb{P}^2}$. We define $d_{\mathbb{P}^2}b$ to be the (partial) exterior derivative in the direction of $U_{\beta,\mathbb{P}^2}$. A priori, this partial exterior derivative is not defined globally on $\mathbb{S}^5$ because it is not the product manifold $\mathbb{P}^2 \times \mathbb{S}^1$. However, the open set $U_{\beta,\mathbb{S}^5}$ is a trivial $\mathbb{S}^1-$bundle over $U_{\beta,\mathbb{P}^2} \subset \mathbb{P}^2$. Employing Formula 2.5 for η, this leads to the following splitting of d_0 in $U_{\beta,\mathbb{S}^5}$.

$$\begin{aligned} d_0 b &= db - \eta \wedge L_\xi b = d_{\mathbb{P}^2} b + d\theta_\beta \wedge L_\xi b - \eta \wedge L_\xi b \\ &= d_{\mathbb{P}^2} b - \frac{1}{2}(d^c \log \phi_\beta) \wedge L_\xi b. \end{aligned} \quad (23)$$

In view of the above decomposition, we have the further splitting

$$d_0 = \partial_0 + \bar{\partial}_0, \quad \text{where}$$
$$\partial_0 = \partial_{\mathbb{P}^2} + \frac{\sqrt{-1}}{2}(\partial_{\mathbb{P}^2} \log \phi_\beta) \wedge L_\xi, \text{ and} \qquad \bar{\partial}_0 = \bar{\partial}_{\mathbb{P}^2} - \frac{\sqrt{-1}}{2}(\bar{\partial}_{\mathbb{P}^2} \log \phi_\beta) \wedge L_\xi. \quad (24)$$

Given an arbitrary semi-basic $(p,q)-$form b, $\partial_0 b = (d_0 b)^{p+1,q}$, $\bar{\partial}_0 b = (d_0 b)^{p,q+1}$. Hence, both of the two operators are globally defined on $\mathbb{S}^5$.

For any β, let $x_i, i = 1,2,3,4$ be the Euclidean coordinate functions on $U_{\beta,\mathbb{S}^5} = \mathbb{S}^1 \times \mathbb{C}^2$. The identity

$$\eta - d\theta_\beta = \eta(\frac{\partial}{\partial x_j})dx^j$$

is verified on the basis ($\xi = \frac{\partial}{\partial \theta_\beta}, \frac{\partial}{\partial x_j},\ j = 1,2,3,4$). Given an endomorphism u, we have

$$d_0 u = [\frac{\partial u}{\partial x_j} - \xi(u)\eta(\frac{\partial}{\partial x_j})]dx^j. \quad (25)$$

Similarly, given a semi-basic endomorphism-valued $1-$form $a_0 = \Sigma_{i=1}^4 a_i dx^i$, we have

$$d_0 a_0 = [\frac{\partial a_i}{\partial x_j} - \xi(a_i)\eta(\frac{\partial}{\partial x_j})]dx^j \wedge dx^i. \quad (26)$$

Remark 2.12. Our calculation for the bundle-valued forms remains true for usual forms, unless the irreducible condition is required. This is because we can simply let it be the trivial line bundle. For example, Definition 2.11 and its subsequent calculations hold for usual forms.

3 The fine splitting for the domain bundle of the linearized operator

3.1 The linearized operator under the model data

Suppose A is not flat on $\mathbb{P}^2$, the pullback connection on $(\mathbb{C}^3 \setminus O) \times \mathbb{S}^1$ has conic singularity along the circle $O \times \mathbb{S}^1$. This is the prototype of what we are interested in. The model linear problem for G_2-instantons with conic singularities along a circle is as follows.

On $\mathbb{C}^3$, let $\omega_{\mathbb{C}^3} = \frac{\sqrt{-1}}{2}(dZ_0 d\bar{Z}_0 + dZ_1 d\bar{Z}_1 + dZ_2 d\bar{Z}_2)$ be the standard Kähler form, and let $\Omega_{\mathbb{C}^3} = dZ_0 dZ_1 dZ_2$ be the standard holomorphic volume form. The standard G_2-structure on $\mathbb{C}^3 \times \mathbb{S}^1$ is defined by

$$\phi_{\mathbb{C}^3 \times \mathbb{S}^1} \triangleq ds \wedge \omega_{\mathbb{C}^3} + Re\Omega_{\mathbb{C}^3}.$$

The standard co-associative $4-$form is $\psi_{\mathbb{C}^3 \times \mathbb{S}^1} \triangleq \frac{\omega^2_{\mathbb{C}^3}}{2} - ds \wedge Im\Omega_{\mathbb{C}^3}$.

Given a Chern connection (E, A) on $\mathbb{P}^2$, the model linearized operator is defined as follows.

$$L_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}\left[\begin{array}{c}\sigma\\ a_{\mathbb{C}^3\times\mathbb{S}^1}\end{array}\right]=\left[\begin{array}{c}d^{\star_{\mathbb{C}^3\times\mathbb{S}^1}}_{A,\mathbb{C}^3\times\mathbb{S}^1}a_{\mathbb{C}^3\times\mathbb{S}^1}\\ d_{A,\mathbb{C}^3\times\mathbb{S}^1}\sigma+\star_{\mathbb{C}^3\times\mathbb{S}^1}(d_{A,\mathbb{C}^3\times\mathbb{S}^1}a_{\mathbb{C}^3\times\mathbb{S}^1}\wedge\psi_{\mathbb{C}^3\times\mathbb{S}^1})\end{array}\right], \tag{27}$$

where $\sigma \in C^\infty[(\mathbb{C}^3 \setminus O) \times \mathbb{S}^1, \Omega^0_{adE}]$ is a section of the pullback adjoint bundle, and $a_{\mathbb{C}^3\times\mathbb{S}^1} \in C^\infty[(\mathbb{C}^3 \setminus O) \times \mathbb{S}^1, \Omega^1_{adE}]$ is a pullback adjoint bundle-valued $1-$form.

Remark 3.1. We do not deal with the global deformation problem for G_2-instantons on a complete manifold, thus we do not need the global linearization. We only address the local model. However, we would calculate the index of the linearization of singular Hermitian Yang-Mills connections on a Calabi-Yau $3-$fold, which certainly requires a global formulation, as in Section 15 below.

A section $a_{\mathbb{C}^3\times\mathbb{S}^1}$ of $\Omega^1_{adE} \to (\mathbb{C}^3 \setminus O) \times \mathbb{S}^1$ can be split into

$$a_{\mathbb{C}^3\times\mathbb{S}^1} = \underline{a}_s ds + a_{\mathbb{C}^3}, \tag{28}$$

where $\underline{a}_s$ is a section of the pullback adE, and $a_{\mathbb{C}^3}$ is a pullback adjoint bundle-valued $1-$form without $ds-$component.

3.2 The fine splitting

We define the finer splitting of a section in the domain of the linearized operator. In view of formulas (27) and (28), let $u \triangleq r\sigma,\ a_s \triangleq r\underline{a}_s$, we find

$$a_{\mathbb{C}^3} = a_r \frac{dr}{r} + (a_\eta)\eta + a_0, \tag{29}$$

where

- a_r and a_η are sections of the pullback adE,
- a_0 is an adjoint bundle-valued semi-basic $1-$form i.e. a $1-$form with no ds, dr, or $\eta-$component.

For further calculation, we let $a_{\mathbb{S}^5} \triangleq a_\eta(\eta) + a_0$. Fixing r and s, both $a_{\mathbb{S}^5}$ and a_0 are forms on $\mathbb{S}^5$. We then obtain the splitting of the domain bundle of the linearized operator.

$$\Omega^0_{adE} \oplus \Omega^1_{adE} = [\pi^\star_{7,4}(adE)]^{\oplus 4} \oplus [\pi^\star_{7,5}(D^\star) \otimes \pi^\star_{7,4}(adE)] :$$

$$\left[\begin{array}{c}\sigma\\ a\end{array}\right]=\begin{bmatrix}\frac{1}{r} & 0 & 0 & 0 & 0\\ 0 & \frac{ds}{r} & \frac{dr}{r} & \eta & 1\end{bmatrix}\begin{bmatrix}u\\ a_s\\ a_r\\ a_\eta\\ a_0\end{bmatrix}. \tag{30}$$

Definition 3.2. Henceforth, let $Dom_{\mathbb{S}^5}$ denote $[\pi^\star_{5,4}(adE)]^{\oplus 4} \oplus [D^\star \otimes (\pi^\star_{5,4}adE)]$, the further pullbacks, and the space of smooth sections of the same bundle on $\mathbb{S}^5$. They are the "domain" of the link operator P, and also of the linearized operator in (27).

4 The fine formula for the link operator

The purpose of this section is to state the formula (Lemma 4.3) for the linearized operator of the G_2-instanton equation under the model data defined in (27).

Formula 4.1. ([33, Proposition 3.13]) In view of the splitting in (28), we have

$$L_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}[1,ds,1]\cdot\begin{bmatrix}\sigma\\ \underline{a}_s\\ a_{\mathbb{C}^3}\end{bmatrix}=[1,ds,1]\cdot\{(\frac{\partial}{\partial s}\begin{bmatrix}0&-1&0\\1&0&0\\0&0&J_{\mathbb{C}^3}\end{bmatrix}+\Box)\begin{bmatrix}\sigma\\ \underline{a}_s\\ a_{\mathbb{C}^3}\end{bmatrix}\}\tag{31}$$

where $\Box\begin{bmatrix}\sigma\\ \underline{a}_s\\ a_{\mathbb{C}^3}\end{bmatrix}=\begin{bmatrix}d^{\star_{\mathbb{C}^3}}a_{\mathbb{C}^3}\\ (d_{\mathbb{C}^3}a_{\mathbb{C}^3})\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3}\\ d_{\mathbb{C}^3}\sigma-J_{\mathbb{C}^3}(d_{\mathbb{C}^3}\underline{a}_s)+(d_{\mathbb{C}^3}a_{\mathbb{C}^3})\lrcorner_{\mathbb{C}^3}Re\Omega_{\mathbb{C}^3}.\end{bmatrix}.$

To obtain a finer splitting, we need to generalize the Sasaki-quaternion structure on $\mathbb{S}^5$ in Lemma 2.8 to the domain bundle on the 7−dimensional manifold $(\mathbb{C}^3\setminus O)\times\mathbb{S}^1$.

Lemma 4.2. *(The "Quaternion" structure on $(\mathbb{C}^3\setminus O)\times\mathbb{S}^1$) Under the setting from* (29) *to* (30) *above, let the column vector* $\begin{bmatrix}u\\a_s\\a_r\\a_\eta\\a_0\end{bmatrix}$ *represents the* 5 *components of* $Dom_{\mathbb{S}^5}$ *respectively.*

Let

$$I=\begin{bmatrix}0&-1&0&0&0\\1&0&0&0&0\\0&0&0&-1&0\\0&0&1&0&0\\0&0&0&0&J_0\end{bmatrix},\ K=\begin{bmatrix}0&0&-1&0&0\\0&0&0&1&0\\1&0&0&0&0\\0&-1&0&0&0\\0&0&0&0&J_H\end{bmatrix},\tag{32}$$

$$T=\begin{bmatrix}0&0&0&-1&0\\0&0&-1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&J_G\end{bmatrix},\ and\ \underline{T}=\begin{bmatrix}0&0&0&-1&0\\0&0&-1&0&0\\0&1&0&0&0\\1&0&0&0&0\\0&0&0&0&-J_G\end{bmatrix}\tag{33}$$

be the isometries of $Dom_{\mathbb{S}^5}$ (and Dom_7) acting on the column vector (by left multiplication). Then $I,K,\underline{T}$ form an quaternion structure. i.e. all the pairwise multiplications anti-commute, and the following is true.

$$K\underline{T}=I,\ IK=\underline{T},\ \underline{T}I=K,\ and\ I^2=K^2=\underline{T}^2=-Id_{Dom_{\mathbb{S}^5}}.\tag{34}$$

Under Convention 2.7 on the tensor contractions, we state our main Lemma.

Lemma 4.3. *Let $(E,A)\to\mathbb{P}^2$ be a Hermitian Yang-Mills bundle. The following formula for the model linearized operator holds true.*

$$L_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}\begin{bmatrix}\frac{1}{r}&0&0&0&0\\0&\frac{ds}{r}&\frac{dr}{r}&\eta&1\end{bmatrix}\cdot\varphi=\begin{bmatrix}\frac{1}{r}&0&0&0&0\\0&\frac{ds}{r}&\frac{dr}{r}&\eta&1\end{bmatrix}[\frac{\partial}{\partial s}\circ I+K\circ(\frac{\partial}{\partial r}-\frac{P}{r})]\cdot\varphi,\tag{35}$$

where P is the following Dirac operator on the bundle $Dom_{\mathbb{S}^5}$ (Definition 3.2).

$$P\begin{bmatrix}u\\a_s\\a_r\\a_\eta\\a_0\end{bmatrix}=\begin{bmatrix}1&-L_\xi&0&0&-(d_0\cdot)\lrcorner H\\L_\xi&1&0&0&(d_0\cdot)\lrcorner G\\0&0&-4&-L_\xi&d_0^{\star_0}\\0&0&L_\xi&-4&-(d_0\cdot)\lrcorner\frac{d\eta}{2}\\J_Hd_0&-J_Gd_0&d_0&J_0d_0&-L_\xi J_0\end{bmatrix}\begin{bmatrix}u\\a_s\\a_r\\a_\eta\\a_0\end{bmatrix},\ and\ \varphi\triangleq\begin{bmatrix}u\\a_s\\a_r\\a_\eta\\a_0\end{bmatrix}\tag{36}$$

denotes a general section in the domain.

Under the tools established in Section 2, and the generalized Quaternion structure in the above Lemma 4.2, the proof of the above Lemma is a routine and fairly tedious tensor calculation. We defer it to Appendix 17.3.

Convention 4.4. A semi-basic pullback $adE-$valued $1-$form a_0 can be viewed as in $Dom_{\mathbb{S}^5}$. The corresponding vector under the basis in (30) is $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_0 \end{bmatrix}$ i.e. the first 4 components are zero.

We note again that Lemma (4.3) does not require the connection A to be Hermitian Yang-Mills, but the Bochner formulas in the following section do i.e. it is not known whether they remain true if A is not Hermitian Yang-Mills.

5 Bochner formula for the link operator

Both $I,\ K$ commute with $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial s}$. Using the quaternion identities (34), we routinely verify the following formula for commutators between the operator P and $I,\ K,\ \underline{T}$.

$$PI = IP,\ KP + PK = -3K,\ \underline{T}P + P\underline{T} = -3\underline{T}. \tag{37}$$

These lead to the formula for the square of the linearized operator.

5.1 The square of the linearized operator

Consequently, we straight-forwardly verify the following formula for the square of $L_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}$.

$$L^2_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}} = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{P^2+2P}{r^2}. \tag{38}$$

There is another formula for $L^2_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}$ than the above. Because the linearized operator only depends on the projective connection induced, we denote the curvature form of the projective connection by F^0_A (or $F^0_{A_0}$).

On the Euclidean space $\mathbb{R}^n \setminus O$ of dimension $n \geq 4$, let A_0 be a pullback connection from $\mathbb{S}^{n-1}$, and let $\boxdot^{dimn}_{A_0} \triangleq \nabla^\star\nabla + 2F^0_{A_0}\otimes_{\mathbb{R}^n}$ be the operator acting on $\Omega^1(EndE) \to \mathbb{R}^n \setminus O$, where $EndE$ is the pullback endomorphism bundle. It is the "linearized" operator for the Yang-Mills equation with gauge fixing (cf. [43, 6.1]). Using formula [42, Lemma 3.2] for the rough Laplacian $\nabla^{\star,\mathbb{R}^n}\nabla^{\mathbb{R}^n}$ on $1-$forms, we find

$$\boxdot^{dimn}_{A_0} = -\frac{\partial^2}{\partial r^2} - \frac{n-3}{r}\frac{\partial}{\partial r} + \frac{\widehat{B}^{dimn}_0}{r^2}, \tag{39}$$

where

$$\widehat{B}_{0,dimn}\begin{bmatrix} a_r \\ a \end{bmatrix} \triangleq \begin{bmatrix} \nabla^{\mathbb{S}^{n-1},\star}\nabla^{\mathbb{S}^{n-1}}a_r - 2d^\star a + 2(n-2)a_r \\ \nabla^{\mathbb{S}^{n-1},\star}\nabla^{\mathbb{S}^{n-1}}a - 2d_s a_r + (n-2)a + 2F^0_{A_0}\otimes_{\mathbb{S}^{n-1}} a \end{bmatrix}. \tag{40}$$

Comparing formulas (38) and (40) yields the desired Bochner formula for the link operator.

Formula 5.1. Let $(E,A) \to \mathbb{P}^2$ be a Hermitian Yang-Mills bundle. In view of formula (4.3), still let $\nabla^\star\nabla$ denote the rough Laplacian on the pullback $adE \to \mathbb{S}^5$. The following identity holds

$$P^2 + 2P = B_{0,dim6},$$

where

$$B_{0,dim6}\begin{bmatrix} u \\ a_s \\ a_r \\ a \end{bmatrix} \triangleq \begin{bmatrix} \nabla^\star\nabla u + 3u \\ \nabla^\star\nabla a_s + 3a_s \\ \nabla^\star\nabla a_r - 2d^\star a + 8a_r \\ \nabla^\star\nabla a - 2da_r + 4a + 2F^0_{A_0}\otimes_{\mathbb{S}^5} a \end{bmatrix}. \tag{41}$$

In relation to the remark under Theorem A, we need the Hermitian Yang-Mills condition in Lemma 5.1 but not in 4.3 because the formula [42, (33)] needs the pullback connection on $(\mathbb{C}^3\setminus O)\times\mathbb{S}^1$ to be a projective G_2-instanton.

Proof of Formula 5.1: The observation is that, using the usual Euclidean coordinates on $\mathbb{C}^3\times\mathbb{S}^1$ (induced from $\mathbb{R}^7$), we have a another way to compute $L^2_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}$ that yields

$$L^2_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}} = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2} - \frac{3}{r}\frac{\partial}{\partial r} + \frac{B_{0,dim6}}{r^2}. \tag{42}$$

The proof is complete comparing (42) with (38).

It remains to show (42). It directly follows from the identities in [42]. In view of the splitting in (28), the Bochner formula [42, (146)], which holds for projective G_2-instantons (because locally the connection form acts on endorphisms-valued forms via Lie-bracket), yields

$$L^2_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}[\frac{1}{r},1]\begin{bmatrix} u \\ a_{\mathbb{C}^3} \end{bmatrix} = \begin{bmatrix} \nabla^{\star,\mathbb{C}^3\times\mathbb{S}^1}\nabla^{\mathbb{C}^3\times\mathbb{S}^1}(\frac{u}{r}) \\ \nabla^{\star,\mathbb{C}^3\times\mathbb{S}^1}\nabla^{\mathbb{C}^3\times\mathbb{S}^1}a_{\mathbb{C}^3} + 2F^0_{A_0}\otimes_{\mathbb{C}^3}\underline{a} \end{bmatrix}. \tag{43}$$

We note that the proof of [42, (146)] is by Euclidean coordinates for the model G_2-structure, thus it also holds in our case as $\mathbb{C}^3\times\mathbb{S}^1$ possesses such coordinates for the standard G_2-structure $\phi_{\mathbb{C}^3\times\mathbb{S}^1}$.

The point is that $\nabla^{\star,\mathbb{C}^3\times\mathbb{S}^1}\nabla^{\mathbb{C}^3\times\mathbb{S}^1} = -\frac{\partial^2}{\partial s^2} + \nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}$, and ds is $\nabla^{\mathbb{C}^3\times\mathbb{S}^1}-$parallel. We then compute

$$\nabla^{\star,\mathbb{C}^3\times\mathbb{S}^1}\nabla^{\mathbb{C}^3\times\mathbb{S}^1}(\frac{a_s ds}{r}) = [\nabla^{\star,\mathbb{C}^3\times\mathbb{S}^1}\nabla^{\mathbb{C}^3\times\mathbb{S}^1}(\frac{a_s}{r})]ds = [-\frac{\partial^2}{\partial s^2}(\frac{a_s}{r}) + \nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}(\frac{a_s}{r})]ds,$$

and the similar identity holds for $\nabla^{\star,\mathbb{C}^3\times\mathbb{S}^1}\nabla^{\mathbb{C}^3\times\mathbb{S}^1}(\frac{u}{r})$ i.e.

$$\nabla^{\star,\mathbb{C}^3\times\mathbb{S}^1}\nabla^{\mathbb{C}^3\times\mathbb{S}^1}(\frac{u}{r}) = -\frac{\partial^2}{\partial s^2}(\frac{u}{r}) + \nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}(\frac{u}{r}).$$

Hence, in view of the splitting in (29), we find

$$L^2_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}[\frac{1}{r},\frac{ds}{r},1]\begin{bmatrix} u \\ a_s \\ a \end{bmatrix} \triangleq -\frac{\partial^2}{\partial s^2}[\frac{1}{r},\frac{ds}{r},1]\begin{bmatrix} u \\ a_s \\ a \end{bmatrix} + [\frac{1}{r},\frac{ds}{r},1]\begin{bmatrix} r\nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}(\frac{u}{r}) \\ r\nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}(\frac{a_s}{r}) \\ \nabla^{\star,\mathbb{C}^3}\nabla^{\star,\mathbb{C}^3}a + 2F^0_{A_0}\otimes_{\mathbb{C}^3} a \end{bmatrix}. \tag{44}$$

Now we view $\mathbb{C}^3\setminus\{O\}$ as the real $6-$dimensional flat cone over $\mathbb{S}^5$. Using the formulas [42, (29) and Lemma 3.2] (let $n=6$ therein) for the rough Laplacians $\nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}a$, $r\nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}(\frac{a_s}{r})$, and $r\nabla^{\star,\mathbb{C}^3}\nabla^{\mathbb{C}^3}(\frac{u}{r})$, the proof for (42) is complete. □

5.2 The 3−forms of the quaternion structure are transverse harmonic

The following formula says the 3 forms yielding the Sasaki-quaternion structure are all d_0-harmonic.

Formula 5.2. The following vanishing holds.

$$d_0(\frac{d\eta}{2}) = d_0 G = d_0 H = d_0\Theta = d_0\bar{\Theta} = 0. \tag{45}$$

Consequently, because they are all $\star_0$ self-dual,

$$d_0^{\star_0}(\frac{d\eta}{2}) = d_0^{\star_0} G = d_0^{\star_0} H = d_0^{\star_0}\Theta = d_0^{\star_0}\bar{\Theta} = 0.$$

Proof of Formula 5.2: Routine calculation shows that the two individual terms in formula (23) for d_0 are

$$d_{\mathbb{P}^2}(Z_0^3 du_1 du_2) = -\frac{3}{2}\bar{\partial}\log\phi_0 \wedge (Z_0^3 du_1 du_2),$$

and

$$-\frac{1}{2}[(d^c_{\mathbb{P}^2})\log\phi_0]\wedge L_\xi(Z_0^3 du_1 du_2) = \frac{3}{2}[\bar{\partial}\log\phi_0]\wedge(Z_0^3 du_1 du_2).$$

Then (23) says that $d_0(Z_0^3 du_1 du_2) = 0$. Taking complex conjugate, we find

$$d_0(\bar{Z}_0^3 d\bar{u}_1 d\bar{u}_2) = 0.$$

Using the expression of G, H, and Θ in (14), (45) holds in $U_{0,\mathbb{S}^5}$. Because they are both smooth forms, by continuity, (45) holds true everywhere on $\mathbb{S}^5$. □

The $\star_0$ self-duality of the forms $\frac{d\eta}{2}$, G, H also yields the following identities.

Formula 5.3. $d_0^{\star_0} J_0(a_0) = d_0(a_0)\lrcorner\omega_0$, $d_0^{\star_0} J_G(a_0) = d_0(a_0)\lrcorner G$, $d_0^{\star_0} J_H(a_0) = d_0(a_0)\lrcorner H$.

Proof of Formula 5.3: We only prove the second identity, the other two are similar. We calculate

$$\begin{aligned}
& d_0^{\star_0}(J_G a_0) \triangleq -\star_0 d_0 \star_0 (a_0\lrcorner G) = -\star_0 d_0 \star_0 \star_0(a_0\wedge G) = \star_0 d_0(a_0\wedge G) \\
= \;& \star_0[(d_0 a_0)\wedge G] \qquad\qquad (\text{by the } d_0-\text{closeness in Formula (5.2)}). \\
= \;& (d_0 a_0)\lrcorner G.
\end{aligned}$$

□

5.3 Bochner formulas and the two binomials

Lemma 5.4. *(Bochner formulas for P) Let $(E, A)\to\mathbb{P}^2$ be a Hermitian Yang-Mills bundle. Still in view of the 5 component separation in* (29) *and Lemma 4.3, the following holds.*

$$(P^2+2P)\begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} = \begin{bmatrix} \nabla^\star\nabla u + 3u \\ \nabla^\star\nabla a_s + 3a_s \\ \nabla^\star\nabla a_r - 2d_0^{\star_0} a_0 + 2L_\xi a_\eta + 8a_r \\ \nabla^\star\nabla a_\eta - 2L_\xi a_r + 8a_\eta + 2d_0^{\star_0} J_0(a_0) \\ \nabla^\star\nabla a_0 - 2d_0 a_r - 2J_0(d_0 a_\eta) + 4a_0 + 2F_A^0 \otimes_{\mathbb{S}^5} a_0 \end{bmatrix}. \tag{46}$$

Consequently,

$$(P^2+4P)\begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} = \begin{bmatrix} \nabla^\star\nabla u + 5u - 2L_\xi a_s - 2d_0^{\star_0} J_H(a_0) \\ \nabla^\star\nabla a_s + 5a_s + 2L_\xi u + 2d_0^{\star_0} J_G(a_0) \\ \nabla^\star\nabla a_r \\ \nabla^\star\nabla a_\eta \\ \nabla^\star\nabla a_0 + 4a_0 + 2F_A^0 \otimes_{\mathbb{S}^5} a_0 + 2J_H(d_0 u) - 2J_G(d_0 a_s) - 2L_\xi(J_0 a_0) \end{bmatrix}. \tag{47}$$

We need the Hermitian Yang-Mills condition because we want the pullback connection to be a projective G_2-instanton. Please see [42, reduction from (32) to (33)].

Proof of Lemma 5.4: We employ the identity $P^2 + 2P = B_{0,dim6}$ and the fine formula for P. Formula (47) is obtained simply by adding twice of P to (46) which we shall prove.

Formula (177) below for the operator $d^{\star_{\mathbb{S}^5}}_{\mathbb{S}^5}$ yields that row 3 of (41) is equal to row 3 of the desired formula (46). On row 4,

- formula 17.2 below says that the η component of $\nabla^\star\nabla a_0$ is $[2d_0^{\star_0}J_0(a_0)]\eta$;
- the $\eta-$component of $-2d_s a_r$ is $-2\eta\wedge L_\xi a_r$, and that of $4a$ is apparently $4a_\eta\eta$;
- Formula 17.3 below says that the $\eta-$component of $\nabla^\star\nabla(\eta a_\eta)$ is $4a_\eta + \nabla^\star\nabla a_\eta$.

The above facts amount to that the $\eta-$component of row 4 in (41) is

$$\eta[\nabla^\star\nabla a_\eta - 2L_\xi a_r + 8a_\eta + 2d_0^{\star_0}J_0(a_0)],$$

which exactly gives row 4 of the desired formula (46) as co-efficient of η.

For row 5, by similar idea, we only have to observe that Formula 17.3 below also says that the semi-basic component of $\nabla^\star\nabla(\eta a_\eta)$ is $-2J_0(d_0 a_\eta)$. This completes the proof of (46). $\square$

6 Sasakian Fourier Series: the expansion with respect to the Reeb vector field

We show that any sufficiently regular endomorphism on the pullback bundle over $\mathbb{S}^5$ admits a global "Fourier series" in terms of the trivializations s_{-k} of the pullback $O(-k)\to\mathbb{S}^5$ (Lemma 6.3). The "Fourier"-co-efficient of s_{-k} is a section of $(EndE)(k)$ over $\mathbb{P}^2$.

Let $\nu\in C^1[S^5, EndE]$. For any $\beta = 0,\ 1$ or 2, the usual Fourier expansion

$$\nu = \Sigma_{k\in\mathbb{Z}}\nu_\beta(k)e^{\sqrt{-1}k\theta_\beta} \tag{48}$$

converges uniformly in $U_{\beta,\mathbb{S}^5}$ under the Hermitian metric (see Lemma 17.7 below). For any k, $v_\beta(k)$ is a section of the pullback $EndE\to U_{\beta,\mathbb{S}^5}$.

On the overlap $U_{\beta,\mathbb{S}^5}\cap U_{\alpha,\mathbb{S}^5}$, the function $\frac{e^{\sqrt{-1}\theta_\beta}}{e^{\sqrt{-1}\theta_\alpha}}$ is equal to $\frac{Z_\beta}{Z_\alpha}\cdot\sqrt{\frac{\phi_\beta}{\phi_\alpha}}$ which is pullback from $U_{\beta,\mathbb{P}^2}\cap U_{\alpha,\mathbb{P}^2}$ in $\mathbb{P}^2$. Therefore, the $k-$th terms of the two Fourier-Series match:

$$\nu_\beta(k)e^{\sqrt{-1}k\theta_\beta} = \nu_\alpha(k)e^{\sqrt{-1}k\theta_\alpha}. \tag{49}$$

Definition 6.1. The standard Hermitian metric on the universal bundle $O(-1)\to\mathbb{P}^2$ is $|Z_0|^2+|Z_1|^2+|Z_2|^2$. It induces uniquely a Hermitian metric $h_{O(l)}$ on $O(l)$ for any integer $l\neq 0$. The Chern connection is called *the standard connection.* Though $O(k),\ k\neq 0$ is not trivial on $\mathbb{P}^2$, the pullback to $\mathbb{S}^5$ are trivial. Let $s_{-1}\triangleq(X_0,X_1,X_2)$ be the standard unitary trivialization of the pullback $O(-1)\to\mathbb{S}^5$. For any integer l, the section

$$s_l \triangleq \begin{cases} s_{-1}^{\otimes -l} \text{ when } l<0\\ 1 \quad \text{when } l=0\\ s_{-1}^{\vee,\otimes l} \text{ when } l>0\end{cases} \tag{50}$$

is the unitary trivialization of the pullback $O(l)\to\mathbb{S}^5$ under the standard metric.

Let $s_k \otimes s_{-k}$ trivializes the tensor product $O(k) \otimes O(-k)$ over $\mathbb{S}^5$, we find

$$\nu_\beta(k)e^{\sqrt{-1}k\theta_\beta} = [\nu_\beta(k)e^{\sqrt{-1}k\theta_\beta}s_k] \otimes s_{-k}. \tag{51}$$

By the transition condition (49), the section ν_k of $(EndE)(k) \to \mathbb{S}^5$, defined piece-wisely by $\nu_\beta(k)e^{\sqrt{-1}k\theta_\beta}s_k$ on $U_{\beta,\mathbb{S}^5}$, is independent of θ_β or the coordinate chosen. Thus it descends to a global section on $\mathbb{P}^2$, and we find the global series:

$$\nu = \Sigma_{k\in\mathbb{Z}}\nu_k \otimes s_{-k}. \tag{52}$$

The negative sign in "$-k$" is to be consistent with the usual local Fourier-Series in (48) i.e. for any integer k, and any β among $0, 1, 2$, the following is true in $U_{\beta,\mathbb{S}^5}$.

$$\nu_\beta(k)e^{\sqrt{-1}k\theta_\beta} = \nu_k \otimes s_{-k}. \tag{53}$$

Definition 6.2. The series (52) is called the *Sasakian-Fourier series* of ν.

The main lemma of this section summarizes our discussion.

Lemma 6.3. *In the general setting of the first sentence in Lemma 8.3, let $\nu \in C^{10}(\mathbb{S}^5, EndE)$. For any $k \in \mathbb{Z}$, there is an unique section ν_k of $(EndE)(k)$ such that the following holds.*

Under the pullback Hermitian metric, the Sasaki-Fourier series $\Sigma_k\nu_k \otimes s_{-k}$ converges uniformly to ν on $\mathbb{S}^5$. Moreover, it can be differentiated term by term by the Reeb Lie derivative L_ξ and the rough Laplacian $\nabla^\star\nabla|_{\mathbb{S}^5}$ i.e.

- *$\Sigma_k L_\xi(\nu_k \otimes s_{-k})$ is the Sasaki-Fourier series of $L_\xi\nu$, and converges uniformly on $\mathbb{S}^5$ to $L_\xi\nu$.*
- *$\Sigma_k\nabla^\star\nabla|_{\mathbb{S}^5}(\nu_k \otimes s_{-k})$ is the Sasaki-Fourier series of $\nabla^\star\nabla\nu$, and converges uniformly on $\mathbb{S}^5$ to $\nabla^\star\nabla\nu$.*

The Sasaki-Fourier co-efficients are traceless if ν is.

Not every operator can differentiate the Series term by term (see above Claim 17.9).

Proof of Lemma 6.3: It is a direct consequence of the uniform convergence in Lemma 17.7 and the term by term-wise differentiation in Claim 17.9 below. $\square$

7 The $P-$invariant subspaces and sheaf cohomologies on $\mathbb{P}^2$

In this section, we study the special class of eigensections of the operator P consisted of pullback $adE-$valued semi-basic $1-$forms i.e. an eigensection of which the first 4 endomorphism components are 0 (regarding the decomposition in (30)). These turn out to be a "building-block" of $SpecP$ (as in Theorem A above, also see Theorem 9.2 below).

Definition 7.1. Let $V_l \triangleq \{a_0 \in C^{10}[\mathbb{S}^5, D^\star \otimes adE]|Pa_0 = la_0\}$. This means

$$\begin{aligned} & V_l \\ = \; & \{a_0 \in C^{10}[\mathbb{S}^5, D^\star \otimes adE]|d_0a_0\lrcorner H = d_0a_0\lrcorner G = d_0^{\star_0}a_0 = d_0a_0\lrcorner\frac{d\eta}{2} = 0, \\ & L_\xi(J_0a_0) = -la_0\}, \\ = \; & \{a_0 \in C^{10}[\mathbb{S}^5, D^\star \otimes adE]|d_0^{\star_0}J_H(a_0) = d_0^{\star_0}J_G(a_0) = d_0^{\star_0}a_0 = d_0^{\star_0}J_0(a_0) = 0, \\ & L_\xi(J_0a_0) = -la_0\}. \end{aligned} \tag{54}$$

The above definition says that V_l is a subspace of the eigenspace $\mathbb{E}_l P$. Elliptic regularity implies that any $a_0 \in V_l$ is smooth.

This subsection is devoted to the proof of the following characterization of V_l.

Proposition 7.2. *In the general setting of the first sentence in Lemma 8.3, for any integer l, the sub space V_l (of the eigenspace) is isomorphic to*

$$\mathcal{H}^{0,1}[\mathbb{P}^2, (End_0 E)(l)] \ \textit{(space of } \bar{\partial}-\textit{harmonic forms).}$$

Consequently, with respect to the complex structure J_0, V_l is complex isomorphic to the sheaf cohomology $H^1[\mathbb{P}^2, (End_0 E)(l)]$.

To prove the above proposition and for other purposes, it is useful to set the following convention.

Notation Convention 7.3. When the two vector spaces are complex vector spaces of sections of a complex vector bundle (like the twisted endomorphism bundles), or when they are sheaf cohomologies etc, the "=" means a complex isomorphism. Otherwise, to say it is a complex isomorphism, the complex structure should be specified in a manner similar to Proposition 7.2.

7.1 The two term Sasaki-Fourier series for elements in V_l

We decompose any pullback $adE-$valued semi-basic $1-$form a_0 into the $(1,0)$ and $(0,1)-$components

$$a_0 = a_0^{1,0} + a_0^{0,1}. \tag{55}$$

Then the condition

$$-L_\xi J_0(a_0) = l a_0 \ \text{(which is part of (54))} \tag{56}$$

yields that

$$l a_0^{1,0} + l a_0^{0,1} = l a_0 = -L_\xi J_0(a_0) = -L_\xi(-\sqrt{-1} a_0^{1,0} + \sqrt{-1} a_0^{0,1}) = \sqrt{-1} L_\xi a_0^{1,0} - \sqrt{-1} L_\xi a_0^{0,1}.$$

Comparing $(1,0)$ and $(0,1)-$part of both sides, we find

$$L_\xi a_0^{0,1} = \sqrt{-1} l a_0^{0,1}, \ L_\xi a_0^{1,0} = -\sqrt{-1} l a_0^{1,0}. \tag{57}$$

Consider the Fourier-expansions

$$a_0^{0,1} = \Sigma_k a_0^{0,1}(k) s_{-k}, \ a_0^{1,0} = \Sigma_k a_0^{1,0}(k) s_{-k}. \tag{58}$$

where the summations are over all integers, and each coefficient is a $End_0 E-$valued $1-$form pulled back from $\mathbb{P}^2$. Since

$$L_\xi s_{-k} = \sqrt{-1} k s_{-k}, \tag{59}$$

the following holds.

$$L_\xi a_0^{0,1} = \Sigma_k \sqrt{-1} k a_0^{0,1}(k) s_{-k}, \quad L_\xi a_0^{1,0} = \Sigma_k \sqrt{-1} k a_0^{1,0}(k) s_{-k}. \tag{60}$$

Compare the Sasaki-Fouriercoefficients of the first equation in (60) with the first identity in (57), we find that the eigenvalue l must be an integer, and $a_0^{0,1}(k) = 0$ if $k \neq l$. Consequently,

$$a_0^{0,1} = a_0^{0,1}(l) s_{-l}. \quad \text{Similarly,} \ a_0^{1,0} = a_0^{1,0}(-l) s_l.$$

In summary, we have found

Claim 7.4. *Suppose $a_0 \in C^{10}[\mathbb{S}^5, D^\star \otimes adE]$ satisfies equation (56) and $a_0 \neq 0$, then the l therein is an integer. Moreover, in view of the $(1,0) \oplus (0,1)-$decomposition (55), we have*

$$a_0^{1,0} = c^{1,0} s_l + c^{0,1} s_{-l}, \tag{61}$$

where $c^{1,0}$ is an $(End_0 E)(-l)-$valued $(1,0)-$form on $\mathbb{P}^2$, and $c^{0,1}$ is an $(End_0 E)(l)-$valued $(0,1)-$form on $\mathbb{P}^2$.

The above claim particularly means that there are only two non-zero terms in the Sasaki-Fourier series of a_0 (if it satisfies (56)).

7.2 Equivalence between the conditions on $d_0 b$ and that $b^{0,1}$ is $\bar{\partial}_0-$harmonic

In this section we prove Lemma 7.6 on the equivalent characterization of the defining conditions in (54). The semi-basic form Θ is $(2,0)$ and nowhere vanishing, and the complex rank of $D^{\star,(1,0)}$ is 2. The following simple algebraic fact inherits that on a Kähler surface.

Fact 7.5. At a point on $\mathbb{S}^5$, et θ_1 and θ_2 be $(0,2)$ and $(2,0)$ semi-basic forms. Then $\theta_1 \lrcorner \Theta = 0$ if and only if $\theta_1 = 0$. $\theta_2 \lrcorner \overline{\Theta}$ if and only if $\theta_2 = 0$.

The contraction between two $(2,0)-$forms ($(0,2)-$forms) vanish.

On semi-basic $1-$forms, we define

$$d_0^{\star_0} \triangleq - \star_0 d_0 \star_0, \quad \partial_0^{\star_0} \triangleq - \star_0 \partial_0 \star_0; \quad \bar{\partial}_0^{\star_0} \triangleq - \star_0 \bar{\partial}_0 \star_0 . \tag{62}$$

Then the adjoint $\bar{\partial}_0$ with respect to the Hermitian inner-product is $\partial_0^{\star_0}$, and that of ∂_0 is $\bar{\partial}_0^{\star_0}$.

Lemma 7.6. *In the general setting of the first sentence in Lemma 8.3, let b be a smooth section of $D^\star \otimes adE \to \mathbb{S}^5$, the following holds true.*

$$d_0 b \lrcorner H = d_0 b \lrcorner G = 0 \;\Leftrightarrow\; \bar{\partial}_0 b^{0,1} = \partial_0 b^{1,0} = 0; \tag{63}$$

$$d_0^{\star_0} b = d_0 b \lrcorner \frac{d\eta}{2} = 0 \;\Leftrightarrow\; \partial_0^{\star_0} b^{0,1} = \bar{\partial}_0^{\star_0} b^{1,0} = 0. \tag{64}$$

Proof of Lemma 7.6: It is a comparison of the types of forms. We split $d_0 b$ into the $(0,2)$, $(2,0)$, and $(1,1)$ components.

$$d_0 b = \bar{\partial}_0 b^{0,1} + \partial_0 b^{1,0} + (\partial_0 b^{0,1} + \bar{\partial}_0 b^{1,0}). \tag{65}$$

Among the 4 terms, only the $(0,2)-$component $\bar{\partial}_0 b^{0,1}$ might have non-zero contraction with Θ, the contractions between the other 3 terms and Θ vanish. Employing Fact 7.5, we find $d_0 b \lrcorner \Theta = 0 \Leftrightarrow \bar{\partial}_0 b^{0,1} = 0$. Similarly, $d_0 b \lrcorner \overline{\Theta} = 0 \Leftrightarrow \partial_0 b^{1,0} = 0$. The proof of (63) is complete.

To prove (64), we first observe that

$$\partial_0^{\star_0} b^{1,0} = 0, \quad \bar{\partial}_0^{\star_0} b^{0,1} = 0. \tag{66}$$

To prove the first identity in (66), it suffices to notice that $\star_0 b^{1,0}$ is a $(2,1)-$form and the complex rank of $D^{\star,(1,0)}$ ($D^{\star,(0,1)}$) is 2, then $\partial_0 \star_0 b^{1,0} = 0$. The other one is similar. Therefore

$$d_0^{\star_0} b = \partial_0^{\star_0} b^{0,1} + \bar{\partial}_0^{\star_0} b^{1,0}. \tag{67}$$

Contracting $d_0 b$ with $\frac{d\eta}{2}$, still using the decomposition (65) and the vanishing (66), we find

$$d_0 b \lrcorner \frac{d\eta}{2} = d_0^{\star_0} J_0 b = d_0^{\star_0}(\sqrt{-1} b^{0,1} - \sqrt{-1} b^{1,0}) = \sqrt{-1}\partial_0^{\star_0} b^{0,1} - \sqrt{-1}\bar{\partial}_0^{\star_0} b^{1,0}. \tag{68}$$

Via the two different identities (67) and (68), the condition $d_0^{\star_0} b = d_0 b \lrcorner \frac{d\eta}{2} = 0$ is equivalent to that $\sqrt{-1}\partial_0^{\star_0} b^{0,1} = 0 = \sqrt{-1}\bar{\partial}_0^{\star_0} b^{1,0}$. The sign difference between (67) and (68) caused by the complex structure J_0 is crucial. The proof for (64) is complete. $\square$

7.3 Transverse parallel of the trivialization of the pullback $O(-1) \to \mathbb{S}^5$, and the proof of Proposition 7.2

We show that the map sending a_0 to $c^{0,1}$ in Claim 7.4 is the desired isomorphism in Proposition 7.2. The trivialization s_{-l} of the pullback $O(l) \to \mathbb{S}^5$ is d_0-closed (parallel). It is not parallel under the (full) connection unless $l = 0$.

Lemma 7.7. *Under the pullback standard connection on the pullback $O(l) \to \mathbb{S}^5$, the standard trivialization s_l (see (50) and Definition 6.1) is d_0-closed i.e.*

$$\partial_0 s_l = \bar{\partial}_0 s_l = 0, \quad \textit{and } d_0 s_l = 0.$$

Proof of Lemma 7.7: We only prove it when $l = -1$. When $l = 1$, it follows by dualizing. For arbitrary integer l, it follows by Leibniz-rule with respect to tensor product.

It suffices to prove $\partial_0 s_{-1} = \bar{\partial}_0 s_{-1} = 0$ in $U_{0,\mathbb{S}^5}$. The vanishing on the whole $\mathbb{S}^5$ follows by continuity. Via formula (10) for Z_0 (which particularly says $L_\xi Z_0 = \sqrt{-1}Z_0$), Fact 2.4 on the Reeb vector field, and formula (24) for ∂_0, $\bar{\partial}_0$ etc, we routinely verify the following.

$$\partial_0 Z_0 = -Z_0 \partial_{\mathbb{P}^2} \log \phi_0, \ \bar{\partial}_0 Z_0 = 0.$$

The trivialization $(1, u_1, u_2)$ of the pullback $O(-1)$ descends to $U_{0,\mathbb{P}^2}$ in $\mathbb{P}^2$. Then, under the Chern-connection of the standard metric, we find

$$\bar{\partial}_0(1, u_1, u_2) = \bar{\partial}_{\mathbb{P}^2}(1, u_1, u_2) = 0,$$

and

$$\partial_0(1, u_1, u_2) = \partial_{\mathbb{P}^2}(1, u_1, u_2) = (\partial_{\mathbb{P}^2} \log \phi_0)(1, u_1, u_2).$$

Therefore, both the $\bar{\partial}_0$ and ∂_0 of s_{-1} vanish.

$$\bar{\partial}_0(Z_0, Z_1, Z_2) = \bar{\partial}_0[Z_0(1, u_1, u_2)] = (\bar{\partial}_0 Z_0)(1, u_1, u_2) + Z_0[\bar{\partial}_{\mathbb{P}^2}(1, u_1, u_2)] = 0.$$

$$\partial_0(Z_0, Z_1, Z_2) = \partial_0[Z_0(1, u_1, u_2)] = (\partial_0 Z_0)(1, u_1, u_2) + Z_0[\partial_{\mathbb{P}^2}(1, u_1, u_2)] = 0.$$

The proof is complete. □

It is helpful, for example to Proposition 7.2 and 10.5 below, to extend the usual conjugate transpose of endomorphisms to twisted endomorphisms.

Notation Convention 7.8. For any integer k, let $\overline{}$ denote the conjugate linear map from the pullback $O(k)$ to $O(-k)$ defined by $\overline{s}_k \triangleq s_{-k}$.

In the general setting of the first sentence in Lemma 8.3, let the transpose only applies to the $EndE-$part but not the line bundle part, the $\overline{}^t$ of any twisted endomorphism is defined.

The conventions and equations established so far are at our disposal to characterize the $P-$invariant subspaces V_l.

***Proof of Proposition** 7.2:* We show that the map

$$\Gamma(b) \triangleq b^{0,1} s_l \tag{69}$$

is the desired isomorphism $V_l \to \mathcal{H}^{0,1}[\mathbb{P}^2, (End_0 E)(l)]$. Because J_0 acts on a $(0,1)$ semi-basic form by the scalar multiplication of $\sqrt{-1}$, the map is complex linear.

Let $a_0 = b$ in Claim 7.4 , $\Gamma(b)$ is the "$c^{0,1}$" in the splitting (61).

Step 1: $\Gamma(b)$ is a priori a $(EndE)(l)$-valued $(0,1)-$form. We show that it is $\bar{\partial}_{\mathbb{P}^2}-$harmonic. By the ∂_0 and $\bar{\partial}_0-$closeness of s_l in Lemma 7.7, we compute

$$\bar{\partial}_0(b^{0,1}s_l) = (\bar{\partial}_0 b^{0,1})s_l + b^{0,1}(\bar{\partial}_0 s_l) = 0, \tag{70}$$

and

$$\begin{aligned} & \partial_0^{\star_0}(b^{0,1}s_l) = -\star_0 \partial_0[s_l \star_0 (b^{0,1})] = -\star_0 (\partial_0 s_l) \wedge \star_0(b^{0,1}) - s_l[\star_0\partial_0 \star_0 (b^{0,1})] \\ = & -s_l[\star_0\partial_0 \star_0 (b^{0,1})] = s_l\partial_0^{\star_0}b^{0,1} \\ = & \ 0. \end{aligned}$$

Because $b^{0,1}s_l$ descends to $\mathbb{P}^2$, the above vanishing implies that

$$\bar{\partial}_{\mathbb{P}^2}(b^{0,1}s_l) = \partial_{\mathbb{P}^2}^{\star_{\mathbb{P}^2}}(b^{0,1}s_l) = 0 \ i.e.\ \Gamma(b) \in \mathcal{H}^{0,1}[\mathbb{P}^2, O(l) \otimes End_0 E].$$

Step 2: We show that the following map $\underline{\Gamma} : \mathcal{H}^{0,1}[\mathbb{P}^2, O(l) \otimes End_0 E] \to V_l$ is the (two-sided) inverse of Γ.

$$\underline{\Gamma}(d^{0,1}) \triangleq d^{0,1} \otimes s_{-l} - \overline{[d^{0,1} \otimes s_{-l}]}^t, \tag{71}$$

where $d^{0,1} \otimes s_{-l}$ is viewed as a pullback End_0E-valued semi-basic $1-$form on $\mathbb{S}^5$.

A pullback endomorphism-valued semi-basic $1-$form α is $adE-$valued if and only if

$$\alpha^{1,0} = -\overline{\alpha^{0,1}}^t (\text{cf. [21, (2.15) VII]}).$$

Hence Γ is injective. By definition, $\Gamma\underline{\Gamma} = Id$ automatically holds true. Then for any $b \in V_l$, we find $(\underline{\Gamma}\Gamma)b - b \in Ker\Gamma = \{0\}$, which means $\underline{\Gamma}\Gamma = Id$ as well. □

8 General spectral reduction for $\nabla^\star\nabla|_{\mathbb{S}^5}$

In this section we reduce the spectrum of the rough Laplacian from $\mathbb{S}^5$ to $\mathbb{P}^2$. This relies on the following formula for the rough Laplacian acting on sections of $adE \to \mathbb{S}^5$:

$$\nabla^\star\nabla u = -L_\xi^2 + d_0^{\star_0}d_0. \tag{72}$$

This is verified by a transverse geodesic frame $(v_i, i = 1,\ 2,\ 3,\ 4)$ on an arbitrary Reeb-orbit together with the Reeb vector field ξ. The vanishing $\nabla_{v_i}v_i = 0$ and $\nabla_\xi\xi = 0$ on the Reeb orbit yields

$$-(\nabla^\star\nabla u)|_{\pi_{5,4}^{-1}p} = [\nabla_{v_i}(\nabla_{v_i}u)]|_{\pi_{5,4}^{-1}p} + (L_\xi^2 u)|_{\pi_{5,4}^{-1}p} = -d_0^{\star_0}d_0 u|_{\pi_{5,4}^{-1}p} + (L_\xi^2 u)|_{\pi_{5,4}^{-1}p}.$$

Now we define the spectrum with multiplicities.

Definition 8.1. Let $Spec^{mul}(\ \cdot\ )$ denote the set of eigenvalues counted with real multiplicity. This means if μ is an eigenvalue and the real dimension of the eigenspace is m_μ, μ appears m_μ times in $Spec^{mul}(\cdot)$.

Similarly, when the operator is $\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5}$ or $\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}$, let $Spec^{mul_{\mathbb{C}}}(\ \cdot\ )$ denote the set of eigenvalues counted with complex multiplicity. This means if μ is an eigenvalue and the complex dimension of the eigenspace is m_μ, μ appears m_μ times in $Spec^{mul_{\mathbb{C}}}(\cdot)$.

Remark 8.2. The complex bundle End_0E is the complexification of adE. Hence, for any $\lambda \in Spec\nabla^\star\nabla|_{\mathbb{S}^5}$, $\mathbb{E}_\lambda\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5}$ is the complexification of

$$\mathbb{E}_\lambda\nabla^\star\nabla|_{adE\to\mathbb{S}^5} \triangleq \mathbb{E}_\lambda\nabla^\star\nabla|_{\mathbb{S}^5}.$$

The real dimension of a vector space is equal to the complex dimension of its complexification. Therefore

$$Spec^{mul}\nabla^\star\nabla|_{adE\to\mathbb{S}^5} = Spec^{mul_{\mathbb{C}}}\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5}. \tag{73}$$

We now proceed to our goal in this section.

Lemma 8.3. *Let $(E, A) \to \mathbb{P}^2$ be a Hermitian Yang-Mills bundle. Then*

$$\begin{aligned} & Spec^{mul}\nabla^\star\nabla|_{\mathbb{S}^5} = Spec^{mul_{\mathbb{C}}}\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5} \\ = \ & \{\alpha_l + l^2 \mid \alpha_l \in Spec^{mul_{\mathbb{C}}}\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}\}. \end{aligned} \tag{74}$$

Consequently, without counting multiplicity,

$$Spec\nabla^\star\nabla|_{\mathbb{S}^5} = \{\alpha_l + l^2 \mid \alpha_l \in Spec\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}\}.$$

Let l_0 be the smallest positive integer such that $H^0[\mathbb{P}^2, (End_0E)(l_0)] \neq \{0\}$. Then for any integer $l \geq l_0$, the integer $4l + l^2$ is an eigenvalue of $\nabla^\star\nabla|_{\mathbb{S}^5}$.

As the spectrum of the Laplace-Beltrami operator on $\mathbb{S}^5$ are integers of the form $4l+l^2$, $l \in \mathbb{Z}_{\geq 0}$, the above means that for a general holomorphic vector bundle over $\mathbb{P}^2$ of rank ≥ 2, the Laplace Beltrami and our bundle rough Laplacian $\nabla^\star\nabla|_{\mathbb{S}^5}$ on $\mathbb{S}^5$ contain infinitely many mutual eigenvalues.

***Proof of Lemma 8.3**:* To prove the first part of the statement, in view of identity (73), it suffices to show

$$Spec^{mul_{\mathbb{C}}}\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5} = \{\alpha_l + l^2 \mid \alpha_l \in Spec^{mul_{\mathbb{C}}}\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}\}. \tag{75}$$

Suppose $u \in C^{10}[\mathbb{S}^5, End_0E]$, the Fourier-expansion $u = \Sigma_{l\in\mathbb{Z}}u_l \otimes s_{-l}$ converges pointwisely on $\mathbb{S}^5$. Moreover, the Laplacian can be taken term by term. Using that $d_0s_{-l} = 0$ and that u_l descends to $\mathbb{P}^2$ so on which the actions of $\star_0$ and d_0 equal the actions of $\star_{\mathbb{P}^2}$ and $d_{\mathbb{P}^2}$ respectively, we find

$$d_0^{\star_0}d_0u = \Sigma_l(\nabla^{\star,\mathbb{P}^2}\nabla^{\mathbb{P}^2}u_l) \otimes s_{-l}.$$

Using $L_\xi u_l = 0$, the decomposition (72) of the rough laplacian on $\mathbb{S}^5$, and $L_\xi^2 s_{-l} = -l^2 s_{-l}$, we find

$$\nabla^\star\nabla u = \Sigma_l[(\nabla^{\star,\mathbb{P}^2}\nabla^{\mathbb{P}^2} + l^2)u_l] \otimes s_{-l}. \tag{76}$$

Then the desired result (74) follows by plugging an arbitrary eigensection u into (76) and comparing the Sasaki-Fourier series of both hand sides. For the reader's convenience, we still provide the full detail.

Let $\lambda \in Spec\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5}$, and u_λ is a (non-zero) eigenvector. Then (76) yields that

$$\lambda u_\lambda = \Sigma_l u_{\lambda,l}s_l = \Sigma_l[(\nabla^{\star,\mathbb{P}^2}\nabla^{\mathbb{P}^2} + l^2)u_{\lambda,l}] \otimes s_{-l}. \tag{77}$$

Then either $u_{\lambda,l} = 0$ or $\nabla^{\star,\mathbb{P}^2}\nabla^{\mathbb{P}^2}u_{\lambda,l} = (\lambda - l^2)u_{\lambda,l}$. Therefore, we obtain a linear injection

$$i: \ \mathbb{E}_\lambda(\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5}) \to \oplus_{l,\lambda-l^2\in Spec\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}}(\mathbb{E}_{\lambda-l^2}\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2})$$

sending u_λ to its Sasaki-Fourier co-efficients. By non-negativity of the rough laplacians on the twisted endomorphism bundles over $\mathbb{P}^2$, only finitely many l yield non-zero eigenspace $\mathbb{E}_{\lambda-l^2}\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}$.

The map i is obviously surjective because for any set of Sasaki-Fourier co-efficients

$$\oplus_l u_{\lambda,l}\in\oplus_{l,\lambda-l^2\in Spec\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}}(\mathbb{E}_{\lambda-l^2}\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}),$$

we have

$$\Sigma_l u_{\lambda,l}s_{-l}\in\mathbb{E}_\lambda(\nabla^\star\nabla|_{End_0E\to\mathbb{S}^5})\text{ and }\ i(\Sigma_l u_{\lambda,l}s_{-l})=\oplus_l u_{\lambda,l}.$$

The proof of (74) is complete.

To prove the second part of the statement, we notice that there is a positive integer l_0 such that for any $l\geq l_0$, $h^0[\mathbb{P}^2,(End_0E)(l)]>0$, and either $l_0=1$ or no positive integer $<l_0$ has this property. This is evident by

- the Riemann Roch (Lemma 17.10),
- the Enriques-Severi-Zariski Lemma [44],
- and that $h^0[\mathbb{P}^2,(End_0E)(l)]>0$ implies $h^0[\mathbb{P}^2,(End_0E)(k)]>0$ for all $k\geq l$. This is because a nontrivial holomorphic section tensoring a nontrivial homogeneous polynomial is nowhere vanishing on a Zariski open set.

Hence the Käher identity i.e. Lemma 17.12 below says $4l$ is an eigenvalue of $\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}$. Therefore $4l+l^2$ is an eigenvalue of $\nabla^\star\nabla|_{\mathbb{S}^5}$. □

9 Describing $SpecP$

9.1 The orthogonal complement of the eigen cohomology space

In this section we find an orthogonal decomposition

$$L^2(\mathbb{S}^5,Dom_{\mathbb{S}^5})=\mathcal{X}\oplus V_{coh}\oplus\mathbb{III}$$

such that each summand is $P-$invariant. Therefore $SpecP$ can be separated.

Definition 9.1. In view of the invariant subspace V_l that is isomorphic to the cohomology (Proposition 7.2), we define the *eigen cohomology space* $V_{coh}\triangleq\oplus_{l\in\mathbb{Z}}V_l$, which is isomorphic to the finite dimensional vector space $\oplus_l H^1[\mathbb{P}^2,(End_0E)(l)]$. We view V_{coh} as a subspace of the Hilbert space $L^2(\mathbb{S}^5,Dom_{\mathbb{S}^5})$.

Let Row^i denote the injection from $L^2[\mathbb{S}^5,adE]$ to the $i-$th row of the domain $L^2[\mathbb{S}^5,Dom_{\mathbb{S}^5}]$ of the link operator. Define another $P-$invariant subspace generated by $Ker\nabla^\star\nabla|_{\mathbb{S}^5}$ with vanishing $5-$row:

$$\mathbb{III}\triangleq\oplus_{i=1}^4 Row^i(Ker\nabla^\star\nabla|_{\mathbb{S}^5})\triangleq\{\begin{bmatrix}u\\a_s\\a_r\\a_\eta\\0\end{bmatrix}\in L^2(\mathbb{S}^5,Dom_{\mathbb{S}^5})|u,a_s,a_r,a_\eta\in Ker\nabla^\star\nabla|_{\mathbb{S}^5}\},\quad(78)$$

Such a section is said to be redundant, and it is thus called the space of redundant sections. Let

$$\mathcal{X}\triangleq(V_{coh}\oplus\mathbb{III})^\perp\quad(79)$$

be the orthogonal complement of the finite-dimensional subspace $V_{coh} \oplus \mathbb{III}$. It is $P-$invariant because $V_{coh} \oplus \mathbb{III}$ is and P is formally self adjoint. The first 4 entries of a section in $\mathbb{X}$ are all perpendicular to $Ker\nabla^{\star}\nabla|_{\mathbb{S}^5}$.

A section with vanishing $5-$th row i.e. vanishing semi-basic $1-$form component is said to be *primitive* if the first $4-$entries are all perpendicular to $Ker\nabla^{\star}\nabla|_{\mathbb{S}^5}$. The space of all such sections (denoted by $\{primitive\ sections\}$) is a closed subspace of $L^2(\mathbb{S}^5, Dom_{\mathbb{S}^5})$ that equals $\oplus_{i=1}^{4} Row^i[(Ker\nabla^{\star}\nabla|_{\mathbb{S}^5})^{\perp}]$. For any integer i among $1,\ 2,\ 3,\ 4$, a primitive section such that every other row than row i vanishes is said to be Row^i-primitive. This means it lies in the range of the map Row^i.

Let $\nabla^{\star}\nabla|_{Ker^{\perp}_{\mathbb{S}^5}}$ denote the restriction of the bundle rough Laplacian $\nabla^{\star}\nabla|_{\mathbb{S}^5}$ to the orthogonal complement in $L^2[\mathbb{S}^5, adE]$ of its kernel. Apparently, the kernel of $\nabla^{\star}\nabla|_{Ker^{\perp}_{\mathbb{S}^5}}$ is trivial.

Apparently, by the fine formula we have

$$SpecP|_{V_{coh}} = S_{coh},\ \text{ and } SpecP|_{\mathbb{III}} = \{1,\ -4\} \text{ if } Ker\nabla^{\star}\nabla|_{\mathbb{S}^5} \text{ is non-trivial.}$$

Moreover, any element in $L^2(\mathbb{S}^5, Dom_{\mathbb{S}^5})$ with vanishing 5th row is perpendicular to $V =_{coh}$.

9.2 The spectrum on $\mathcal{X}$

We determine the spectrum of P restricted to $\mathcal{X}$.

Theorem 9.2. *Let (E, A) be a Hermitian Yang-Mills bundle over $\mathbb{P}^2$ with rank ≥ 2. let $S^0_{\nabla^{\star}\nabla}$ be all the numbers generated by the nonzero eigenvalues of $\nabla^{\star}\nabla|_{\mathbb{S}^5}$ (via (3) or the first bullet point above Theorem A, equivalently). Then $SpecP|_{\mathcal{X}} = S^0_{\nabla^{\star}\nabla}$.*

We do not ask for non-projective flatness at this point, and aim at a general result. The numbers -1 and -2 are not eigenvalues if and only if it is projectively flat.

Before proving it, to intuitively understand the feature of the Bochner formulas of P (Lemma 5.4), we introduce the notion of "autonomous".

Definition 9.3. For any $i = 1,\ 2,\ 3$, or 4, let $v_i \in L^2[\mathbb{S}^5, adE]$ be the independent variable in the $i-$th row of the domain $Dom_{\mathbb{S}^5}$. A linear differential operator L on $Dom_{\mathbb{S}^5}$ is said to be autonomous with respect to row i, if row i of L is

$$(\nabla^{\star}\nabla + kId)v_i$$

for some real constant k, and the other rows does not depend on v_i.

The crucial observation is that by the Bochner formulas in Lemma 5.4, $P^2 + 2P$ is autonomous with respect to row $1,\ 2$, *and $P^2 + 4P$ is autonomous with respect to row* $3,\ 4$. We also need the Fourier expansion with respect to the eigen-basis of the restricted P.

Definition 9.4. Let $\{\phi_{\mu},\ \mu \in Spec^{mul}(P|_{\mathcal{X}})\}$ be an eigenbasis with respect to $P|_{\mathcal{X}}$. The eigen expansion of any section in $\mathcal{X}$ is called the $P|_{\mathcal{X}}-$*eigen Fourier expansion.* Similar term applies to P itself (no restriction) and the other rough Laplacians. *This is not the same as the Sasaki-Fourier series (Definition 6.2).*

Proof. We first prove the direction

$$SpecP|_{\mathcal{X}} \subseteq S^0_{\nabla^{\star}\nabla}. \tag{80}$$

The idea is to inject an eigen-space of the rough Laplacian on the pullback adjoint bundle to each among row 1–4 of the domain of the link operator. Let λ be an eigenvalue of $\nabla^{\star}\nabla|_{Ker^{\perp}_{\mathbb{S}^5}}$, for any non-zero eigen-section u_λ, $Row^1 u_\lambda$ is in $\mathcal{X}$ i.e. perpendicular to $\mathbb{III}$ and V_{coh}. The $P|_{\mathcal{X}}-$eigen Fourier expansion is

$$Row^1 u_\lambda = \Sigma_{\mu\in Spec^{mul}(P|_{\mathcal{X}})} u_{\lambda,\mu}\phi_\mu.$$

Using the Bochner formula (46), we calculate that

$$\begin{aligned}
& \Sigma_{\mu\in Spec^{mul}(P|_{\mathcal{X}})}(\mu^2+2\mu)u_{\lambda,\mu}\phi_\mu = [P^2+2P]Row^1 u_\lambda = Row^1(\nabla^\star\nabla u_\lambda + 3u_\lambda) \\
= \ & (\lambda+3)Row^1 u_\lambda \\
= \ & \Sigma_{\mu\in Spec^{mul}(P|_{\mathcal{X}})}(\lambda+3)u_{\lambda,\mu}\phi_\mu. \qquad (81)
\end{aligned}$$

Comparing the non-zero coefficients $u_{\lambda,\mu}$ (which must exist because $u_\lambda \neq 0$), we find

$$\mu^2+2\mu = \lambda+3.$$

The above means that for any $\lambda \in Spec(\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})$,

$$Row^1[\mathbb{E}_\lambda(\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})] \subseteq \oplus_{\mu^2+2\mu-3=\lambda}\mathbb{E}_\mu P|_{\mathcal{X}}.$$

Because u_λ is an arbitrary non-zero eigenvector, and that the eigenbasis with respect to $\nabla^\star\nabla$ is complete in $L^2[\mathbb{S}^5, adE]$, the following holds.

$$\begin{aligned}
& Row^1[(Ker\nabla^\star\nabla|_{\mathbb{S}^5})^\perp] \subseteq \oplus^{L^2}_{\lambda\in Spec(\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})} Row^1(\mathbb{E}_\lambda \nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}}) \\
\subseteq \ & \oplus^{L^2}_{\mu^2+2\mu-3\in Spec(\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})}\mathbb{E}_\mu P|_{\mathcal{X}}.
\end{aligned}$$

Similarly, using (46) again, we verify that

$$Row^2[(Ker\nabla^\star\nabla|_{\mathbb{S}^5})^\perp] \subseteq \oplus^{L^2}_{\mu^2+2\mu-3\in Spec(\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})}\mathbb{E}_\mu P|_{\mathcal{X}}.$$

Using the Bochner formula (47) instead of (46), we verify

$$Row^3[(Ker\nabla^\star\nabla|_{\mathbb{S}^5})^\perp] \oplus Row^4[(Ker\nabla^\star\nabla|_{\mathbb{S}^5})^\perp] \subseteq \oplus^{L^2}_{\mu^2+4\mu\in Spec(\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})}\mathbb{E}_\mu P|_{\mathcal{X}}.$$

In summary, by definition of $S_{\nabla^\star\nabla}$, we find

$$\{primitive\ sections\} \subseteq \oplus^{L^2}_{\mu\in S^0_{\nabla^\star\nabla}}\mathbb{E}_\mu P|_{\mathcal{X}}$$

Consequently, related to the orthogonal complement of the above, we find

$$\begin{aligned}
& \oplus^{L^2}_{\mu\notin S^0_{\nabla^\star\nabla}}\mathbb{E}_\mu P|_{\mathcal{X}} \subseteq \{primitive\ sections\}^\perp. \\
= \ & \{\begin{bmatrix} \underline{u} \\ \underline{a}_s \\ \underline{a}_r \\ \underline{a}_\eta \\ a_0 \end{bmatrix} | a_0 \in L^2[\mathbb{S}^5, D^\star\otimes adE],\ u, a_s, a_r, a_\eta \in Ker\nabla^\star\nabla|_{\mathbb{S}^5}\}. \qquad (82)
\end{aligned}$$

The equality simply follows from definition of primitive sections under (79). The above relation means if μ is an eigenvalue of $P|_{\mathcal{X}}$ and $\mu \notin S^0_{\nabla^\star\nabla}$, row 1–4 of any eigensection of $P|_{\mathcal{X}}$ must be in $Ker\nabla^\star\nabla|_{\mathbb{S}^5}$. Again by the fine formula (36), because $Ker\nabla^\star\nabla|_{\mathbb{S}^5}$ is perpendicular to range of $d_0^{\star 0}$, if a_0 does not vanish it must belong to V_{coh}. Then the eigen-section is in

$V_{coh} \oplus \mathbb{III}$, but it is assumed to be in the orthogonal complement. Therefore it must vanish. This means μ is not an eigenvalue of $P|_{\mathcal{X}}$ if $\mu \notin S^0_{\nabla^\star\nabla}$. The direction (80) is proved.

Now we prove the other direction:

$$SpecP|_{\mathcal{X}} \supseteq S^0_{\nabla^\star\nabla}. \tag{83}$$

It suffices to read the $P|_{\mathcal{X}}-$eigen Fourier expansion (81) deeper. Namely, it means

$$Row^1 u_\lambda = \Sigma_{\mu\in Spec^{mul}(P|_{\mathcal{X}}),\mu=-1+\sqrt{4+\lambda}} u_{\lambda,\mu}\phi_\mu + \Sigma_{\mu\in Spec^{mul}(P|_{\mathcal{X}}),\mu=-1-\sqrt{4+\lambda}} u_{\lambda,\mu}\phi_\mu.$$

Because u_λ is non-zero, the above means at least one of $-1-\sqrt{4+\lambda}$ and $-1+\sqrt{4+\lambda}$ is an eigenvalue of $P|_{\mathcal{X}}$. We show by contradiction that both of them must be eigenvalues. If not, $Row^i u_\lambda$ is an eigensection of P, thus $d_0 u_\lambda = L_\xi u_\lambda = 0$, which is equivalent to that $u_\lambda \in Ker\nabla^\star\nabla|_{\mathbb{S}^5}$. But $u_\lambda \perp Ker\nabla^\star\nabla|_{\mathbb{S}^5}$, then $u_\lambda = 0$. This contradicts the hypothesis that $u_\lambda \neq 0$. Likewise, regarding the other polynomial $\mu^2+4\mu$, corresponding to the 3rd and 4th row of $Dom_{\mathbb{S}^5}$, $Row^i u_\lambda$ yields the two eigenvalues $-2-\sqrt{4+\lambda}$ and $-2+\sqrt{4+\lambda}$. Thus all 4 numbers in $S^0_{\nabla^\star\nabla}$ generated by $\lambda \neq 0$ must be eigenvalues of $P|_{\mathcal{X}}$. The proof of the direction (83) is complete. $\square$

A direct consequence of the spectral decomposition Lemma 8.3 is that the connection A is irreducible on $\mathbb{P}^2$ if and only if the pullback A is irreducible on $\mathbb{S}^5$. Moreover, the poly-stable bundle E is stable if and only if the Hermitian Yang-Mills connection A is irreducible on $\mathbb{P}^2$ ([21, VII, Proposition 4.14]).

Then we proceed to the final task of this section.

Proof of Theorem *A spectrum part:* The eigenvalues of P in $S^0_{\nabla^\star\nabla}$ must be either >0 or <3. Those on $\mathbb{III}$ are 1 and -4. Hence $\mathcal{X}$ and $\mathbb{III}$ contribute

$$S_{\nabla^\star\nabla} \setminus \{0,-3\}$$

to $SpecP$ as the Laplacian eigenvalue 0 generates the four numbers $0,\ -3,\ 1,\ -4$ via the two binomials. This holds even if $\mathbb{III}$ is trivial i.e. E is stable. Moreover, any eigenvalue in $[0,3]$ must be induced by V_{coh} thus is an integer. Assuming E is not projective flat (but polystable with rank ≥ 2), the strict Bogomolov Chern number inequality, Riemann-Roch formula (Lemma 17.10) imply

$$h^1[\mathbb{P}^2,\ (EndE)(-1)] = h^1[\mathbb{P}^2,\ (EndE)(-2)] \geq c_2(EndE) > 0. \tag{84}$$

Therefore -1 and -2 must be eigenvalues of P.

Please see Uhlenbeck-Yau [38, Theorem 8.1] and Kobayashi [21, IV, Theorem 4.7] about why $c_2(EndE)=0$ implies projective flatness in our case.

An eigenvalue $4l+l^2,\ l \geq l_0$ for the bundle rough Laplacian in Lemma 8.3, through (3) or equivalently the first bullet point above Theorem A, generates the four integer eigenvalues of P. The proof of Theorem A is complete. $\square$

10 The multiplicities

In this section we determine the multiplicities of the eigenvalues of the link operator.

10.1 The definition of the projection map

For any $\mu \in S^0_{\nabla^\star\nabla} \subseteq SpecP$, at least one of

$$\lambda_1 \triangleq \mu^2 + 2\mu - 3, \quad \lambda_2 = \mu^2 + 4\mu$$

belongs to $\{0, -3\}$. In this section, we define the projection and give a formula for it.

Definition 10.1. Let $(E, A) \to \mathbb{P}^2$ be a Hermitian Yang-Mills bundle. Suppose

$$\mu \in SpecP|_{\mathcal{X}} = S^0_{\nabla^\star\nabla}.$$

Let $\|_{\mathbb{E}_\mu P|_{\mathcal{X}}}$ denote the orthogonal projection

$$(\mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2} \to \mathbb{E}_\mu P|_{\mathcal{X}}$$

that is factored as follows.

$$(\mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2} \to \mathcal{X} \to L^2(\mathbb{S}^5, Dom_{\mathbb{S}^5}) \to \mathbb{E}_\mu P|_{\mathcal{X}}.$$

From here to the end of Section 10, let $\varphi \in \mathbb{E}_\mu P|_{\mathcal{X}}$, and ζ denote a section of the form $\begin{bmatrix} v \\ h \\ g \\ w \\ 0 \end{bmatrix} \in (\mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2}.$

We start from the first entry. Suppose $v \in \mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}}$, still by the $P|_{\mathcal{X}}-$eigen Fourier expansion, there is an (unique) orthogonal splitting

$$\begin{bmatrix} v \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} + \begin{bmatrix} \widetilde{u} \\ -a_s \\ -a_r \\ -a_\eta \\ -a_0 \end{bmatrix} \triangleq \varphi + \widetilde{\varphi}, \tag{85}$$

such that $\varphi \in \mathbb{E}_{\mu_{\lambda_1,+}}P|_{\mathcal{X}}$ and $\widetilde{\varphi} \in \mathbb{E}_{\mu_{\lambda_1,-}}P|_{\mathcal{X}}$. By the fine formula (36) for P, the former condition gives the left column and the latter gives the right column in the following.

$$\begin{aligned}
u - L_\xi a_s - (d_0 a_0)\lrcorner H &= \mu_{\lambda_1,+}u \quad , \quad \widetilde{u} + L_\xi a_s + (d_0 a_0)\lrcorner H = \mu_{\lambda_1,-}\widetilde{u}. \\
L_\xi u + a_s + (d_0 a_0)\lrcorner G &= \mu_{\lambda_1,+}a_s \quad , \quad L_\xi \widetilde{u} - a_s - (d_0 a_0)\lrcorner G = -\mu_{\lambda_1,-}a_s. \\
-4a_r - L_\xi a_\eta + d_0^{\star_0} a_0 &= \mu_{\lambda_1,+}a_r \quad , \quad 4a_r + L_\xi a_\eta - d_0^{\star_0} a_0 = -\mu_{\lambda_1,-}a_r. \\
L_\xi a_r - 4a_\eta - (d_0 a_0)\lrcorner\frac{d\eta}{2} &= \mu_{\lambda_1,+}a_\eta \quad , \quad -L_\xi a_r + 4a_\eta + (d_0 a_0)\lrcorner\frac{d\eta}{2} = -\mu_{\lambda_1,-}a_\eta. \\
\left.\begin{matrix} J_H d_0 u - J_G d_0 a_s + d_0 a_r \\ +J_0 d_0 a_\eta - L_\xi J_0(a_0) \end{matrix}\right\} &= \mu_{\lambda_1,+}a_0 \quad , \quad \left.\begin{matrix} J_H d_0 \widetilde{u} + J_G d_0 a_s - d_0 a_r \\ -J_0 d_0 a_\eta + L_\xi J_0(a_0) \end{matrix}\right\} = -\mu_{\lambda_1,-}a_0.
\end{aligned} \tag{86}$$

Then the last row of (86) simply becomes the following two equations.

$$J_H d_0 u - J_G d_0 a_s - L_\xi J_0(a_0) = \mu_{\lambda_1,+}a_0, \quad J_H d_0 \widetilde{u} + J_G d_0 a_s + L_\xi J_0(a_0) = -\mu_{\lambda_1,-}a_0. \tag{87}$$

Summing them up and using $v = u + \widetilde{u}$ (which is evident from (85)), we find

$$J_H d_0 v = 2\sqrt{4+\lambda_1}a_0 \ \text{ i.e. } \ a_0 = \frac{J_H d_0 v}{2\sqrt{4+\lambda_1}}.$$

On the other hand, summing up the two equations in the first row of (86), we find

$$v = \mu_{\lambda_1,+} u + \mu_{\lambda_1,-} \widetilde{u}.$$

Using $v = u + \widetilde{u}$ again, we conclude that

$$u = \frac{1 - \mu_{\lambda_1,-}}{\mu_{\lambda_1,+} - \mu_{\lambda_1,-}} v = (\frac{1}{\sqrt{4+\lambda_1}} + \frac{1}{2})v, \ \widetilde{u} = \frac{\mu_{\lambda_1,+} - 1}{\mu_{\lambda_1,+} - \mu_{\lambda_1,-}} v = (-\frac{1}{\sqrt{4+\lambda_1}} + \frac{1}{2})v.$$

Next, summing up the two equations in the second row of (86), we obtain

$$a_s = \frac{L_\xi v}{2\sqrt{4+\lambda_1}}.$$

Then

$$\begin{bmatrix} v \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\mu_{\lambda_1,+}}} P|_{\mathcal{X}}} = \begin{bmatrix} (\frac{1}{\sqrt{4+\lambda_1}} + \frac{1}{2})v \\ \frac{L_\xi v}{2\sqrt{4+\lambda_1}} \\ 0 \\ 0 \\ \frac{J_H d_0 v}{2\sqrt{4+\lambda_1}} \end{bmatrix}, \begin{bmatrix} v \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\mu_{\lambda_1,-}}} P|_{\mathcal{X}}} = \begin{bmatrix} (-\frac{1}{\sqrt{4+\lambda_1}} + \frac{1}{2})v \\ -\frac{L_\xi v}{2\sqrt{4+\lambda_1}} \\ 0 \\ 0 \\ -\frac{J_H d_0 v}{2\sqrt{4+\lambda_1}} \end{bmatrix}. \quad (88)$$

The same method as above yields the following projection formulas for the other rows of $Dom_{\mathbb{S}^5}$.

- Suppose $\lambda_1 \in Spec(\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}})$, for any $h \in \mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}}$ such that $h \neq 0$,

$$\begin{bmatrix} 0 \\ h \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\mu_{\lambda_1,+}}} P|_{\mathcal{X}}} = \begin{bmatrix} -\frac{L_\xi h}{2\sqrt{4+\lambda_1}} \\ (\frac{1}{\sqrt{4+\lambda_1}} + \frac{1}{2})h \\ 0 \\ 0 \\ -\frac{J_G d_0 h}{2\sqrt{4+\lambda_1}} \end{bmatrix}, \begin{bmatrix} 0 \\ h \\ 0 \\ 0 \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\mu_{\lambda_1,-}}} P|_{\mathcal{X}}} = \begin{bmatrix} \frac{L_\xi h}{2\sqrt{4+\lambda_1}} \\ (-\frac{1}{\sqrt{4+\lambda_1}} + \frac{1}{2})h \\ 0 \\ 0 \\ \frac{J_G d_0 h}{2\sqrt{4+\lambda_1}} \end{bmatrix}. \quad (89)$$

- Suppose $\lambda_2 \in Spec(\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}})$, for any $g \in \mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}}$ such that $g \neq 0$,

$$\begin{bmatrix} 0 \\ 0 \\ g \\ 0 \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\underline{\mu}_{\lambda_2,+}}} P|_{\mathcal{X}}} = \begin{bmatrix} 0 \\ 0 \\ (-\frac{1}{\sqrt{4+\lambda_2}} + \frac{1}{2})g \\ \frac{L_\xi g}{2\sqrt{4+\lambda_2}} \\ \frac{d_0 g}{2\sqrt{4+\lambda_2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ g \\ 0 \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\underline{\mu}_{\lambda_2,-}}} P|_{\mathcal{X}}} = \begin{bmatrix} 0 \\ 0 \\ (\frac{1}{\sqrt{4+\lambda_2}} + \frac{1}{2})g \\ -\frac{L_\xi g}{2\sqrt{4+\lambda_2}} \\ -\frac{d_0 g}{2\sqrt{4+\lambda_2}} \end{bmatrix}. \quad (90)$$

- Suppose $\lambda_2 \in Spec(\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}})$, for any $w \in \mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}}$ such that $w \neq 0$,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\underline{\mu}_{\lambda_2,+}}} P|_{\mathcal{X}}} = \begin{bmatrix} 0 \\ 0 \\ -\frac{L_\xi w}{2\sqrt{4+\lambda_2}} \\ (-\frac{1}{\sqrt{4+\lambda_2}} + \frac{1}{2})w \\ \frac{J_0 d_0 w}{2\sqrt{4+\lambda_2}} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_{\underline{\mu}_{\lambda_2,-}}} P|_{\mathcal{X}}} = \begin{bmatrix} 0 \\ 0 \\ \frac{L_\xi w}{2\sqrt{4+\lambda_2}} \\ (\frac{1}{\sqrt{4+\lambda_2}} + \frac{1}{2})w \\ -\frac{J_0 d_0 w}{2\sqrt{4+\lambda_2}} \end{bmatrix}. \quad (91)$$

Summing up formulas (88)–(91), we arrive on target.

Lemma 10.2. *In the setting of Definition 10.1, for any* $\mu \in S^0_{\nabla^\star\nabla} \subset SpecP$*, let*

$$\lambda_1 \triangleq \mu^2 + 2\mu - 3 \text{ and } \lambda_2 \triangleq \mu^2 + 4\mu.$$

Suppose $v,\ h \in \mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}}$ *and* $g,\ w \in \mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}}$*, the following projection formula is true.*

$$\begin{bmatrix} v \\ h \\ g \\ w \\ 0 \end{bmatrix}^{\|_{\mathbb{E}_\mu P|_{\mathcal{X}}}} = \begin{bmatrix} -\frac{L_\xi h}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})v \\ \frac{L_\xi v}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})h \\ -\frac{L_\xi w}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})g \\ \frac{L_\xi g}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})w \\ -\frac{J_G d_0 h}{2(\mu+1)} + \frac{J_H d_0 v}{2(\mu+1)} + \frac{d_0 g}{2(\mu+2)} + \frac{J_0 d_0 w}{2(\mu+2)} \end{bmatrix}. \tag{92}$$

If $\lambda_1 \notin Spec\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}}$, then $v,\ h$ must be 0. This is an advantage in defining eigenspaces for all real numbers. The rationale is that if it is not an eigenvalue, then all "eigensections" are 0. The same applies to λ_2.

10.2 The Kernel and Co-kernel

The purpose of this section is to prove the following.

Proposition 10.3. *In the setting of Definition 10.1,* $\|_{\mathbb{E}_\mu P|_{\mathcal{X}}}$ *is surjective.*

Remark 10.4. The surjectivity particularly means that any eigensection of the restricted P must be of the form (92), respectively.

Proof of Proposition 10.3: It turns out that the co-kernel can be directly shown to vanish, using the projection formula. The condition $\varphi \perp \zeta^{\|_{\mathbb{E}_\mu P|_{\mathcal{X}}}}$ is equivalent to that

$$\begin{aligned} 0 \;=\;& \langle u, -\frac{L_\xi h}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})v\rangle + \langle a_s, \frac{L_\xi v}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})h\rangle \\ &+\langle a_r, -\frac{L_\xi w}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})g\rangle + \langle a_\eta, \frac{L_\xi g}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})w\rangle \\ &+\langle a_0, -\frac{J_G d_0 h}{2(\mu+1)} + \frac{J_H d_0 v}{2(\mu+1)} + \frac{d_0 g}{2(\mu+2)} + \frac{J_0 d_0 w}{2(\mu+2)}\rangle. \end{aligned}$$

Using that the adjoint of L_ξ is $-L_\xi$, and that $J_0 d_0$, $J_H d_0$, $J_G d_0$ are adjoint to $-d_0^{\star_0} J_0$, $-d_0^{\star_0} J_H$, $-d_0^{\star_0} J_G$ respectively, we find

$$\begin{aligned} 0 \;=\;& < h, \frac{L_\xi u}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})a_s + \frac{d_0^{\star_0} J_G a_0}{2(\mu+1)} > \\ &+ < v, -\frac{L_\xi a_s}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})u - \frac{d_0^{\star_0} J_H a_0}{2(\mu+1)} > \\ &+ < w, \frac{L_\xi a_r}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})a_\eta - \frac{d_0^{\star_0} J_0 a_0}{2(\mu+2)} > \\ &+ < g, -\frac{L_\xi a_\eta}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})a_r + \frac{d_0^{\star_0} a_0}{2(\mu+2)} > . \end{aligned} \tag{93}$$

Therefore, the condition $\varphi \in Coker\ \|_{\mathbb{E}_\mu P|_{\mathcal{X}}}$ is equivalent to that (93) holds for all $\zeta \in (\mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}})^{\oplus 2}$ and $\varphi \in \mathbb{E}_\mu P|_{\mathcal{X}}$.

The eigensection condition (the left of comma in system (86)) again says that

$$\begin{aligned} -L_\xi a_s - (d_0 a_0) \lrcorner H = (\mu-1)u, \qquad & L_\xi u + (d_0 a_0) \lrcorner G = (\mu-1)a_s. \\ -L_\xi a_\eta + d_0^{\star_0} a_0 = (\mu+4)a_r, \qquad & L_\xi a_r - (d_0 a_0) \lrcorner \frac{d\eta}{2} = (\mu+4)a_\eta. \end{aligned} \tag{94}$$

Plugging (94) into (93), we find

$$< h, a_s > + < v, u > + < w, a_\eta > + < g, a_r >= 0. \tag{95}$$

Because $\zeta \in (\mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2}$ is arbitrary and the first 4 entries of φ is in the same space, they must vanish i.e. $u = a_s = a_r = a_\eta = 0$. That $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_0 \end{bmatrix} \in \mathcal{X}$ implies $a_0 = 0$ as well, therefore $\varphi = 0$. This means the co-kernel is trivial. □

10.3 Kernel of the projection $\|_{\mathbb{E}_\mu P|_\mathcal{X}}$

In the setting of Definition 10.1, the following holds.

Proposition 10.5. *1. If μ is not an integer, the orthogonal projection $\|_{\mathbb{E}_\mu P|_\mathcal{X}}$ is injective.*

2. When μ is an integer,

$$Ker \|_{\mathbb{E}_\mu P|_\mathcal{X}}= H^0[\mathbb{P}^2, (End_0E)(\mu)] \oplus H^0[\mathbb{P}^2, (End_0E)(-\mu - 3)]. \tag{96}$$

The observation is that a primitive section (92) being in the kernel of the projection is equivalent to that

- the second row h and the fourth row w are two term Sasakian-Fourier series
- the Fourier co-efficient of w with respect to $s_{-\mu}$ and that of h with respect to $s_{\mu+3}$ are holomorphic sections of $(End_0E)(mu)$ and $(End_0E)(-mu - 3)$ respectively.
- the other two entries v and g are determined linearly by h and w respectively.

For the reader's convenience, we still provide the detail.

Proof of Proposition 10.5 : It might be fairly surprising that the projection formulas also force the kernel to be holomorphic sections. Suppose

$$\zeta \in (\mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})^{\oplus 2}, \text{ and } \zeta^{\|_{\mathbb{E}_\mu P|_\mathcal{X}}} = 0. \tag{97}$$

By the projection formula (92), the vanishing (97) is equivalent to

$$\begin{aligned}
&-\frac{L_\xi h}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})v = 0 \quad , \quad \frac{L_\xi v}{2(\mu+1)} + (\frac{1}{\mu+1} + \frac{1}{2})h = 0,\\
&-\frac{L_\xi w}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})g = 0 \quad , \quad \frac{L_\xi g}{2(\mu+2)} + (-\frac{1}{\mu+2} + \frac{1}{2})w = 0,\\
&\text{and} \quad -\frac{J_G d_0 h}{2(\mu+1)} + \frac{J_H d_0 v}{2(\mu+1)} \quad + \quad \frac{d_0 g}{2(\mu+2)} + \frac{J_0 d_0 w}{2(\mu+2)} = 0.
\end{aligned} \tag{98}$$

We note again that Theorem A says none of $\mu,\ \mu+1,\ \mu+2,\ \mu+3$ is 0. Hence row 1 of (98) is equivalent to

$$L^2_\xi h = -(\mu+3)^2 h, \quad v = \frac{L_\xi h}{\mu+3}. \tag{99}$$

Similarly, row 2 of (98) is equivalent to

$$L^2_\xi w = -\mu^2 w, \quad g = \frac{L_\xi w}{\mu}. \tag{100}$$

Part I: Suppose μ is not an integer.

Because the eigenvalues of $-L_\xi^2$ are squares of integers, but μ is not an integer, we find by (99) and (100) that $\zeta = 0$.

Part II: Suppose μ is an integer.

In this case, (99) and (100) do not force the eigensection to vanish. We show that this is how the space of holomorphic sections come into play.

Similarly to the proof of the two term expansion in Claim 7.4, equation (99) implies that the Sasaki-Fourier series of h only has $2-$terms i.e.

$$h = h_{\mu+3}s_{-(\mu+3)} + h_{-(\mu+3)}s_{\mu+3}. \tag{101}$$

Moreover, because $L_\xi s_k = -\sqrt{-1}ks_k$, we find

$$v = \frac{L_\xi h}{\mu+3} = \sqrt{-1}[h_{\mu+3}s_{-(\mu+3)} - h_{-(\mu+3)}s_{\mu+3}]. \tag{102}$$

Consequently, the transverse differential of h is

$$d_0 h = [d_{\mathbb{P}^2}h_{\mu+3}]s_{-(\mu+3)} + [d_{\mathbb{P}^2}h_{-(\mu+3)}]s_{\mu+3}, \tag{103}$$

and that of v is

$$d_0 v = \sqrt{-1}\{[d_{\mathbb{P}^2}h_{\mu+3}]s_{-(\mu+3)} - [d_{\mathbb{P}^2}h_{-(\mu+3)}]s_{\mu+3}\}. \tag{104}$$

Hence,

$$\begin{aligned} J_0 d_0 v &= \sqrt{-1}\{[J_{\mathbb{P}^2}d_{\mathbb{P}^2}h_{\mu+3}]s_{-(\mu+3)} - [J_{\mathbb{P}^2}d_{\mathbb{P}^2}h_{-(\mu+3)}]s_{\mu+3}\}. \\ &= [\partial_{\mathbb{P}^2}h_{\mu+3} - \bar{\partial}_{\mathbb{P}^2}h_{\mu+3}]s_{-(\mu+3)} - [\partial_{\mathbb{P}^2}h_{-(\mu+3)} - \bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)}]s_{\mu+3}. \end{aligned} \tag{105}$$

Equation (103) and (105) amount to

$$d_0 h + J_0 d_0 v = 2[(\partial_{\mathbb{P}^2}h_{\mu+3})s_{-(\mu+3)} + (\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)})s_{\mu+3}]. \tag{106}$$

Because G is minus the imaginary part of the form Θ i.e. $G = \frac{\overline{\Theta}-\Theta}{2\sqrt{-1}}$, we calculate that

$$\begin{aligned} & -J_G d_0 h + J_H d_0 v = -J_G(d_0 h + J_0 d_0 v) \\ &= -2\{[(\partial_{\mathbb{P}^2}h_{\mu+3}) \lrcorner G]s_{-(\mu+3)} + [(\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)}) \lrcorner G]s_{\mu+3}\} \\ &= \sqrt{-1}\{[(\partial_{\mathbb{P}^2}h_{\mu+3}) \lrcorner \overline{\Theta}]s_{-(\mu+3)} - [(\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)}) \lrcorner \Theta]s_{\mu+3}\}. \end{aligned} \tag{107}$$

In the above, we used again that the contraction between a $(1,0)-$form and a $(2,0)-$form vanishes, and that the contraction between a $(0,1)-$form and a $(0,2)-$form vanishes.

Using (100), the following two term expansions hold as well.

$$w = w_\mu s_{-\mu} + w_{-\mu}s_\mu, \quad g = \sqrt{-1}(w_\mu s_{-\mu} - w_{-\mu}s_\mu). \tag{108}$$

By similar derivation as of (106), we find

$$d_0 g + J_0 d_0 w = 2\sqrt{-1}[(\bar{\partial}_{\mathbb{P}^2}w_\mu)s_{-\mu} - (\partial_{\mathbb{P}^2}w_{-\mu})s_\mu]. \tag{109}$$

In the light of (107) and (109), the last equation in (98) reads

$$\begin{aligned} & \frac{\sqrt{-1}}{2(\mu+1)}\{[(\partial_{\mathbb{P}^2}h_{\mu+3}) \lrcorner \overline{\Theta}]s_{-(\mu+3)} - [(\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)}) \lrcorner \Theta]s_{\mu+3}\} \\ & + \frac{\sqrt{-1}}{(\mu+2)}[(\bar{\partial}_{\mathbb{P}^2}w_\mu)s_{-\mu} - (\partial_{\mathbb{P}^2}w_{-\mu})s_\mu] \\ &= 0. \end{aligned} \tag{110}$$

The $(1,0)$ and $(0,1)$ part of (110) should both vanish. This yields the following.

$$-[(\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)})\lrcorner\Theta]s_{\mu+3}-\frac{2(\mu+1)}{(\mu+2)}(\partial_{\mathbb{P}^2}w_{-\mu})s_{\mu}=0; \tag{111}$$

$$[(\partial_{\mathbb{P}^2}h_{\mu+3})\lrcorner\bar{\Theta}]s_{-(\mu+3)}+\frac{2(\mu+1)}{(\mu+2)}(\bar{\partial}_{\mathbb{P}^2}w_{\mu})s_{-\mu}=0. \tag{112}$$

Using that

- $d_0\Theta=d_0\bar{\Theta}=0$ (see Formula 5.2),
- any power of s_{-1} or its conjugation is d_0-closed (Lemma 7.7),
- and that the curvature form F_A is $(1,1)$,

we find that

$$[(\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)})\lrcorner\Theta]s_{\mu+3}=\star_0[(\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)})\wedge\Theta]s_{\mu+3}\text{ is }\bar{\partial}_0^{\star_0}-\text{closed},$$

and

$$[(\partial_{\mathbb{P}^2}h_{\mu+3})\lrcorner\bar{\Theta}]s_{-(\mu+3)}=\star_0[(\partial_{\mathbb{P}^2}h_{\mu+3})\wedge\bar{\Theta}]s_{-(\mu+3)}\text{ is }\partial_0^{\star_0}-\text{closed}.$$

Plugging the above into (111) and (112), we see that

$$(\partial_{\mathbb{P}^2}w_{-\mu})s_{\mu}\text{ is }\bar{\partial}_0^{\star_0}-\text{closed, and }(\bar{\partial}_{\mathbb{P}^2}w_{\mu})s_{-\mu}\text{ is }\partial_0^{\star_0}-\text{closed}.$$

Because both $\partial_{\mathbb{P}^2}w_{-\mu}$ and $\bar{\partial}_{\mathbb{P}^2}w_{\mu}$ are forms on $\mathbb{P}^2$, and $\partial_0^{\star_0}$ $(\bar{\partial}_0^{\star_0})$ are equal to the usual $\partial_{\mathbb{P}^2}^{\star_{\mathbb{P}^2}}$ $(\bar{\partial}_{\mathbb{P}^2}^{\star_{\mathbb{P}^2}})$ on such forms, we find $\bar{\partial}_{\mathbb{P}^2}^{\star_{\mathbb{P}^2}}\partial_{\mathbb{P}^2}w_{-\mu}=0$ and $\partial_{\mathbb{P}^2}^{\star_{\mathbb{P}^2}}\bar{\partial}_{\mathbb{P}^2}w_{\mu}=0$. Integrating by parts on $\mathbb{P}^2$ shows

$$\partial_{\mathbb{P}^2}w_{-\mu}=0\text{ and }\bar{\partial}_{\mathbb{P}^2}w_{\mu}=0. \tag{113}$$

Plugging (113) back into (111) and (112), because the $(2,0)-$form Θ is no-where vanishing (its real and imaginary parts are both complex structures), we find $\bar{\partial}_{\mathbb{P}^2}h_{-(\mu+3)}=0=\partial_{\mathbb{P}^2}h_{\mu+3}$. Then $w_{\mu}\in H^0[\mathbb{P}^2,(End_0E)(\mu)]$ and $h_{-(\mu+3)}\in H^0[\mathbb{P}^2,(End_0E)(-\mu-3)]$. The derivation so far has given a map

$$Q: Ker\ \|_{\mathbb{E}_{\mu}P|_{\mathcal{X}}}\rightarrow H^0[\mathbb{P}^2,(End_0E)(\mu)]\oplus H^0[\mathbb{P}^2,(End_0E)(-\mu-3)]$$

defined by $Q\zeta\triangleq(w_{\mu},\ h_{-(\mu+3)})$.

In a similar manner to (71), reversing the above arguments yields the obvious inverse of Q. For the reader's convenience, we still give the detail. For any

$$w_{\mu}\in H^0[\mathbb{P}^2,(End_0E)(\mu)]\text{ and }h_{-(\mu+3)}\in H^0[\mathbb{P}^2,(End_0E)(-\mu-3)],$$

let

$$w_{-\mu}\triangleq\overline{w}_{\mu}^t,\quad h_{\mu+3}\triangleq\overline{h}_{-(\mu+3)}^t.$$

The inverse $Q^{-1}(w_{\mu},\ h_{-(\mu+3)})$ is simply the ζ defined by conditions (101), (102), and (108):

- the spectral reduction in Formula 8.3 and the identification in Lemma 17.12 below says that
$$v\text{ and }h\in\mathbb{E}_{\mu^2+2\mu-3}(\nabla^{\star}\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})=\mathbb{E}_{\lambda_1}(\nabla^{\star}\nabla|_{Ker^{\perp}_{\mathbb{S}^5}}),$$
$$g\text{ and }w\in\mathbb{E}_{\mu^2+4\mu}(\nabla^{\star}\nabla|_{Ker^{\perp}_{\mathbb{S}^5}})=\mathbb{E}_{\lambda_2}(\nabla^{\star}\nabla|_{Ker^{\perp}_{\mathbb{S}^5}});$$
When $\mu\neq1$ or -4 the Laplacian eigenvalues λ_1 and λ_2 are never 0, thus the above four Laplacian eigen-sections are all perpendicular to the kernel. But there is an interesting subtlety when μ is indeed 1 or -4. Assume it is -1, then vanishing of holomorphic section of a Hermitian Yang-Mills bundle of negative degree already says $h_{-\mu+3}$ therefore also h and v are 0. The claim that they are perpendicular to kernel of Laplacian still holds. Similar argument applies to -4.

- ζ satisfies conditions (99) and (100), therefore it satisfies the whole system (98) which means $\zeta \in Ker \parallel_{\mathbb{E}_\mu P|_{\mathcal{X}}}$.

The desired identification (96) is proved. $\square$

We are ready to achieve the goal.

Proof of Theorem A *multiplicities:* According to the commutators (37), that P commutes with I makes I a complex structure of the eigenspaces of P. For any eigenvalue μ of P, $-\mu-3$ is also an eigenvalue, and both K and $\underline{T}$ are isomorphisms

$$Eigen_\mu(P) \to Eigen_{-(\mu+3)}(P)$$

that anti-commute with the complex structure I. This is consistent with that the Serre-duality map is conjugate linear.

The idea for multiplicities is to separately consider P on each of the invariant subspaces $\mathcal{X}$, V_{coh}, and $\mathbb{III}$. The multiplicity of an eigenvalue $\mu \in S^0_{\nabla^\star\nabla}$ is completely determined by the surjective map

$$\parallel_{\mathbb{E}_\mu P|_{\mathcal{X}}}\colon\ (\mathbb{E}_{\lambda_1}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2}\nabla^\star\nabla|_{Ker^\perp_{\mathbb{S}^5}})^{\oplus 2} \to \mathbb{E}_\mu P|_{\mathcal{X}} \text{ in Definition 10.1.} \tag{114}$$

When μ is not an integer, the vanishing of $Ker \parallel_{\mathbb{E}_\mu P|_{\mathcal{X}}}$ (Proposition 10.5) says it is an isomorphism. *Because λ_1 and λ_2 should both be non-zero if μ is not an integer, the restricted Laplacian coincides with the Laplacian on the eigenspaces. This rationale applies below as well for non-zero eigenvalue of the Laplacian.* The first bullet point is proved.

In addition to our characterizations of $P|_{\mathcal{X}}$ and $P|_{V_{coh}}$, if $Ker\nabla^\star\nabla_\nabla|_{\mathbb{S}^5}$ is non-trivial, the eigenvalues of $P|_{\mathbb{III}}$ are 1 and -4. In view of Definition 9.1 and the characterization of V_l in Proposition 7.2, $SpecP|_{V_{coh}}$ are those integers l such that $V_l = H^1[\mathbb{P}^2, (End_0E)(l)]$ is nonzero, and the eigenspace is V_l. It suffices to determine the multiplicities of $P|_{\mathcal{X}}$. The contribution of $\mathbb{III}$ to $Mult_1P$ is $2dimKer\nabla^\star\nabla|_{\mathbb{S}^5}$. The V_{coh} component contributes $2h^1[\mathbb{P}^2, (EndE)(1)]$. The $\mathcal{X}-$component contributes

$$2Mult_5(\nabla^\star\nabla|_{\mathbb{S}^5}) - 2h^0[\mathbb{P}^2, (End_0E)(1)].$$

Then $Mult_1P$ equals

$$2dimKer\nabla^\star\nabla|_{\mathbb{S}^5} + 2Mult_5(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2h^1[\mathbb{P}^2, (EndE)(1)] - 2h^0[\mathbb{P}^2, (End_0E)(1)].$$

Using Riemann-Roch (Lemma 17.10) and vanishing of holomorphic section to Hermitian Yang-Mills bundle of negative degree, the last two terms above equals -2 times the Hilbert polynomial of the bundle $(End_0E)(1)$. The multiplicity of -4 is the same. The second bullet point is proved .

The four integers $0,\ -1,\ -2,\ -3$ can never be generated by the restrictions $P|_{\mathcal{X}}$ or $P|_{\mathbb{III}}$. They can only be generated by V_{coh} isomorphic to the cohomologies. The proof of the third bullet point is complete by the identification in Proposition 7.2 between V_l and the subspace of $Eigen_lP$.

That μ is not among the aforementioned six integers implies that it can never be generated by $\mathbb{III}$. Consequently, saying that it is in or not in $S_{\nabla^\star\nabla}$ is equivalent to saying that it is in or not in $S^0_{\nabla^\star\nabla}$, respectively. If $\mu \notin S_{\nabla^\star\nabla}$ i.e. not generated by $\mathcal{X}$, then it can only be generated by V_{coh}. By the same reason as the above paragraph the eigenspace must be isomorphic to the cohomology.

In the other way, if it is not generated by V_{coh} (i.e. $\mu \notin S_{coh}$) but by $\mathcal{X}$ (i.e. $\mu \in S_{\nabla^\star\nabla}$), the characterization (96) of $Ker \parallel_{\mathbb{E}_\mu P|_{\mathcal{X}}}$ and the surjectivity of the projection establishes

$$\begin{aligned} Mult_\mu P &= 2Mult_{\mu^2+2\mu-3}(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2Mult_{\mu^2+4\mu}(\nabla^\star\nabla|_{\mathbb{S}^5}) \\ &\quad -2h^0[\mathbb{P}^2,(End_0E)(\mu)] - 2h^0[\mathbb{P}^2,(End_0E)(-\mu-3)]. \end{aligned}$$

If μ is in both S_{coh} and $S_{\nabla^\star\nabla}$ i.e. generated by both V_{coh} and $\mathcal{X}$, we find by similar reason that

$$\begin{aligned} Mult_\mu P &= 2Mult_{\mu^2+2\mu-3}(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2Mult_{\mu^2+4\mu}(\nabla^\star\nabla|_{\mathbb{S}^5}) + 2h^1[\mathbb{P}^2,(EndE)(\mu)] \\ &\quad -2h^0[\mathbb{P}^2,(End_0E)(\mu)] - 2h^0[\mathbb{P}^2,(End_0E)(-\mu-3)]. \end{aligned}$$

The contribution from the H^1 is non-trivial. The last three terms still gives -2 times the Euler characteristic of $(End_0E)(\mu)$ so again Riemann-Roch says it equals the Hilbert polynomial. All three cases in the third bullet point are proved. □

11 Rough Laplacian on a homogeneous vector bundle, and the interplay of the two Casimirs

In this section we determine the spectrum of the bundle rough Laplacian on $\mathbb{S}^5$ for the homogeneous $T^{1,0}\mathbb{P}^2(k)$. We call the Levi-Civita connection of the Fubini-Study metric on $T'\mathbb{P}^2$ *the Fubini-Study connection*, and denote it by ∇^{FS}. Consequently, on the twisted endomorphism bundles $(EndT'\mathbb{P}^2)(l)$, the tensor product of the Fubini-Study connection (metric) and the standard connection (metric) on $O(l)$ is called *the twisted Fubini-Study connection (metric)*, respectively. The same terms also apply to $T'\mathbb{P}^2(k)$. The required representation theoretic method has been well recorded in literature. For example, see [29] and [7].

Theorem D. *In the setting of Theorem A and Corollary C, for any integer l, let the rough Laplacian $\nabla^\star\nabla|_{(End_0T'\mathbb{P}^2)(l)\to\mathbb{P}^2}$ be defined by the twisted Fubini-Study connection and the Fubini-Study metric $\frac{d\eta}{2}$. Then the following holds.*

- *(Spectrum)*

$$\begin{aligned} Spec\nabla^\star\nabla|_{(End_0T'\mathbb{P}^2)(l)\to\mathbb{P}^2} &= \{\frac{4}{3}(a^2+b^2+ab+3a+3b) - \frac{4}{3}l^2 - 8 \mid a,b\in\mathbb{Z};\ a,b\geq 0; \\ &\quad \max(3-a-2b, b-a-3) \leq l \leq \min(2a+b-3, 3+b-a)\}. \end{aligned} \tag{115}$$

Consequently, in the associated data setting,

$$\begin{aligned} Spec\nabla^\star\nabla|_{\mathbb{S}^5} &= \{\frac{4}{3}(a^2+b^2+ab+3a+3b) - \frac{l^2}{3} - 8 \mid a,b,l\in\mathbb{Z};\ a,b\geq 0; \\ &\quad \max(3-a-2b, b-a-3) \leq l \leq \min(2a+b-3, 3+b-a)\}. \end{aligned} \tag{116}$$

- *(Multiplicities) for any number $\lambda \in Spec\nabla^\star\nabla|_{(End_0T'\mathbb{P}^2)(l)}$, let the set S^l_λ be defined by*

$$\begin{aligned} S^l_\lambda \triangleq &\quad \{(a,b)\in\mathbb{Z}^{\geq 0}\times\mathbb{Z}^{\geq 0} | \frac{4}{3}(a^2+b^2+ab+3a+3b) - \frac{4}{3}l^2 - 8 = \lambda,\ \textit{and} \\ &\quad \max(3-a-2b, b-a-3) \leq l \leq \min(2a+b-3, 3+b-a)\}. \end{aligned}$$

The (complex) multiplicity of any $\lambda_l \in Spec\nabla^\star\nabla|_{(End_0 T'\mathbb{P}^2)(l)}$ *is*

$$\Sigma_{S^l_{\lambda_l}} \frac{(a+1)(b+1)(a+b+2)}{2}. \tag{117}$$

The (real) multiplicity of any $\lambda \in Spec\nabla^\star\nabla|_{\mathbb{S}^5}$ *is*

$$\Sigma_{\{l|\lambda-l^2\in Spec\nabla^\star\nabla|_{(End_0 T'\mathbb{P}^2)(l)}\}}\Sigma_{S^l_{\lambda-l^2}} \frac{(a+1)(b+1)(a+b+2)}{2}. \tag{118}$$

In view of the spectral reduction in Lemma 8.3, it suffices to show the eigenvalues (115) on $\mathbb{P}^2$. Because $T'\mathbb{P}^2$ and $O(l)$ are both $SU(3)-$homogeneous, Peter-Weyl formulation and representation theory are applicable.

- The numbers $\frac{4}{3}(a^2+b^2+ab+3a+3b)$ and -8 therein arise from the Casimir operators of $su(3)$ and $su(2)$ on certain irreducible representations respectively. The number $-\frac{4l^2}{3}$ arises from the action of a certain element in the Cartan sub-algebra of $su(3)$. Please see (138) and Formula 13.5, 14.1 below. We have to add l^2 to the eigenvalue on $\mathbb{P}^2$ to obtain the corresponding eigenvalue on $\mathbb{S}^5$.

  These 3 terms are all the contributions from the representation theoretic quantities.

- The condition on $a,\ b$ in (115) is the equivalence condition of that a certain irreducible $SU(3)-$representation appears as a summand in a certain infinite dimensional representation (see Fact 13.9 below).

Some geometric Laplacians (not the rough Laplacian) corresponds to a single Casimir operator. For example, see the Hodge Laplacian in [18, Proposition 2.3], and the "$\overline{\Delta}$" in [29, Lemma 5.2].

To be self-contained, we recall the pedestrian background tailored for our purpose.

11.1 Killing reductive homogeneous spaces

Our references for this section are [23, Section 2: geometry of homogeneous spaces] and [35, Section 5, page 13]. All the group actions below will be smooth left actions unless otherwise specified. The "$\cdot$" between a group element and a vector in a representation space means the underlying action, which should be clear from the context.

Definition 11.1. (Reductive homogeneous space) Let G be a compact semi-simple matrix Lie group, and K be a closed matrix Lie subgroup of G. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. Let $\mathfrak{m}$ be a subspace of g such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$. The manifold $M = G/K$ is called a *reductive homogenous space with respect to* $\mathfrak{m}$ if $Ad_K\mathfrak{m} \subseteq \mathfrak{m}$ (which means that for any $k \in K$ and $X \in \mathfrak{m}$, $Ad_k X \in \mathfrak{m}$).

In practice, we suppress the "$\mathfrak{m}$" and abbreviate it to *reductive homogeneous space.*

At an arbitrary point $gK \in M$, any $X \in \mathfrak{g}$ generates a tangent vector $X^\star$ in the following way.

$$X^\star(gK) = \frac{d}{dt}|_{t=0}(\exp tX)gK. \tag{119}$$

Let E be a homogeneous bundle over a reductive homogeneous space $M = G/K$. *Let e denote the identity element in G (and K).* Let the base point $o \in M$ be eK. The natural map $\rho:\ G\times_K E_o \to E$ defined by $\rho(g, v) = g\cdot v$ is a $G-$equivariant isomorphism (covering identity diffeomorphism of M).

On the tangent bundle, let τ denote the natural isomorphism $G \times_{K,ad} \mathfrak{m} \to TM$ defined by

$$\tau(g, X) \triangleq g_\star[X^\star|_{eK}] = (Ad_g X)^\star|_{gK}. \tag{120}$$

The tautological isomorphism $\tau_{taut}: \ \mathfrak{m} \to T_o M$ is defined by $\tau_{taut}(X) = X^\star|_o$.

The set of $G-$invariant Riemannian metrics on M is bijective to the set of Ad_K-invariant inner products on $\mathfrak{m}$. For example, under the semi-simple condition on G, the restriction to $\mathfrak{m}$ of a negative scalar multiple of the Killing form of G yields a $G-$invariant metric on M.

Definition 11.2. A reductive homogeneous space $M = G/K$ with a $G-$invariant Riemannian metric $(\ , \ )$ is called a *Killing reductive homogeneous space* with respect to $(\ , \ )$ and $\mathfrak{m}$ if the following holds.

- Let B be the Killing form on $\mathfrak{g}$. With respect to the inner product $-B$, $\mathfrak{m}$ is perpendicular to the Lie algebra $\mathfrak{k}$ of K.
- The restriction of $-B$ on $\mathfrak{m}$ is a (constant) real scalar multiple of the inner product $\langle , \rangle_{\mathfrak{m}}$ corresponding to $(\ , \ )$.

We usually abbreviate it to *Killing reductive homogeneous space*, and denote it by $[M, \ \langle , \rangle_{\mathfrak{m}}]$.

The following frames relate the bundle rough Laplacian to a proper Casimir operator.

Lemma 11.3. *Let $[M, \ \langle , \rangle_{\mathfrak{m}}]$ be a Killing reductive homogeneous space equipped with the Levi-Civita connection, and let $(e_i, \ i = 1, ..., dimM)$ be an orthonormal basis of $\mathfrak{m}$. Then for any i,*

$$\nabla_{e_i^\star} e_i^\star = 0 \ \ at \ o = eK. \tag{121}$$

Consequently, for any $g \in G$, $[Ad_g(e_i)]^\star = g_\star(e_i^\star)$ is an orthonormal frame at gK such that

$$\nabla_{[Ad_g(e_i)]^\star}[Ad_g(e_i)]^\star = 0 \ \ at \ gK. \tag{122}$$

We do not know whether this holds without the Killing condition. The proof is deferred to Appendix 17.7 below.

11.2 Homogeneous vector bundles

We briefly recall the homogeneous bundles.

Definition 11.4. Let $M = G/K$ be a reductive homogeneous space. A (smooth) vector-bundle $E \to M$ is said to be $G-$homogeneous if the left action of G on G/K can be lifted to a compatible action of G on E.

We usually suppress the "G" and call it a homogeneous vector bundle.

The space of smooth sections of a homogeneous vector bundle can be identified to an $\infty-$dimensional $G-$representation.

Definition 11.5. (Sections of a homogeneous bundle) Let $\rho_{\mathcal{E}}: \ K \to GL(\mathcal{E})$ be a (real or complex) $K-$representation. We consider the associated vector bundle $E = G \times_{K,\rho_{\mathcal{E}}} \mathcal{E}$.

Let $C^\infty(G, \mathcal{E})$ denote the space of all smooth $\mathcal{E}-$valued functions on G, and let $C^\infty_{K,\rho_{\mathcal{E}}}(G, \mathcal{E})$ be the subspace of $K-$invariant functions i.e. the functions f such that

$$f(gk) = \rho_{\mathcal{E}}(k^{-1})f(g) \triangleq k^{-1} \cdot f(g). \tag{123}$$

We sometimes suppress the representation $\rho_{\mathcal{E}}$ in the notation $C^\infty_{K,\rho_{\mathcal{E}}}(G, \mathcal{E})$.

A section u of E defines uniquely a $K-$invariant function in $C^{\infty}_K(G,\mathcal{E})$, denoted by $\widetilde{u}$. The converse is also true. The correspondence between u and $\widetilde{u}$ is given by

$$u(gK)=(g,\widetilde{u}) \ \text{ for any } \ g\in G. \tag{124}$$

The same correspondence also holds pointwisely: for any $u\in E|_{gK}$, there is an unique $K-$invariant function $\widetilde{u}$ defined on the $K-$orbit passing through g such that $u=(g,\widetilde{u})$.

The left regular representation of G on $C^{\infty}_K(G,\mathcal{E})$ is defined by

$$[L(a)\cdot f](g)\triangleq f(a^{-1}g) \text{ for any } a,\ g\in G.$$

Similarly, the right regular representation of G on $C^{\infty}(G,\mathcal{E})$ is $[R(a)\cdot f](g)=f(ga)$.

We expect that the subspace $C^{\infty}_K(G,\mathcal{E})$ in $C^{\infty}(G,\mathcal{E})$ is not necessarily invariant under the right regular representation, though it is under the left. For further references, see [29, Section 5.1] and [6, III.6].

We keep the following routine convention in mind.

Convention on the $G-$equivariant isomorphism: from here to the end of Section 14.2, the equal signs "=" between reductive homogeneous spaces or sections of homogeneous bundles, and any correspondence/identification between homogeneous connections, or between sections of homogeneous bundles, or between reductive homogeneous spaces are via the underlying $G-$equivariant isomorphism or diffeomorphism.

11.3 A useful identity

For any $X\in\mathfrak{m}$, we calculate the invariant function corresponding to the vector field $X^{\star}$. Let $[\,\cdot\,]_{\mathfrak{m}}$ be the projection to $\mathfrak{m}$ with respect to the directly sum $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{k}$.

Lemma 11.6. *In the setting of Definition 11.1, let $\widetilde{\mathfrak{m}}$ denote the linear map $\mathfrak{m}\to C^{\infty}(G,\mathfrak{m})$ defined by $[\widetilde{\mathfrak{m}}(X)](g)\triangleq[g^{-1}Xg]_{\mathfrak{m}}$. Then for any $X\in\mathfrak{m}$, $\widetilde{\mathfrak{m}}(X)$ is $AdK-$invariant i.e. $\widetilde{\mathfrak{m}}$ is a linear map $\mathfrak{m}\to C^{\infty}_{K,Ad}(G,\mathfrak{m})$. Via the isomorphism (120), $\widetilde{\mathfrak{m}}(X)$ corresponds to the vector field $X^{\star}$ i.e. $X^{\star}(gK)=[Ad_g\widetilde{\mathfrak{m}}(X)]^{\star}(gK)=\tau(g,[\widetilde{\mathfrak{m}}(X)](g))$ at any point $gK\in M$.*

Proof of Lemma 11.6: Because $(Ad_K)\mathfrak{m}\subseteq\mathfrak{m}$, Ad_K preserves the splitting $Y=Y_{\mathfrak{m}}+Y_{\mathfrak{k}}$ i.e.

$$[(Ad_k)Y]_{\mathfrak{m}}=(Ad_k)[Y]_{\mathfrak{m}} \text{ for any } k\in K. \tag{125}$$

Thus $[\widetilde{\mathfrak{m}}(X)](gk)=[k^{-1}g^{-1}Xgk]_{\mathfrak{m}}=Ad_{k^{-1}}[g^{-1}Xg]_{\mathfrak{m}}$.

To prove the second part, for any $Y\in\mathfrak{g}$, let $Y=[Y]_{\mathfrak{m}}+[Y]_{\mathfrak{k}}$ where $[Y]_{\mathfrak{k}}$ is the $\mathfrak{k}-$component of Y, we calculate

$$\{Ad_g[\widetilde{\mathfrak{m}}(X)]\}(g)=g[g^{-1}Xg]_{\mathfrak{m}}g^{-1}=gg^{-1}Xgg^{-1}-g[g^{-1}Xg]_{\mathfrak{k}}g^{-1}=X-g[g^{-1}Xg]_{\mathfrak{k}}g^{-1}.$$

Because $[g^{-1}Xg]_{\mathfrak{k}}\in\mathfrak{k}$, the tangent vector $\{g[g^{-1}Xg]_{\mathfrak{k}}g^{-1}\}^{\star}$ at gK is equal to 0. The proof is complete $\square$

11.4 Formula for the Rough Laplacians

The purpose of this section is to show Formula 11.10 of the rough Laplacian in terms of the Casimir operator.

Definition 11.7. (Casimir operator associated with a basis) Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{B} = (e_i,\ i = 1...dim\mathfrak{g})$ be a basis of $\mathfrak{g}$. Let $\rho:\ \mathfrak{g} \to gl(\mathcal{E})$ be a representation of $\mathfrak{g}$. Then we define the Casimir operator with respect to the basis $\mathcal{B}$ by

$$Cas^{\mathcal{B}}_{\mathfrak{g},\rho} \triangleq \Sigma_{i=1}^{dim\mathfrak{g}}\rho(e_i)\rho(e_i).$$

We do not require G to be semi-simple, though it indeed is in the case of interest. We do not need the Killing form either. All we need is a basis of the Lie algebra.

Definition 11.8. Let $M = G/K$ be a reductive homogeneous space with respect to $\mathfrak{m}$. We view G as a $K-$principal bundle over M. We define the *left invariant principal connection of* $\mathfrak{m}$ to be the connection of which the horizontal distribution at $g \in G$ is $g\mathfrak{m}$ (viewed as subspace of left invariant vector fields). On an associated bundle, the connection given by this horizontal distribution is called *the connection induced by* $\mathfrak{m}$, or simply *the induced connection.*

Definition 11.9. Let $(M,\ \langle,\rangle_{\mathfrak{m}})$ be a Killing reductive homogeneous space. A basis

$$B_{\mathfrak{g}} = (e_i,\ i = 1...dim\mathfrak{g})$$

for the Lie algebra $\mathfrak{g}$ is called triply orthonormal if

- $B_{\mathfrak{g}}$ is orthonormal with respect to a negative real scalar multiple of the Killing-form;
- the set of vectors $B_{\mathfrak{k}} = (e_i,\ i = 1 + dim\mathfrak{m}, ..., dim\mathfrak{g})$ form a basis for the Lie algebra $\mathfrak{k}$ of K;
- the set of vectors $B_{\mathfrak{m}} = (e_i,\ i = 1, ..., dim\mathfrak{m})$ form an orthonormal basis of $\mathfrak{m}$ with respect to $\langle,\rangle_{\mathfrak{m}}$.

In view of the above 3 definitions and the notation $\widetilde{\cdot}$ in Definition 11.5 for the invariant function in terms of a section, we prove the formula for the rough Laplacian.

Formula 11.10. Let $(M,\ \langle,\rangle_{\mathfrak{m}})$ be a Killing reductive homogeneous space with a triply orthonormal basis $B_{\mathfrak{g}}$ for the Lie algebra $\mathfrak{g}$. Let $\rho : K \to GL(\mathcal{E})$ be a (real or complex) representation of K. On the homogeneous bundle $G \times_{K,\rho} \mathcal{E}$, the following holds with respect to induced connection.

$$-(\widetilde{\nabla^\star\nabla u}) = (Cas^{\mathcal{B}_{\mathfrak{g}}}_{\mathfrak{g},L} - Cas^{\mathcal{B}_{\mathfrak{k}}}_{\mathfrak{k},\rho})\widetilde{u}.$$

Remark 11.11. *The second operator $Cas^{\mathcal{B}_{\mathfrak{k}}}_{\mathfrak{k},\rho}$ acts on the value of $\widetilde{u}$.*

Proof of Formula 11.10: Let $\widetilde{Y}$ denote the horizontal lift of a tangent vector Y, the Kobayashi-Nomizu formula [22, Vol I, Chap III, page 115] says that

$$[\widetilde{Y}(\widetilde{u})](g) = [\widetilde{\nabla_Y u|_{gK}}](g). \tag{126}$$

For any $e_i \in \mathfrak{m}$, the vector ge_i (the value of the left invariant vector field) at g is the horizontal lift of $[Ad_g(e_i)]^\star$ at gK, using the vanishing (122), we compute

$$\begin{aligned} &-(\widetilde{\nabla^\star\nabla u})(g) = \Sigma_{i=1}^{dimM}\{[Ad_g(e_i)]^\star \cdot [Ad_g(e_i)]^\star]\widetilde{u}\}(g) = \Sigma_{i=1}^{dimM}[R_\star(e_i)R_\star(e_i)\widetilde{u}](g) \\ = \ & \Sigma_{i=1}^{dimG}[R_\star(e_i)R_\star(e_i)\widetilde{u}](g) - \Sigma_{i=1+dimM}^{dimG}[R_\star(e_i)R_\star(e_i)\widetilde{u}](g) \\ = \ & [Cas^{\mathcal{B}_{\mathfrak{g}}}_{\mathfrak{g},R}\widetilde{u}](g) - [Cas^{\mathcal{B}_{\mathfrak{k}}}_{\mathfrak{k},R}\widetilde{u}](g). \end{aligned} \tag{127}$$

For any g and $\widetilde{u} \in C^{\infty}(G,\mathcal{E})$, we compute

$$\begin{aligned}
& (Cas^{\mathcal{B}_{\mathfrak{g}}}_{\mathfrak{g},R}\cdot \widetilde{u})(g) = [\Sigma_{i=1}^{dimG} R_{\star}(e_i)R_{\star}(e_i)\widetilde{u}](g) = \frac{d^2}{dsdt}|_{t=s=0}\widetilde{u}(g\exp^{se_i}\exp^{te_i}) \\
= \ & \frac{d^2}{dsdt}|_{t=s=0}\widetilde{u}(g\exp^{se_i}g^{-1}\cdot g\exp^{te_i}g^{-1}\cdot g) = \Sigma_{i=1}^{dimG}\{[L_{\star}(Ad_g e_i)L_{\star}(Ad_g e_i)]\widetilde{u}\}(g) \\
= \ & \Sigma_{i=1}^{dimG}\{[L_{\star}(e_i)L_{\star}(e_i)]\widetilde{u}\}(g) \text{ (because } Ad_g \text{ is an orthogonal transformation of } \mathfrak{g}) \\
= \ & (Cas^{\mathcal{B}_{\mathfrak{g}}}_{\mathfrak{g},L}\widetilde{u})(g) \qquad (128)
\end{aligned}$$

This means that on $C^{\infty}(G,\mathcal{E})$ (not requiring $K-$invariance), the Casimir operator of the right regular representation coincides with the Casimir of the left regular representation.

Because of the $K-$invariance of $\widetilde{u}$, we have $R(e_i)\widetilde{u} = -\rho(e_i)\widetilde{u}$ (acting on the value of $\widetilde{u}$). Hence

$$Cas^{\mathcal{B}_{\mathfrak{k}}}_{\mathfrak{k},R}\cdot\widetilde{u} = \Sigma_{i=1}^{dimK}R_{\star}(e_i)R_{\star}(e_i)\widetilde{u} = \Sigma_{i=1}^{dimK}\rho_{\star}(e_i)\rho_{\star}(e_i)\widetilde{u} = Cas^{\mathcal{B}_{\mathfrak{k}}}_{\mathfrak{k},\rho}\widetilde{u}. \qquad (129)$$

Applying (128) and (129) to the two individual terms in (127), the desired formula is proved.

□

12 The standard connection on the homogeneous bundle $[EndT^{1,0}(\mathbb{P}^2)](l)$

The purpose of this section is to interpret $EndT'\mathbb{P}^2(l)$ as a homogeneous bundle over the homogeneous space $\mathbb{P}^2$, and show that the twisted Fubini-Study connection corresponds to the standard horizontal distribution $\mathfrak{m}_{\mathbb{P}^2}$ defined in Section 12.0.1. Please see Proposition 12.3 for the main statement of this section.

12.0.1 The horizontal distribution $\mathfrak{m}_{\mathbb{P}^2}$

Recall that $\mathbb{P}^2 = SU(3)/S[U(1)\times U(2)]$. Let the subspace $\mathfrak{m}_{\mathbb{P}^2} \subset su(3)$ be spanned by the following 4 matrices.

$$e_1 \triangleq X_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ e_2 \triangleq Y_1 \triangleq \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (130)$$

$$e_3 \triangleq X_3 \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ e_4 \triangleq Y_3 \triangleq \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}.$$

It admits a natural complex structure

$$JX_1 = -Y_1,\ JY_1 = X_1,\ JX_3 = -Y_3,\ JY_3 = X_3. \qquad (131)$$

Then $\mathfrak{m}_{\mathbb{P}^2}$ is naturally isomorphic to $\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}$ (the $(1,0)-$part of the complexification of $\mathfrak{m}_{\mathbb{P}^2}$). The isomorphism is given by the natural injection $\mathfrak{m}_{\mathbb{P}^2} \to \mathfrak{m}_{\mathbb{P}^2}\otimes\mathbb{C}$ composed by the projection to the $(1,0)-$part. $\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}$ is spanned by the vectors

$$s_1 = \frac{1}{2}(X_1 + iY_1),\ s_2 = \frac{1}{2}(X_3 + iY_3). \qquad (132)$$

It is routine to verify that $\mathfrak{m}_{\mathbb{P}^2}$ is preserved by $Ad_{S[U(1)\times U(2)]}$. Thus, with respect to the Fubini-Study metric and $\mathfrak{m}_{\mathbb{P}^2}$, $\mathbb{P}^2$ is a Killing reductive homogeneous space.

As complex vector bundles, the homogeneous bundle $SU(3)\times_{S[U(1)\times U(2)],ad}\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}$ is $SU(3)-$equivariantly isomorphic to the holomorphic tangent bundle $T^{1,0}(\mathbb{P}^2)$.

12.1 Interpreting the line bundle $O(l) \to \mathbb{P}^n$ as an associated bundle

$SU(n+1)$ acts on $\mathbb{C}^{n+1} \setminus O$ i.e. the total space of $O(-1)$. Let $S[U(1) \times U(n)]$ denote the subgroup of block-diagonal matrices of the form $\begin{bmatrix} e^{\sqrt{-1}\theta} & 0 & ... & 0 \\ 0 & . & . & . \\ \vdots & . & . & . \\ 0 & . & . & . \end{bmatrix}$. A natural group homorphism $\tau_S: \; S[U(1)\times U(n)] \to U(1)$ maps a matrix in $S[U(1)\times U(n)]$ to its $(1,1)-$entry.

For any integer l, let ρ_l denote the $1-$dimensional complex representation of $U(1)$ i.e. $\rho_l(e^{\sqrt{-1}\theta}) = e^{\sqrt{-1}l\theta} \in GL(1,\mathbb{C})$. Abusing notation, we still let ρ_l denote the $S[U(1) \times U(n)$ representation $\rho_l \cdot \tau_S$.

As a homogeneous space, $SU(n+1)/S[U(1)\times U(n)]$ is $\mathbb{P}^n$, via the action of $SU(n+1)$ on the base point $o \triangleq \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{P}^n$. The universal bundle $O(-1) \to \mathbb{P}^n$ is $SU(n+1)-$equivariantly isomorphic to the homogeneous bundle

$$SU(n+1) \times_{S[U(1)\times U(n)],\rho_1} \mathbb{C}.$$

For any integer l, $O(l) \to \mathbb{P}^n$ is $SU(n+1)-$equivariantly isomorphic to

$$SU(n+1) \times_{S[U(1)\times U(n)],\rho_{-l}} \mathbb{C}.$$

12.2 Characterizing the connection of interest

The main proposition of Section 12 is a direct corollary of the following two lemmas addressing the horizontal distribution corresponding to the standard $SU(3)-$invariant connections.

Lemma 12.1. *Via the $SU(3)-$equivariant isomorphism*

$$SU(3) \times_{S[U(1)\times U(2)],\rho_1} \mathbb{C} = O(-1) \to \mathbb{P}^2,$$

the connection induced by $\mathfrak{m}_{\mathbb{P}^2}$ corresponds to the standard connection (see Definition 6.1).

Lemma 12.2. *Via the $SU(3)-$equivariant isomorphism*

$$SU(3) \times_{S[U(1)\times U(2)],Ad} \mathfrak{m}^{(1,0)}_{\mathbb{P}^2} = T'\mathbb{P}^2,$$

the connection induced by $\mathfrak{m}_{\mathbb{P}^2}$ corresponds to the Fubini-Study connection.

The proof of the above two Lemmas is deferred to Appendix 17.8 and 17.9. We are ready for our main proposition about the standard connection on $[EndT^{1,0}(\mathbb{P}^2)](l)$, which indicates that $\mathbb{P}^2$ is the natural one to work on.

Proposition 12.3. *On the homogeneous bundle $[EndT^{1,0}(\mathbb{P}^2)](l)$, the tensor product of the Fubini-Study connections (on $T^{1,0}(\mathbb{P}^2)$ and its dual) and the standard connection on $O(l)$ is induced by the horizontal distribution $\mathfrak{m}_{\mathbb{P}^2}$.*

Proof of Proposition 12.3: Using Lemma 12.1, the standard connection on $O(l) \to \mathbb{P}^2$, obtained by the dual and/or tensor product of the standard connection on $O(-1)$, is induced by $\mathfrak{m}_{\mathbb{P}^2}$. Using Lemma 12.2, the associated connection on the holomorphic co-tangent bundle $\Omega^1_{\mathbb{P}^2}$, therefore the (tensor product) Fubini-Study connection on $[EndT^{1,0}(\mathbb{P}^2)]$, are induced by $\mathfrak{m}_{\mathbb{P}^2}$. Then the tensor product connection on $[EndT^{1,0}(\mathbb{P}^2)](l)$ is induced by $\mathfrak{m}_{\mathbb{P}^2}$. $\square$

13 Representation theory of $SU(3)$ and $S[U(1)\times U(2)]$

From here to the end of Section 13, given a vector space V, the symbol $|_V$ means "as an endomorphism of V" or "as a representation on V".

13.1 The representation of $S[U(1)\times U(2)]$ on $End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}$ and its "Casimir" operator

The purpose of this section is to prove Formula 13.5 on the Casimir operator of the subgroup $S[U(1)\times U(2)]$ of $SU(3)$ (see Section 12.1 for the definition of the subgroup).

As a subgroup of $SU(3)$, the Lie algebra of $S[U(1)\times U(2)]$ is spanned by

$$e_5 \triangleq \widehat{H}_1 \triangleq \frac{1}{\sqrt{3}}\begin{bmatrix} 2i & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & -i\end{bmatrix},\ e_6 \triangleq H_2 \triangleq \begin{bmatrix} 0 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{bmatrix}, \tag{133}$$
$$e_7 \triangleq X_2 \triangleq \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix},\ e_8 \triangleq Y_2 \triangleq \begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & i\\ 0 & i & 0\end{bmatrix}.$$

Definition 13.1. Let $B_{s[u(1)\times u(2)]} \triangleq \{e_5, e_6, e_7, e_8\}$ be the basis for the Lie algebra $s[u(1)\times u(2)]$ of $S[U(1)\times U(2)]$.

Remark 13.2. $SU(2)$ is isomorphic to the subgroup in $S[U(1)\times U(2)]$ of block diagonal matrices with $(1,1)-$entry equal to 1. Henceforth, let $SU(2)$ denote this subgroup, whose Lie algebra is spanned by $H_2,\ X_2,\ Y_2$ $(e_6,\ e_7,\ e_8)$. We denote this basis by $\mathcal{B}_{su(2)}$.

The first columns of $(e_1,\ e_2,\ e_3,\ e_4)$ form an orthonormal set of vectors in $\mathbb{R}^6$ (see (130)). Thus in view of the formula for the Euclidean metric (Kähler form) $\omega_{\mathbb{C}^3}$ in Table (173), it is straight forward to verify that the quadruple $(e_1,\ e_2,\ e_3,\ e_4)$ is an orthonormal basis of the inner product $\langle\ ,\ \rangle_{\mathfrak{m}_{\mathbb{P}^2}}$ induced by the Fubini-Study form $\frac{d\eta}{2}$.

Let $\mathcal{B}_{su(3)}$ denote the basis $(e_i,\ i=1...8)$ of $su(3)$. According to the previous paragraph, it is triply orthonormal on $\mathbb{P}^2$. That $\widehat{H}_1$ is of the form in (133) is important for this triple orthogonality.

Let V_d be the space of all degree 2 homogeneous polynomials of $2-$complex variables. Let $\rho_{V_d}:\ su(2)\to gl(V_d)$ be the irreducible representation of $su(2)$ on V_d. With respect to the notation convention in Definition 11.7, the Casimir operator obeys the following formula

$$Cas^{\mathcal{B}_{su(2)}}_{su(2),V_d} = -(d^2+2d)Id|_{V_d}.\quad \text{When } d=2,\ Cas^{\mathcal{B}_{su(2)}}_{su(2),V_2} = -8Id|_{V_2}. \tag{134}$$

We routinely verify the following identities on $ad_{su(2)}|_{\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}$.

$$[H_2,s_1]=is_1,\ [H_2,s_2]=-is_2,\ [X_2,s_1]=-s_2,\ [X_2,s_2]=s_1,\ [Y_2,s_1]=is_2,\ [Y_2,s_2]=is_1. \tag{135}$$

Therefore, under the basis (s_1,s_2) of $\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}$, the representation $ad_{su(2)}$ is given by

$$ad_{H_2}\cdot(s_1,s_2)=(s_1,s_2)\begin{bmatrix} i & 0\\ 0 & -i\end{bmatrix},\ ad_{X_2}\cdot(s_1,s_2)=(s_1,s_2)\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix} \tag{136}$$
$$ad_{Y_2}\cdot(s_1,s_2)=(s_1,s_2)\begin{bmatrix} 0 & i\\ i & 0\end{bmatrix}.$$

Let an element in $su(2)$ be represented by its lower block 2×2 (which is exactly the standard form of $su(2)$), the above identities mean that under the basis (s_1,s_2), $ad_{su(2)}|_{\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}$ is the standard representation of $su(2)$.

Based on the above discussion, we can characterize $End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}$ as an $su(2)-$representation.

Lemma 13.3. *In view of Remark 13.2, the $su(2)$ representation on $End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}$ inherited from $s[u(1)\times u(2)]$ is 3−dimensional and irreducible. Consequently, it is equivalent to ρ_{V_2}.*

Proof of Lemma 13.3: Because $ad_{su(2)}|_{\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}$ is equivalent to the standard representation of $su(2)$ (see (136)), it suffices to show that the $su(2)$−representation on $End_0\mathbb{C}^2$ induced by the standard representation is irreducible. Firstly, it extends complex linearly to the adjoint representation of $sl(2,\mathbb{C})$ which is simple. Thus the adjoint action must be irreducible, and is so as a $su(2)$−representation. □

To find the Casimir operator of $S[U(1)\times U(2)]$, it remains to understand the adjoint action of $e_5=\widehat{H}_1$ on $\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}$.

Lemma 13.4. $ad_{\widehat{H}_1}|_{\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}=-\sqrt{3}iId|_{\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}$.

Proof of Lemma 13.4: We straight-forwardly verify the following.

$$[\widehat{H}_1,X_1]=\sqrt{3}Y_1,\ [\widehat{H}_1,Y_1]=-\sqrt{3}X_1,\ [\widehat{H}_1,X_3]=\sqrt{3}Y_3,\ [\widehat{H}_1,Y_3]=-\sqrt{3}X_3.$$

Then $[\widehat{H}_1,s_1]=-\sqrt{3}is_1$, $[\widehat{H}_1,s_2]=-\sqrt{3}is_2$. □

The representation of $su(2)$ on $\mathbb{C}$ is trivial, we compute the $su(2)$−Casimir on the representation of interest.

$$\Sigma^8_{i=6}[(ad\otimes\rho_{-l})_\star(e_i)(ad\otimes\rho_{-l})_\star(e_i)]|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}}=-8Id|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}}.$$

Elementary calculation yields the action of e_5 via ρ_l:

$$\rho_{-l}(e_5)|_{\mathbb{C}}=-\frac{2li}{\sqrt{3}}Id|_{\mathbb{C}},\ \text{consequently}\ [\rho_{-l}(e_5)\rho_{-l}(e_5)]|_{\mathbb{C}}=-\frac{4l^2}{3}Id|_{\mathbb{C}}. \tag{137}$$

On the other hand, Lemma 13.4 says that $ad_{e_5}|_{\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}$ is a (complex) scalar multiple of the identity. Thus $ad_{e_5}|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}=0$. We obtain

$$\begin{aligned}&[(ad\otimes\rho_{-l})_\star(e_5)(ad\otimes\rho_{-l})_\star(e_5)]|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}}=Id_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}}\otimes[\rho_{-l,\star}(e_5)\rho_{-l,\star}(e_5)\mathbb{C}]\\ =\ &-\frac{4l^2}{3}Id|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}}.\end{aligned} \tag{138}$$

Combining (137) and (138), we arrive at the desired Casimir.

Formula 13.5.

$$\begin{aligned}&Cas^{\mathcal{B}_{s[u(1)\times u(2)]}}_{s[u(1)\times u(2)],ad\otimes\rho_{-l}}|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}}\triangleq\Sigma^8_{i=5}[(ad\otimes\rho_{-l})_\star(e_i)(ad\otimes\rho_{-l})_\star(e_i)]|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}}\\ =\ &(-8-\frac{4l^2}{3})Id|_{End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}}.\end{aligned} \tag{139}$$

13.2 The translation between two conventions of $SU(3)$−representations

Let $W_{1,0}$ be the standard representation of $su(3)$ on $\mathbb{C}^3$, and $W_{0,1}$ be the dual representation of $W_{1,0}$. Let $W_{a,b}$ be the irreducible representation generated by the highest weight vector in the tensor product representation $W^{\otimes a}_{1,0}\otimes W^{\otimes b}_{0,1}$ (see [16, II.5]). Any irreducible representation of $su(3)$ is equivalent to $W_{a,b}$ for some integer $a,\ b\geq 0$ ($W_{0,0}$ is the 1−dimensional trivial representation).

Notation Convention 13.6. In [18], a $SU(3)$ irreducible representation is labelled by an integer linear combination of the two weights $x_1^\star,\ x_2^\star$. We denoted it by $V^{SU(3)}_{m_1x_1^\star+m_2x_2^\star}$, and this is said to be the *Ikeda-Taniguchi convention*. In [18], such integer linear combinations also label the irreducible $S[U(1)\times U(2)]-$representations. We denote it by $V^{S[U(1)\times U(2)]}_{k_1x_1^\star+k_2x_2^\star}$.

We need the following translation from the Ikeda-Taniguchi convention to the (usual) $W_{a,b}-$convention.

Lemma 13.7. *The irreducible representation $W_{a,b}$ of $SU(3)$ is isomorphic to Ikeda-Taniguchi's $V^{SU(3)}_{(a+b)x_1^\star+bx_2^\star}$.*

Proof of Lemma 13.7: It is an algebra exercise to verify that in Ikeda-Taniguchi convention, the standard representation $W_{1,0}$ of $su(3)$ has highest weight $x_1^\star$, the dual representation $W_{0,1}$ has highest weight $-x_3^\star$, which is equal to $x_1^\star+x_2^\star$. The highest weight of a (possibly multiple) tensor product of irreducible $su(3)-$representations is the sum of the highest weight of each one. Thus, the highest weight of $W_{1,0}^{\otimes a}\otimes W_{0,1}^{\otimes b}$ (in Ikeda-Taniguchi convention) is $ax_1^\star-bx_3^\star$, which equals $(a+b)x_1^\star+bx_2^\star$. Because $W_{a,b}$ is the irreducible representation generated by the highest weight vector in $W_{1,0}^{\otimes a}\otimes W_{0,1}^{\otimes b}$, the highest weight of $W_{a,b}$ is the same i.e. $(a+b)x_1^\star+bx_2^\star$. $\square$

13.3 The irreducible $S[U(1)\times U(2)]-$representation $End_0\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}\otimes\mathbb{C}$ in Ikeda-Taniguchi convention

The Cartan sub-algebra $\Upsilon_{su(3)}$ of $su(3)$ consists of diagonal traceless matrices with purely imaginary diagonal entries. Let $x_i^\star$ maps any matrix in $\Upsilon_{su(3)}$ to its $i-$th diagonal entry. Then $x_1^\star,\ x_2^\star,\ x_3^\star$ are roots. They are subject to the relation $x_1^\star+x_2^\star+x_3^\star=0$. According to [18, Section 5, Page 529], the partial ordering is determined by

$$x_1^\star>x_2^\star>0>x_3^\star.$$

Moreover, $\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}$ has highest weight $x_2^\star-x_1^\star$, and the dual $\mathfrak{m}_{\mathbb{P}^2}^{(1,0),\star}$ has highest weight $x_1^\star-x_3^\star$ (see [18, page 532, (iii)]). Then the highest weight of $End\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}=\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}\otimes\mathfrak{m}_{\mathbb{P}^2}^{(1,0),\star}$ is $x_1^\star+2x_2^\star$: the sum of the highest weights of $\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}$ and $\mathfrak{m}_{\mathbb{P}^2}^\star$.

Lemma 13.8. *The tensor product representation*

$$Ad\otimes\rho_{-l}:\ S[U(1)\times U(2)]\to GL(End_0\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}\otimes\mathbb{C})$$

is equivalent to $V^{S[U(1)\times U(2)]}_{(-l+1)x_1^\star+2x_2^\star}$.

Proof of Lemma 13.8: Because $S[U(1)\times U(2)]$ is a subgroup of $SU(3)$ having the same Cartan sub-algebra $\Upsilon_{su(3)}$, the highest weight on $End_0\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}$ is the same as the highest weight as a $SU(3)-$representation, which is equal to $x_1^\star+2x_2^\star$. Because $\mathbb{C}$ is $1-$dimensional, the only weight for ρ_{-l} is $-lx_1^\star$. Then in Ikeda-Taniguchi convention, the representation $Ad\otimes\rho_{-l}$ is denoted by $V^{S[U(1)\times U(2)]}_{(-l+1)x_1^\star+2x_2^\star}$. $\square$

13.4 The infinite dimensional $SU(3)-$representation of invariant functions

Using the translation in (13.7) between two different conventions, we re-state the result of Ikeda-Taniguchi in the following.

Fact 13.9. (Ikeda-Taniguchi [18, Proposition 5.1, Proposition 1.1])
Let l be an integer, and let a, b be nonnegative integers. $W_{a,b}$ appears as an irreducible summand in $C^{\infty}_{S[U(1)\times U(2)],Ad\otimes\rho_{-l}}(SU(3), End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C})$ if and only if

$$\max(3-a-2b, b-a-3) \leq l \leq \min(2a+b-3, 3+b-a). \tag{140}$$

Proof of Fact 13.9: The representation $SU(3)-$representation $W_{a,b}$ is also a representation of the subgroup $S[U(1)\times U(2)]$ by restriction. The Frobenius reciprocal theorem (for example, see [18, Proposition 1.1])) implies that the following two conditions are equivalent.

- As $SU(3)-$representations, $W_{a,b}$ appears as an irreducible summand in $C^{\infty}_{S[U(1)\times U(2)],Ad\otimes\rho_{-l}}(SU(3), End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C})$.
- As $S[U(1)\times U(2)]-$representations, $(End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C}, Ad\otimes\rho_{-l})$ appears as an irreducible summand in $W_{a,b}$.

It suffices to determine for which a, b the latter happens.

[18, Proposition 5.1] states that $V^{S[U(1)\times U(2)]}_{k_1x_1^\star+k_2x_2^\star}$ appear as an irreducible summand in $V^{SU(3))}_{m_1x_1^\star+m_2x_2^\star}$ if and only if the following holds.

$$m_1 \geq k_2+k \geq m_2 \geq k \geq 0, \text{ and } k_1 = m_1+m_2-k_2-3k. \tag{141}$$

Because of Lemma 13.7 and 13.8, to verify the second bullet point above, it suffices to let $m_1 = a+b$, $m_2 = b$, $k_1 = -l+1$, $k_2 = 2$. Then the second bullet point holds if and only if

$$a+b \geq 2+k \geq b \geq k \geq 0, \text{ and } 3k = a+2b-3+l. \tag{142}$$

Elementary calculation shows (142) is equivalent to (140). □

14 Proof of Theorem C and D

In conjunction with the notation convention in Definition 11.7 above, the known formula for the quadratic Casimir operator of $su(3)$ states:

Formula 14.1. $Cas^{\mathcal{B}_{su(3)}}_{su(3),W_{a,b}} = \Sigma_{i=1}^8 (e_i|_{W_{a,b}})\cdot(e_i|_{W_{a,b}}) = (-\frac{4}{3}a^2-\frac{4}{3}b^2-4a-4b-\frac{4}{3}ab)Id.$

The tools at our disposal now can be assembled to achieve our goal.

14.1 Theorem D

Proof of Theorem D. We first prove Theorem D.(115). It is a direct corollary of Fact 13.9 on the irreducible summand of the infinite dimensional representation, Formula 14.1 for the Casimir operator of $su(3)$, Formula 13.5 for $Cas^{\mathcal{B}_{s[u(1)\times u(2)]}}_{s[u(1)\times u(2)],ad\otimes\rho_{-l}}$, and the general Formula 11.10 for rough Laplacian on a homogeneous bundle over a Killing reductive homogeneous space.

Because of the $SU(3)-$equivariant isomorphism:

$$(End_0T'\mathbb{P}^2)(l) \to SU(3)\times_{S[U(1)\times U(2)],Ad\otimes\rho_{-l}}[(End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2})\otimes\mathbb{C}],$$

the general formula 11.10 for $G = SU(3)$, $K = S[U(1)\times U(2)]$, and $\rho = Ad\otimes\rho_{-l}$ says that the spectrum of the rough Laplacian is equal to the spectrum of

$$-Cas^{\mathcal{B}_{su(3)}}_{su(3),L} + Cas^{\mathcal{B}_{s[u(1)\times u(2)]}}_{s[u(1)\times u(2)],Ad\otimes\rho_{-l}} \tag{143}$$

on the space $C^\infty_{S[U(1)\times U(2)],Ad\otimes\rho_{-l}}(SU(3), End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C})$ of invariant functions.

On the whole $C^\infty_{S[U(1)\times U(2)],Ad\otimes\rho_{-l}}(SU(3), End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C})$, by Formula 13.5, $Cas^{\mathcal{B}_{s[u(1)\times u(2)]}}_{s[u(1)\times u(2)],Ad\otimes\rho_{-l}}$ acts by $-(\frac{4}{3}l^2+8)Id$. In the Peter-Weyl formulation (see the presentation in [29, Section 5.1]), as $SU(3)-$representations, on each irreducible summand $W_{a,b}$ of $C^\infty_{S[U(1)\times U(2)],Ad\otimes\rho_{-l}}(SU(3), End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C})$, Formula 14.1 says that the action of $-Cas^{\mathcal{B}_{su(3)}}_{su(3),L}$ is the scalar multiplication by $\frac{4}{3}(a^2+b^2+ab+3a+3b)Id$. Then on the irreducible summand $W_{a,b}$, the action of the Casimir (143) is the scalar multiplication by

$$\frac{4}{3}(a^2+b^2+ab+3a+3b)-\frac{4}{3}l^2-8.$$

Fact 13.9 says that $W_{a,b}$ appears as an irreducible summand if and only if the condition on the right side of (115) holds. The proof of Theorem D.(115) is complete.

Hence, Theorem D.(116) directly follows from the spectral splitting in Formula 8.3 and Theorem D.(115): we only need to add l^2 to the $\frac{4}{3}(a^2+b^2+ab+3a+3b)-\frac{4}{3}l^2-8$ in (115).

Next, we address the multiplicities. It is evident from the first 4 paragraphs in the underlying proof that the eigenspace of any $\lambda_l\in Spec\nabla^\star\nabla|_{(End_0T'\mathbb{P}^2)(l)\to\mathbb{P}^2}$ is isomorphic to the direct sum of all those $W_{a,b}$ such that

- $W_{a,b}$ is a summand in $C^\infty_{S[U(1)\times U(2)],Ad\otimes\rho_{-l}}(SU(3), End_0\mathfrak{m}^{(1,0)}_{\mathbb{P}^2}\otimes\mathbb{C})$ i.e. the conditions for a, b, on the right side of (115) holds;
- the value of $\frac{4}{3}(a^2+b^2+ab+3a+3b)-\frac{4}{3}l^2-8$ is equal to λ_l.

In the terminology of Theorem D , the above means that $(a,b)\in S^l_{\lambda_l}$. The proof of Theorem D .(117) is complete.

Hence, Theorem D .(118) follows by (117), and the spectral splitting (74) counting multiplicity. □

14.2 Theorem C

Theorem C can be proved using Theorem D, A, and the following Lemma on cohomology.

Lemma 14.2. $h^1[\mathbb{P}^2,\ (EndT'\mathbb{P}^2)(l)]=\begin{cases}3 & if\ l=-1\ or\ -2,\\ 0 & otherwise.\end{cases}$

Consequently, $h^0[\mathbb{P}^2,\ (End_0T'\mathbb{P}^2)(l)]=\begin{cases}\frac{3l(l+3)}{2} & if\ l>0,\\ 0 & if\ l\leq 0.\end{cases}$

The proof of Lemma 14.2 is completely routine via Euler sequence and Bott formula for sheaf cohomology on $\mathbb{P}^n$ (see [31]). We defer it to Appendix 17.5.

Proof of Theorem C:. Under the setting of Theorem A, $EndE=End(T'\mathbb{P}^2)$ (as pullbacks) is equipped with the pullback Fubini-Study connection. Lemma 14.2 means that except when $l\neq -1$ or -2, the sheaf cohomology has no contribution to $SpecP$. On the other hand, Theorem D addresses the source $Spec\nabla^\star\nabla|_{\mathbb{S}^5}$ of the other part of $SpecP$.

We need the fact that any $W_{a,b}$ appears in the infinite-dimensional representation at most once. This is because the representation of the associated bundle is irreducible (see Lemma 13.8). Please see the Frobenius reciprocal theorem (stated in [18, Proposition 1.1]), and also [18, Proposition 5.1].

We seek for those eigenvalues of $\nabla^\star\nabla|_{\mathbb{S}^5}$ that is strictly less than 8. When $l\geq 3$, because of the "$+l^2$" in Formula 8.3.(74), the eigenvalues of $\nabla^\star\nabla|_{\mathbb{S}^5}$ generated are ≥ 9. Thus, it suffices to assume $-2 \leq l \leq 2$ and seek for those eigenvalues of $\nabla^\star\nabla|_{(End_0 T'\mathbb{P}^2)(l)\to\mathbb{P}^2}$ that is strictly less than 8.

Under the conditions on $a,\ b$ in Theorem D.(115), elementary calculation shows that this can only happen for the following values of $l,\ a,\ b$.

- $l = 0,\ (a, b) = (1, 1)$. In this case, the corresponding eigenvalue of $\nabla^\star\nabla|_{End_0 T'\mathbb{P}^2}$ is 4, the eigenspace is isomorphic to $W_{1,1}$.

- $l = 1,\ (a, b) = (2, 0)$. In this case, the corresponding eigenvalue of $\nabla^\star\nabla|_{(End_0 T'\mathbb{P}^2)(1)}$ is 5, the eigenspace is isomorphic to $W_{2,0}$.

- $l = -1,\ (a, b) = (0, 2)$. In this case, the corresponding eigenvalue of $\nabla^\star\nabla|_{(End_0 T'\mathbb{P}^2)(-1)}$ is 5, the eigenspace is isomorphic to $W_{0,2}$.

Then, still according to Formula 8.3.(74), the above three cases generate the numbers 4 and 5 in $Spec\nabla^\star\nabla|_{\mathbb{S}^5}$. The multiplicity of 4 is equal to $dimW_{1,1} = 8$, the multiplicity of 5 is equal to $dimW_{2,0} + dimW_{0,2} = 12$. The former generates the following values in $SpecP$.

$$2\sqrt{2} - 1,\ 2\sqrt{2} - 2,\ -1 - 2\sqrt{2},\ -2 - 2\sqrt{2},$$

The latter generates the following values in $SpecP$.

$$1,\ 2,\ -4,\ -5.$$

Among the above 8 numbers, $2\sqrt{2} - 2$ and 1 are the only ones in the interval $(0, 1]$. Because $2\sqrt{2}-2$ is not an integer, its multiplicity is 16 i.e. twice of the multiplicity of $4 \in Spec\nabla^\star\nabla|_{\mathbb{S}^5}$. The other number 1 is an integer, in view of Lemma 14.2, the multiplicity is

$$2dim\mathbb{E}_5(\nabla^\star\nabla|_{\mathbb{S}^5}) - 2h^0[\mathbb{P}^2,\ (End_0 T'\mathbb{P}^2)(1)] = 24 - 12 = 12.$$

The proof of Table (4) is complete. This is a sample of how the multiplicity of each eigenvalue of P is determined. $\square$

15 Index of the deformation of a stable reflexive sheave on a Calabi-Yau 3−fold

The operator P and Hermitian Yang-Mills connections with isolated singularities

In general dimensions, we do not know whether the linearization of Hermitian Yang-Mills connection is formally self-adjoint. Neverthelessr, there is a self adjoint formulation on a compact Calabi-Yau 3−fold (X, ω, Ω).

On a Hermitian vector bundle over the Calabi-Yau, a triple (A, σ, u) consisted of a smooth unitary connection A and two smooth sections σ and u of the adjoint bundle is called a Hermitian Yang-Mills monopole if it satisfies the following equations.

$$F_A \lrcorner Re\Omega + d_A\sigma - J(d_A u) = 0,\ F_A \lrcorner \omega = 0.$$

The global linearized operator is

$$\square_{CY^3} \begin{bmatrix} \sigma \\ u \\ a \end{bmatrix} = \begin{bmatrix} d_A^\star a \\ d_A a \lrcorner \omega \\ d_A\sigma - J(d_A u) + (d_A a) \lrcorner Re\Omega \end{bmatrix},$$

which is apparently self-adjoint because Ω is closed.

Let $\mathcal{F}$ be an admissible stable reflexive sheaf over the Calabi-Yau $3-$fold, such that that near any possible singular point, under a punched coordinate neighborhood biholomorphic to $B_O(1)\setminus O$, the sheaf is locally free and is the pullback of a non-projective flat polystable holomorphic vector bundle E over $\mathbb{P}^2$. Bando-Siu [2] proved the existence of an admissible Hermitian Yang-Mills connection on such a sheaf. Jacob-Sá Earp-Walpuski [19] showed that the induced projective connection is asymptotic to a projective Hermitian Yang-Mills connection on $E\to\mathbb{P}^2$.

Near a (nontrivial) singular point defined as above, with gauge fixing and two monopole terms and under model data, the model linearized operator on $EndE\to\mathbb{C}^3\setminus O$ is

$$\square = K\circ(\frac{\partial}{\partial r}-\frac{P}{r})\quad \text{(see (4.1))}.$$

This is part of the linearization of G_2-instantons (see Formula 4.1 and (181)). Under the cylindrical coordinate

$$t=-\log r,\ r \text{ is the distance to the origin } O,$$

we find

$$\square = -e^t K\circ(\frac{\partial}{\partial t}+P).$$

This fits into the setting of translation-invariant operators considered classically by Atiyah-Patodi-Singer [1].

Index of the operator between weighted Schauder spaces

Under the usual weighted Schauder spaces on $adE^{\oplus 2}\oplus\Omega^1(adE)\to CY^3\setminus\cup_j O_j$ (see [42]), let $\delta>0$, the linearized operator

$$\square:\ C^{1,\frac{1}{2}}_{1-\delta}\to C^{0,\frac{1}{2}}_{2-\delta}\ \text{ (suppressing the bundles) is bounded.} \tag{144}$$

Then convention is that $|\xi|_{euc}=O(\frac{1}{r^{1-\delta}})$ if ξ belongs to the domain $C^{1,\frac{1}{2}}_{1-\delta}$. This deformation is supposed to preserve the tangent connection at the singular point.

The inner product between $1-$forms under the Euclidean metric on $\mathbb{C}^3$ is $\frac{1}{r^2}$ times that under the cylindrical metric $dt^2+g_{\mathbb{S}^5}$ on $\mathbb{C}^3\setminus O$. Hence, the domain $C^{1,\frac{1}{2}}_{1-\delta}$ is equal to $C^{1,\frac{1}{2}}_{\delta,cyl}$ under cylindrical coordinate and norm in which sections decay as $O(e^{-\delta t})$. The conformal changed target space $r\cdot C^{0,\frac{1}{2}}_{2-\delta}$ is equal to $C^{0,\frac{1}{2}}_{\delta,cyl}$ in which sections decay also as $O(e^{-\delta t})$. Comparing to the weight change in (144), the advantage of cylindrical theory is that the weight for the domain and target Schauder spaces are the same. Please see the Definition in [43, Section 2]. Namely,

$$r\square = e^{-t}\square:\ C^{1,\frac{1}{2}}_{\delta,cyl}\to C^{\frac{1}{2}}_{\delta,cyl}\ \text{ is bounded under cylindrical coordinate} \tag{145}$$

with an unanimous weight $e^{\delta t}$.

Using the regularity for harmonic sections [43, Proposition 4.5], the index of (145) therefore also of (144) is the same as that of the following simple weighted Sobolev-theory

$$e^{-t}\square:\ W^{1,2}_{\delta,cyl}\to L^2_{\delta,cyl}. \tag{146}$$

The space of $W^{k,2}_{\delta,cyl}$ is simply the conformal change $e^{\delta t}W^{k,2}_{cyl}$ if we extend t to the interior by 0 (r by 1). Let g_{CY^3} be the metric of the Kähler form ω. The metric of conformal change

$$g_{cyl}\triangleq r^2 g_{CY^3}=e^{-2t}g_{CY^3}$$

is asymptotic cylindrical at the singular points of the sheaf. It has volume form

$$-e^{-6t}dvol_{g_{cyl}} = dvol_{g_{CY^3}}.$$

Proof of Theorem *B.* We calculate via the cylindrical coordinate change (144) that

$$\int_{CY^3\setminus\cup_j O_j} \langle \Box_{CY^3}\cdot, \cdot\rangle_{g_{CY^3}} dvol_{g_{CY^3}} = \int_{CY^3\setminus\cup_j O_j} \langle e^{-t}\Box_{CY^3}\cdot, \cdot\rangle_{g_{cyl}} e^{-3t} dvol_{g_{cyl}} \tag{147}$$

That $\Box_{CY^3}$ is formally self-adjoint is equivalent to that the conformal change $e^{-t}\Box_{CY^3}$ is formally self-adjoint with respect to the inner product

$$\int_{CY^3\setminus\cup_j O_j} \langle \cdot, \cdot\rangle_{g_{cyl}} e^{-3t} dvol_{g_{cyl}}.$$

This means that in the cylindrical setting, the weight $-\frac{3}{2}$ is the one under which $e^{-t}\Box_{CY^3}$ is formally self adjoint. We notice that $Spec^{mul}P$ is indeed symmetric with respect to $-\frac{3}{2}$ in the sense of Theorem A multiplicities. Therefore, by the Lockhart-McOwen index change formula [25, Theorem 6.2] and the observation for formally self adjoint asymptotic cylindrical operators [25, proof of Theorem 7.4], if δ is positive and small enough with respect to the gaps of $SpecP$, we find

$$Index|_{(146)} = \Sigma_j\{-2h^1[\mathbb{P}^2, EndE(-1)] - 2h^1[\mathbb{P}^2, EndE]\}. \tag{148}$$

Namely, formal self adjointness of $e^{-t}\Box$ with respect the measure $e^{-3t}dvol_{g_{cyl}}$ says

$$\text{Index for weight } e^{\delta t} = -\text{ Index for weight } e^{(-3-\delta)t}.$$

The index change formula says their difference

$$\text{Index for weight } e^{(-3-\delta)t} - \text{Index for weight } e^{\delta t}$$

is the sum of the multiplicities of all eigenvalues between the two weights $-3-\delta$ and δ. Therefore $Index|_{(146)}$ is minus the half sum. Our Theorem A says the only possible eigen values in $(-3-\delta, \delta)$ are -1, -2 and $0, -3$. Theorem A multiplicities says the $\mathbb{R}-$multiplicities of the former two are both $\Sigma_j 2h^1[\mathbb{P}^2, (EndE_j)(-1)]$, and those of the latter two are $\Sigma_j 2h^1[\mathbb{P}^2, (EndE_j)]$ ($\mathbb{R}-$dimension of the deformation space of $E \to \mathbb{P}^2$). The sum of the multiplicities of these 4 eigenvalues is $\Sigma_j\{4h^1[\mathbb{P}^2, EndE(-1)] + 4h^1[\mathbb{P}^2, EndE]\}$. Divide it by 2, the proof of (148) is complete. Recall that

$$Index|_{(146)} = Index|_{(144)}.$$

The proof of Theorem B is then complete. □

16 Some remark on the projectively flat case

In our main Theorems A—D, non-projective flatness is assumed. However, in the generating force Theorem 9.2, we do not assume it. We are more interested in the non-projective flatness of (E, A), because they represent nontrivial singularities.

If E, A is projectively flat and with rank ≥ 2, $V_{coh} = \{0\}$ i.e. the co-homologies vanish. Moreover, $EndE$ is just the trivial bundle with rank equal to the square of that of E, the induced connection is trivial. Therefore, $Spec\nabla^\star\nabla|_{\mathbb{S}^5}$ are just the spectrum of the usual Laplace-Beltrami on $\mathbb{P}^2$ with Fubini-Study metric: the integers of the form

$$\iota^2 + 4\iota,\ \iota \in \mathbb{Z}_{\geq 0}.$$

We can obtain this formula via the tools herein, applied to the trivial bundle (connection) and the one dimensional $S[U(1) \times U(2)]$ representations ρ_{-l} that acts through $U(1)$.

17 Appendix

17.1 Elementary Sasakian geometry

Formula 2.5 and Lemma 2.6

Elementary calculations establish the formulas for $\eta,\ G,\ H$.

Proof of Formula *2.5:* It suffices to check it in $U_{0,\mathbb{C}^3}$, the proof is similar in $U_{1,\mathbb{C}^3}$ and $U_{2,\mathbb{C}^3}$. We first have $Z_0\frac{\partial}{\partial Z_1}=\frac{Z_0\bar{Z}_1}{2r}\frac{\partial}{\partial r}+\frac{\partial}{\partial u_1}$ and $Z_0\frac{\partial}{\partial Z_2}=\frac{Z_0\bar{Z}_2}{2r}\frac{\partial}{\partial r}+\frac{\partial}{\partial u_2}$. Using

$$\eta=\frac{1}{2}d^c\log(|Z_0|^2+|Z_1|^2+|Z_2|^2)\ \ \text{(see definition (6))},$$

we find

$$\eta(\frac{\partial}{\partial u_1})=\eta(Z_0\frac{\partial}{\partial Z_1})=-\frac{\sqrt{-1}\bar{u}_1}{2\phi_0}=-\frac{\sqrt{-1}}{2}\frac{\partial\log\phi_0}{\partial u_1}. \tag{149}$$

Similarly, we have $\eta(\frac{\partial}{\partial u_2})=-\frac{\sqrt{-1}}{2}\frac{\partial\log\phi_0}{\partial u_2}$. Taking conjugation, we then obtain

$$\eta(\frac{\partial}{\partial\bar{u}_1})=\frac{\sqrt{-1}}{2}\frac{\partial\log\phi_0}{\partial\bar{u}_1},\quad \eta(\frac{\partial}{\partial\bar{u}_2})=\frac{\sqrt{-1}}{2}\frac{\partial\log\phi_0}{\partial\bar{u}_2}.$$

The proof is complete by observing that η coincides with $d\theta_0+\frac{d^c\log\phi_0}{2}$ on the basis

$$\frac{\partial}{\partial\theta_0},\ \frac{\partial}{\partial u_1},\ \frac{\partial}{\partial u_2},\ \frac{\partial}{\partial\bar{u}_1},\ \frac{\partial}{\partial\bar{u}_2}\quad\text{for } T^{\mathbb{C}}\mathbb{S}^5.$$

□

Proof of Lemma *2.6:* We routinely verify in $U_{0,\mathbb{C}^3}$ that

$$\begin{aligned}\frac{dZ_0}{Z_0}&=\frac{1}{Z_0}d(\frac{re^{\sqrt{-1}\theta_0}}{\sqrt{\phi_0}})=\frac{dr}{r}-\frac{d\log\phi_0}{2}+\sqrt{-1}d\theta_0\\ &=\frac{dr}{r}-\frac{d\log\phi_0}{2}+\sqrt{-1}\eta-\frac{\sqrt{-1}}{2}(d^c\log\phi_0)\ \text{by Formula 2.5.}\end{aligned}$$

When $i=1,2$, we calculate $\frac{dZ_i}{Z_0}=\frac{d(Z_0u_i)}{Z_0}=du_i+u_i(\frac{dZ_0}{Z_0})$. Then

$$\begin{aligned}\Omega_{\mathbb{C}^3}&=dZ_0dZ_1dZ_2=Z_0^3\cdot\frac{dZ_0}{Z_0}\frac{dZ_1}{Z_0}\frac{dZ_2}{Z_0}=Z_0^3\cdot\frac{dZ_0}{Z_0}\wedge[du_1+u_1(\frac{dZ_0}{Z_0})]\wedge[du_2+u_2(\frac{dZ_0}{Z_0})].\\ &=Z_0^3\cdot\frac{dZ_0}{Z_0}\wedge du_1\wedge du_2\\ &=Z_0^3\cdot\frac{dr}{r}\wedge du_1\wedge du_2+\sqrt{-1}Z_0^3\cdot\eta\wedge du_1\wedge du_2.\end{aligned}$$

The last inequality above uses that $d\log\phi_0,\ d^c\log\phi_0$ are both pulled back from $U_{0,\mathbb{P}^2}\subset\mathbb{P}^2$, therefore $d\log\phi_0\wedge du_1\wedge du_2=d^c\log\phi_0\wedge du_1\wedge du_2=0$.

In $U_{0,\mathbb{C}^3}$, the proof of (11) and the first row in (14) is complete. The proof is similar in $U_{1,\mathbb{C}^3}$ and $U_{2,\mathbb{C}^3}$. □

Transverse geodesic frame

For any point $p \in \mathbb{S}^5$ and $X,\ Y \in D|_p$, the following is true.

$$\begin{aligned}&\nabla_X \xi = J_0(X),\ \nabla_X \eta = [J_0(X)]^{\sharp},\ (\nabla^2 \xi)(X,Y) = - < X,Y > \xi,\\ &(\nabla^2 \eta)(X,Y) = - < X,Y > \eta. \text{ Consequently, } \nabla^{\star}\nabla \eta = 4\eta.\end{aligned} \tag{150}$$

For the point-wise calculations in the proof of Lemma 4.3 and others, it is helpful to have a transverse geodesic frame in the following sense.

Lemma 17.1. *(Properties of a transverse geodesic frame) Let* $(x_i,\ i = 1, ..., 4)$ *be a Kähler geodesic coordinate with respect to the Fubini-Study metric induced by* $\frac{d\eta}{2}$ *at (near) an arbitrary point* $[p] \in \mathbb{P}^2$*. Then, for any* β *among* $0,\ 1,\ 2$ *such that* $[p] \in U_{\beta,\mathbb{P}^2}$*, the following vector fields*

$$[\xi;\ v_i \triangleq \frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})\xi, i = 1, 2, 3, 4]$$

is a frame near the Reeb orbit $\pi_{5,4}^{-1}[p]$*, and is orthonormal on* $\pi_{5,4}^{-1}[p]$*. Moreover, the following holds on the Reeb orbit.*

$$(\nabla_{v_i} v_i)|_{\pi_{5,4}^{-1}[p]} = 0,\ [\nabla_{v_i}(J_0 v_i)]|_{\pi_{5,4}^{-1}[p]} = -\xi.$$

Near the Reeb orbit, we call the $(v_i,\ i = 1, 2, 3, 4)$ above a transverse geodesic frame. It is generated by the geodesic coordinate on $\mathbb{P}^2$.

Proof of Lemma 17.1: We first show

$$[v_i, v_j] = [d\eta(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i})]\xi. \tag{151}$$

Because the Reed vector-field ξ is a coordinate vector field in U_{β,θ_β} for any $\beta = 0, 1, \text{or } 2$, we have the vanishing

$$[\xi, \frac{\partial}{\partial x_i}] = 0 \text{ for any } i. \tag{152}$$

Then we calculate

$$\begin{aligned}[v_i, v_j] &= [\frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})\xi,\ \frac{\partial}{\partial x_j} - \eta(\frac{\partial}{\partial x_j})\xi] = [-\frac{\partial}{\partial x_i}\eta(\frac{\partial}{\partial x_j}) + \frac{\partial}{\partial x_j}\eta(\frac{\partial}{\partial x_i})]\xi\\ &= [d\eta(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i})]\xi.\end{aligned} \tag{153}$$

The proof of (151) is complete.

The vanishing (152) above implies the following vanishing.

$$[\xi, v_i] = 0 \text{ for any } i. \tag{154}$$

The identity (151) implies that the Lie bracket of v_i and v_j is perpendicular to both v_i and v_j. Then, using the Koszul formula [32, page 25], we find

$$2\langle \nabla_{v_i} v_j, v_k \rangle = v_i \langle v_j, v_k \rangle - v_k \langle v_i, v_j \rangle + v_j \langle v_k, v_i \rangle. \tag{155}$$

We recall the following formula for the standard metric on $\mathbb{S}^5$.

$$g_{\mathbb{S}^5} = \pi_{5,4}^{\star} g_{FS} + \eta \otimes \eta. \tag{156}$$

Then (155) implies

$$
\begin{aligned}
& 2\langle\nabla_{v_i}v_j, v_k\rangle = \frac{\partial}{\partial x_i}\langle\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle_{\mathbb{P}^2} - \frac{\partial}{\partial x_k}\langle\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle_{\mathbb{P}^2} + \frac{\partial}{\partial x_j}\langle\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i}\rangle_{\mathbb{P}^2} \\
= \; & 2\langle\nabla^{FS}_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle_{\mathbb{P}^2} = 2\langle\nabla^{FS}_{\pi_{5,4,\star}v_i}\pi_{5,4,\star}v_j, \pi_{5,4,\star}v_k\rangle_{\mathbb{P}^2} \\
= \; & 2\langle(\pi^\star_{5,4}\nabla^{FS})_{v_i}v_j, v_k\rangle \qquad (157)
\end{aligned}
$$

Moreover, the Lie bracket identity (154) and the Koszul formula yield that

$$2\langle\nabla_{v_i}v_j, \xi\rangle =< [v_i, v_j], \xi >= d\eta(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}). \qquad (158)$$

Thus the identities (157) and (158) yield

$$\nabla_{v_i}v_j = (\pi^\star_{5,4}\nabla^{FS})_{v_i}v_j + \xi[\frac{d\eta}{2}(v_j, v_i)]. \qquad (159)$$

Via the tangent map $\pi_{5,4,\star}D \to T\mathbb{P}^2$ which is an isometry, we verify that $J_0 = \pi^\star_{5,4}J_{\mathbb{P}^2}$. This implies that $J_0v_1 = v_2$, $J_0v_3 = v_4$, because $J_{\mathbb{P}^2}\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}$, $J_{\mathbb{P}^2}\frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4}$. Using that $(\nabla^{FS}_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j})|_{[p]} = 0$, the following holds true.

$$
\begin{aligned}
& \nabla_{v_i}(J_0v_i) = (\pi^\star_{5,4}\nabla^{FS})_{v_i}J_0v_i + \xi[\frac{d\eta}{2}(J_0v_i, v_i)] = \nabla^{FS}_{\frac{\partial}{\partial x_i}}(J_{\mathbb{P}^2}\frac{\partial}{\partial x_i}) + \xi[\frac{d\eta}{2}(J_0v_i, v_i)] \\
= \; & -\xi \quad \text{on the Reeb orbit } \pi^{-1}_{5,4}[p]. \qquad (160)
\end{aligned}
$$

Similarly, we compute

$$(\nabla_{v_i}v_i)|_q = 0 + \xi[\frac{d\eta}{2}(v_i, v_i)]|_q = 0.$$

The proof is complete. □

The transverse geodesic frame helps in proving the following two formulas which are applied in the proof of the Bochner formulas (see Lemma 5.4 above).

Formula 17.2. In the setting of Lemma 5.4, $(\nabla^\star\nabla a_0)(\xi) = 2d_0^{\star_0}J_0(a_0)$.

Proof of Formula 17.2. : Using that a_0 is semi-basic i.e. $a_0(\xi) = 0$, Leibniz-rule yields

$$0 = (\nabla^\star\nabla)[a_0(\xi)] = (\nabla^\star\nabla a_0)(\xi) - 2tr(\nabla a_0 \otimes \nabla\xi) + a_0(\nabla^\star\nabla\xi). \qquad (161)$$

Because $\nabla^\star\nabla\xi = 4\xi$ (see (151)), we find $a_0(\nabla^\star\nabla\xi) = 0$, hence

$$(\nabla^\star\nabla a_0)(\xi) = 2tr(\nabla a_0 \otimes \nabla\xi). \qquad (162)$$

Therefore, at an arbitrary $p \in \mathbb{S}^5$, let v_i be a transverse geodesic frame given by Lemma 17.1, using $\nabla_\xi\xi = 0$, we compute

$$
\begin{aligned}
& 2tr(\nabla a_0 \otimes \nabla\xi)|_p = 2\Sigma^4_{i=1}(\nabla_{v_i}a_0)(\nabla_{v_i}\xi)|_p = 2\Sigma^4_{i=1}(\nabla_{v_i}a_0)(J_0v_i)|_p \\
= \; & (2\Sigma^4_{i=1}\nabla_{v_i}[a_0(J_0v_i)] - 2a_0[\Sigma^4_{i=1}\nabla_{v_i}(J_0v_i)])|_p = 2\Sigma^4_{i=1}\nabla_{v_i}[a_0(J_0v_i)]|_p \\
= \; & -2\Sigma^4_{i=1}\nabla_{v_i}[(J_0a_0)(v_i)]|_p \\
= \; & 2d_0^{\star_0}J_0(a_0)|_p. \qquad (163)
\end{aligned}
$$

The proof is complete by the above two identities. □

Formula 17.3. In the setting of Lemma 5.4, $[\nabla^\star\nabla(\eta a_\eta)] = \eta(4a_\eta + \nabla^\star\nabla a_\eta) - 2J_0(d_0 a_\eta)$.

Proof of formula 17.3: We still work with a transverse geodesic frame v_i $(i = 1, 2, 3, 4)$ at an arbitrary $p \in \mathbb{S}^5$. Using the fundamental identities (151), we calculate

$$\begin{aligned} & \nabla^\star\nabla(a_\eta\eta) = (\nabla^\star\nabla a_\eta)\eta + a_\eta\nabla^\star\nabla\eta - 2tr(\nabla a_\eta \otimes \nabla\eta) \\ = & \ (\nabla^\star\nabla a_\eta)\eta + 4a_\eta\eta - 2tr(\nabla a_\eta \otimes \nabla\eta). \end{aligned} \tag{164}$$

In view of the local formula (25) for $d_0 a_\eta$, the transverse geodesic frame yields

$$(\nabla_{v_i} a_\eta)dx^i|_p = (d_0 a_\eta)|_p. \tag{165}$$

Using the vanishing $\nabla_\xi\eta = 0$ and the formula $\nabla_X\eta = [J_0(X^{\|_0})]^\sharp$, the trace term in the above identity can be additionally analyzed as follows.

$$\begin{aligned} & tr(\nabla a_\eta \otimes \nabla\eta)|_p = [\Sigma_{i=1}^4 \nabla_{v_i} a_\eta \otimes \nabla_{v_i}\eta + L_\xi a_\eta \otimes \nabla_\xi\eta]|_p = [\Sigma_{i=1}^4 \nabla_{v_i} a_\eta \otimes \nabla_{v_i}\eta]|_p \\ = & \ J_0(dx^i)(\nabla_{v_i} a_\eta)|_p \\ = & \ J_0(d_0 a_\eta)|_p \quad \text{by (165)}. \end{aligned}$$

The desired identity follows. □

17.2 The usual separation of variable: proof of Formula 4.1

Proof of Formula 4.1. It is completely routine. To be self-contained, we still show the detail. In view of the splitting (28), we find

$$d_{\mathbb{C}^3\times\mathbb{S}^1} a_{\mathbb{C}^3\times\mathbb{S}^1} = (d_{\mathbb{C}^3}\underline{a}_s) \wedge ds + d_{\mathbb{C}^3} a_{\mathbb{C}^3} - \frac{\partial a_{\mathbb{C}^3}}{\partial s} \wedge ds \tag{166}$$

and $d^{\star_{\mathbb{C}^3\times\mathbb{S}^1}}_{\mathbb{C}^3\times\mathbb{S}^1} a_{\mathbb{C}^3\times\mathbb{S}^1} = -\frac{\partial \underline{a}_s}{\partial s} + d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3} a_{\mathbb{C}^3}$. Using

$$\star_{\mathbb{C}^3\times\mathbb{S}^1}[d_{\mathbb{C}^3\times\mathbb{S}^1} a_{\mathbb{C}^3\times\mathbb{S}^1} \wedge \psi_{\mathbb{C}^3\times\mathbb{S}^1}] = (d_{\mathbb{C}^3\times\mathbb{S}^1} a_{\mathbb{C}^3\times\mathbb{S}^1})\lrcorner_{\mathbb{C}^3\times\mathbb{S}^1}\phi_{\mathbb{C}^3\times\mathbb{S}^1},$$

it suffices to calculate the right side of (166) term-wisely as follows.

$$(d_{\mathbb{C}^3\times\mathbb{S}^1} a_{\mathbb{C}^3\times\mathbb{S}^1})\lrcorner_{\mathbb{C}^3\times\mathbb{S}^1}(\omega_{\mathbb{C}^3} \wedge ds) = -(d_{\mathbb{C}^3}\underline{a}_s)\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3} + (d_{\mathbb{C}^3} a_{\mathbb{C}^3}\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3})ds + \frac{\partial a_{\mathbb{C}^3}}{\partial s}\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3}. \tag{167}$$

$$(d_{\mathbb{C}^3\times\mathbb{S}^1} a_{\mathbb{C}^3\times\mathbb{S}^1})\lrcorner_{\mathbb{C}^3\times\mathbb{S}^1} Re\Omega = (d_{\mathbb{C}^3} a_{\mathbb{C}^3})\lrcorner_{\mathbb{C}^3} Re\Omega. \tag{168}$$

The contraction $\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3}$ is the complex- structure $J_{\mathbb{C}^3}$ on $\Omega^1[ad(E)]$. Summing (167) and (168) up, we arrive at the following.

$$\begin{aligned} & (d_{\mathbb{C}^3\times\mathbb{S}^1} a_{\mathbb{C}^3\times\mathbb{S}^1})\lrcorner_{\mathbb{C}^3\times\mathbb{S}^1}\phi_{\mathbb{C}^3\times\mathbb{S}^1} \\ = & \ -J_{\mathbb{C}^3}(d_{\mathbb{C}^3}\underline{a}_s) + (d_{\mathbb{C}^3} a_{\mathbb{C}^3}\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3})ds + J_{\mathbb{C}^3}(\frac{\partial a_{\mathbb{C}^3}}{\partial s}) + (d_{\mathbb{C}^3} a_{\mathbb{C}^3})\lrcorner_{\mathbb{C}^3} Re\Omega. \end{aligned}$$

Using the above, the easy identity $d_{\mathbb{C}^3\times\mathbb{S}^1}\sigma = (\frac{\partial\sigma}{\partial s})ds + d_{\mathbb{C}^3}\sigma$, and definition (27) of $L_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}$, we obtain the following.

$$L_{A,\phi_{\mathbb{C}^3\times\mathbb{S}^1}}\begin{bmatrix} \sigma \\ \underline{a}_s ds + a_{\mathbb{C}^3} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \underline{a}_s}{\partial s} + d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3} a_{\mathbb{C}^3} \\ \{\frac{\partial\sigma}{\partial s} + (d_{\mathbb{C}^3} a_{\mathbb{C}^3})\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3}\}ds + \\ J_{\mathbb{C}^3}(\frac{\partial a_{\mathbb{C}^3}}{\partial s}) + d_{\mathbb{C}^3}\sigma - J_{\mathbb{C}^3}(d_{\mathbb{C}^3}\underline{a}_s) + (d_{\mathbb{C}^3} a_{\mathbb{C}^3})\lrcorner_{\mathbb{C}^3} Re\Omega \end{bmatrix}.$$

The desired formula follows. □

17.3 The fine separation of variable: proof of Lemma 4.3

We prove Lemma 4.3 by computing each row in the operator $\square$ (see Formula 4.1). We first recall the following splitting.

$$a_{\mathbb{C}^3} = a_0 + (a_\eta \eta) + a_r \frac{dr}{r}. \tag{169}$$

Given a section a of $\wedge^p T^\star \mathbb{S}^5$ and a section b of $\wedge^q T^\star \mathbb{S}^5$ such that $5 \geq q \geq p$, we need the following identity.

$$a \lrcorner_{\mathbb{C}^3} b = \frac{1}{r^{2p}} a \lrcorner_{\mathbb{S}^5} b.$$

Employing the splitting

$$d_{\mathbb{C}^3} = d_0 + \eta \wedge L_\xi + dr \wedge L_{\frac{\partial}{\partial r}}, \tag{170}$$

we find

$$\begin{aligned} d_{\mathbb{C}^3} a_{\mathbb{C}^3} &= d_0 a_0 + (2a_\eta)\frac{d\eta}{2} + \eta \wedge (L_\xi a_0 - d_0 a_\eta) + \frac{dr}{r} \wedge (r\frac{\partial a_0}{\partial r} - d_0 a_r) \\ &\quad + (\frac{dr}{r} \wedge \eta)(r\frac{\partial a_\eta}{\partial r} - L_\xi a_r). \end{aligned} \tag{171}$$

Via the splitting (170), using $\sigma = \frac{u}{r}$, $\underline{a}_s = \frac{a_s}{r}$, we routinely verify the following two identities.

$$d_{\mathbb{C}^3}\sigma = \frac{d_0 u}{r} + \frac{\eta \wedge L_\xi u}{r} + (\frac{\partial u}{\partial r} - \frac{u}{r})\frac{dr}{r};\ d_{\mathbb{C}^3}\underline{a}_s = \frac{d_0 a_s}{r} + \frac{\eta \wedge L_\xi a_s}{r} + (\frac{\partial a_s}{\partial r} - \frac{a_s}{r})\frac{dr}{r}. \tag{172}$$

Employing the table for the forms:

$\omega_{\mathbb{C}^3} = r dr \wedge \eta + \frac{r^2 d\eta}{2}$	
$dVol_{\mathbb{P}^2} = \frac{1}{2}(\frac{d\eta}{2})^2$	
$dVol_{\mathbb{S}^5} = \eta \wedge dVol_{\mathbb{P}^2}$	(173)
$dVol_{\mathbb{C}^3} = r^5 dr \wedge \eta \wedge dVol_{\mathbb{P}^2}$	

via (172), the contraction is

$$(d_{\mathbb{C}^3}\underline{a}_s)\lrcorner_{\mathbb{C}^3}\omega_{\mathbb{C}^3} = \frac{1}{r} d_0 a_s \lrcorner \frac{d\eta}{2} - \frac{L_\xi a_s}{r}\frac{dr}{r} + (\frac{\partial a_s}{\partial r} - \frac{a_s}{r})\eta.$$

Employing the commutators (21) and formula (171) for $d_{\mathbb{C}^3} a_{\mathbb{C}^3}$, we verify

$$\begin{aligned} & d_{\mathbb{C}^3} a_{\mathbb{C}^3} \lrcorner_{\mathbb{C}^3} Re\Omega_{\mathbb{C}^3} \\ = \ & \frac{dr}{r} \cdot \frac{d_0 a_0 \lrcorner H}{r} + \eta \cdot \frac{d_0 a_0 \lrcorner G}{r} + \frac{1}{r}[L_\xi(J_G a_0) + 3J_H(a_0) - J_G(d_0 a_\eta)] + (\frac{\partial}{\partial r} J_H a_0) - \frac{J_H(d_0 a_r)}{r}. \end{aligned} \tag{174}$$

Assembling i (172), (174), (174), we describe row 3 of the operator $\square$.

Formula 17.4. In view of Formula 4.1, on the third row of the operator $\square$, we have

$$\begin{aligned} & d_{\mathbb{C}^3}\sigma - (d_{\mathbb{C}^3}\underline{a}_s)\lrcorner\omega_{\mathbb{C}^3} + d_{\mathbb{C}^3} a_{\mathbb{C}^3} \lrcorner_{\mathbb{C}^3} Re\Omega_{\mathbb{C}^3} \\ = \ & \frac{dr}{r} \cdot [\frac{\partial u}{\partial r} - \frac{u}{r} + \frac{L_\xi a_s}{r} + \frac{d_0 a_0 \lrcorner H}{r}] + \eta \cdot [-\frac{\partial a_s}{\partial r} + \frac{a_s}{r} + \frac{L_\xi u}{r} + \frac{d_0 a_0 \lrcorner G}{r}] \\ & + \frac{1}{r}[d_0 u - J_0(d_0 a_s) + L_\xi(J_G a_0) + 3J_H(a_0) - J_G(d_0 a_\eta) + r\frac{\partial}{\partial r}(J_H a_0) - J_H(d_0 a_r)]. \end{aligned}$$

Next, we calculate the first and second row of $\square$.

Formula 17.5. The following two identities hold.

$$d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3} a_{\mathbb{C}^3} = -\frac{1}{r}\frac{\partial a_r}{\partial r} - \frac{4a_r}{r^2} - \frac{L_\xi a_\eta}{r^2} + \frac{d_0^{\star_0} a_0}{r^2}. \tag{175}$$

$$d_{\mathbb{C}^3} a_{\mathbb{C}^3} \lrcorner_{\mathbb{C}^3} \omega_{\mathbb{C}^3} = \frac{1}{r^2} d_0 a_0 \lrcorner \frac{d\eta}{2} - \frac{L_\xi a_r}{r^2} + \frac{1}{r}\frac{\partial a_\eta}{\partial r} + \frac{4a_\eta}{r^2}. \tag{176}$$

In particular, under the splitting $a_{\mathbb{S}^5} = a_\eta \eta + a_0$,

$$d^{\star_{\mathbb{S}^5}}_{\mathbb{S}^5} a_{\mathbb{S}^5} = -L_\xi a_\eta + d_0^{\star_0} a_0. \tag{177}$$

Proof of Formula 17.5: The volume forms in table (173) imply the following two identities.

$$\star_{\mathbb{C}^3}\eta = -r^3 dr \wedge dVol_{\mathbb{P}^2}, \quad \star_{\mathbb{C}^3} a_0 = r^3 dr \wedge \eta \wedge \star_0 a_0. \tag{178}$$

Since $d\eta$ is a section of $\wedge^{(1,1)} \otimes D^\star$, but G is a section of $[\wedge^{(2,0)} \oplus \wedge^{(0,2)}] \otimes D^\star-$valued, we find the following vanishing

$$(d\eta)\lrcorner(G \wedge \eta) = 0. \tag{179}$$

Using the above 3 elementary identities, we calculate $d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3} a_{\mathbb{C}^3}$ according to the 3−terms in the fine splitting (169). First, we verify

$$d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3}(a_r \frac{dr}{r}) = -\star_{\mathbb{C}^3} d_{\mathbb{C}^3} \star_{\mathbb{C}^3} (a_r \frac{dr}{r}) = -\frac{1}{r}\frac{\partial a_r}{\partial r} - \frac{4a_r}{r^2}.$$

On the contact form component, we verify

$$d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3}(a_\eta \eta) = \star_{\mathbb{C}^3} d_{\mathbb{C}^3}(a_\eta r^3 dr \wedge dVol_{\mathbb{P}^2}) = L_\xi a_\eta \star_{\mathbb{C}^3} (r^3 \eta \wedge dr \wedge dVol_{\mathbb{P}^2}) = -\frac{L_\xi a_\eta}{r^2}.$$

On the semi-basic 1−form a_0, we verify

$$d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3} a_0 = -\star_{\mathbb{C}^3} d_{\mathbb{C}^3}(r^3 dr \wedge \eta \wedge \star_0 a_0) = -\star_{\mathbb{C}^3} (r^3 dr \wedge \eta \wedge d_0 \star_0 a_0) = \frac{d_0^{\star_0} a_0}{r^2}.$$

Identity (175) follows simply by summing up the above 3.

Because $d_{\mathbb{C}^3} a_{\mathbb{C}^3} \lrcorner_{\mathbb{C}^3} \omega_{\mathbb{C}^3} = d^{\star_{\mathbb{C}^3}}_{\mathbb{C}^3} J_{\mathbb{C}^3} a_{\mathbb{C}^3}$, using

$$J_{\mathbb{C}^3}(a_r \frac{dr}{r} + a_\eta \eta + a_0) = a_r \eta - a_\eta \frac{dr}{r} + J_0 a_0,$$

identity (176) follows from (175) replacing a_η by a_r, a_r by $-a_\eta$, and a_0 by $J_0 a_0$ therein. $\square$

The formulas established so far can be assembled into the desired formula of P.

Proof of Lemma 4.3: Still in view of Formula 4.1, it is natural to classify the terms in the fine splitting of $\square$ into 3 kinds of terms: those only involving $\frac{\partial}{\partial r}$ (derivative in r), those only involving L_ξ (derivative along the Reeb vector field), and those only involving d_0.

We carry out the above scheme. Using

- the formula for the isometries K and T in Lemma 4.2,
- Formula 17.5 for the first and second row of $\square$,
- Formula 17.4 for the third row of $\square$,

we find the following fine splitting $\Box = \frac{\partial}{\partial r}K + \frac{L_\xi T}{r} + \frac{\underline{B}_0}{r}$, where

$$\underline{B}_0 \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 & 0 & d_0^{\star 0} \\ 0 & 0 & 0 & 4 & (d_0 \cdot) \lrcorner \frac{d\eta}{2} \\ -1 & 0 & 0 & 0 & (d_0 \cdot) \lrcorner H \\ 0 & 1 & 0 & 0 & (d_0 \cdot) \lrcorner G \\ d_0 & -J_0 d_0 & -J_H d_0 & -J_G d_0 & 3J_H \end{bmatrix}. \tag{180}$$

It is then routine to verify, with the help of the second commutator identity in (21), that $P \triangleq K(\underline{B}_0 + L_\xi T)$ is equal to the one given by (36). Hence

$$\Box = K(\frac{\partial}{\partial r} - \frac{P}{r}). \tag{181}$$

The proof is complete. $\Box$

17.4 Fourier series re-visited

We recall an elementary fact on uniform convergence of the usual Fourier series.

Lemma 17.6. *There exists a positive function $[\epsilon(N),\ N \in \mathbb{Z}^+]$ such that $\lim_{N\to\infty} \epsilon(N) = 0$ and the following holds. Let $f \in W^{1,2}(\mathbb{S}^1)$ and let its Fourier series be $\Sigma_k f_k e^{\sqrt{-1}k\theta}$, then*

$$\Sigma_{|k|\geq N} |f_k e^{\sqrt{-1}k\theta}| \leq \epsilon(N) + \frac{1}{\sqrt{N}} |f|^2_{W^{1,2}(\mathbb{S}^1)}. \tag{182}$$

Proof of Lemma 17.6: Cauchy-Schwartz inequality implies that

$$|f_k e^{\sqrt{-1}k\theta}| \leq \frac{1}{k^{\frac{3}{2}}} + \frac{k^{\frac{3}{2}} f_k^2}{2}. \tag{183}$$

Then, $\xi(N) \triangleq \Sigma_{N\geq 1} \frac{1}{k^{\frac{3}{2}}}$ satisfies the desired conditions. The other term in (183) satisfies $\Sigma_{N\geq 1} k^{\frac{3}{2}} f_k^2 \leq \frac{1}{\sqrt{N}} \Sigma_{N\geq 1} k^2 f_k^2 \leq \frac{1}{\sqrt{N}} |f|^2_{W^{1,2}(\mathbb{S}^1)}$. The desired estimate (182) follows. $\Box$

Under the assumption $f \in W^{1,2}(\mathbb{S}^1)$, it is well known that the Fourier-series converges uniformly to f. Based on the above bound on the remainder, we provide an ingredient for Lemma 6.3.

Lemma 17.7. *In the setting of Lemma 6.3, let $\nu \in C^1(\mathbb{S}^5, \pi^\star_{5,4} EndE)$. Under the pull-back Hermitian metric on $\pi^\star_{5,4} EndE$, for any $\beta = 0, 1,$ or 2, the Sasaki-Fourier Series $\Sigma_k v_\beta(k) e^{\sqrt{-1}k\theta_\beta}$ converges uniformly to ν on $U_{\beta,\mathbb{S}^5}$. The equivalent global series $\Sigma_k \nu_k \otimes s_{-k}$ converges uniformly to ν on $\mathbb{S}^5$.*

Proof of Lemma 17.7: Under the pullback connection from $EndE \to \mathbb{P}^2$, $L_\xi = \nabla_\xi$ on the sections of $\pi^\star_{5,4} EndE$. Thus the C^1-condition implies that $L_\xi \nu \in C^0(\mathbb{S}^5, EndE)$. Because $\xi = \frac{\partial}{\partial\theta_\beta}$ in $U_{\beta,\mathbb{S}^5}$, fixing $u_1,\ u_2$ in the Sasakian coordinate, under a unitary trivialization, $\nu \in W^{1,2}(\theta_\beta)$ (as a function of $\theta_\beta \in \mathbb{S}^1$). The estimate in Lemma 17.6 says that

$$\Sigma_{|k|\geq N} |\nu_\beta(k) e^{\sqrt{-1}k\theta_\beta}|_{\pi^\star_{5,4} EndE} \leq \epsilon(N) + \frac{|\nu|^2_{W^{1,2}(\theta_\beta)}}{\sqrt{N}} \leq \epsilon(N) + \frac{[|\nu|^2_{C^0(\theta_\beta)} + |\nabla_\xi \nu|^2_{C^0(\theta_\beta)}]}{\sqrt{N}}.$$

This means the "remainder" $\Sigma_{|k|\geq N} |\nu_\beta(k) e^{\sqrt{-1}k\theta_\beta}|_{\pi^\star_{5,4} EndE}$ is bounded uniformly in β and $p \in U_{\beta,\mathbb{S}^5}$. The desired uniform convergence is proved. $\Box$

Remark 17.8. Let $(\nu)_{-k}$ denote the $-k-$th term $\nu_k \otimes s_{-k}$ in the Fourier-series. The value of $(\nu)_{-k}$ on an arbitrary Reeb orbit only depends on the value of ν on the same Reeb orbit.

In the setting of Lemma 17.6, let $c(\theta)$ be a smooth function, the operator $c(\theta)\frac{\partial}{\partial\theta}$ in general can not differentiate the Sasaki-Fourierseries terms by term i.e. in general

$$[c(\theta)\frac{\partial f}{\partial\theta}]_k \neq [c(\theta)\frac{\partial}{\partial\theta}]f_k,$$

where the subscript $\cdot_k$ means the $k-$th Fourier-coefficient. The next result shows that this is not the case for the two operators we are interested in.

Claim 17.9. *Still in the setting of Lemma 6.3, for any $\nu \in C^3(\mathbb{S}^5, \pi^\star_{5,4}EndE)$, in view of the notation $(,)_{-k}$ in Remark 17.8 for the Sasaki-Fourier coefficients,*

$$(\nabla^\star\nabla\nu)_k = \nabla^\star\nabla(\nu)_k, \text{ and } (L_\xi\nu)_k = L_\xi(\nu)_k.$$

Proof of Claim 17.9: It suffices to prove the two identities under the local Fourier-Series i.e. the left hand side of (52). We only need to work near each Reeb orbit.

The identity for L_ξ holds because $\xi = \frac{\partial}{\partial\theta_\beta}$ in $U_{\beta,\mathbb{S}^5}$, and the usual Fourier Series in θ_β can be differentiated term by term with respect to θ_β.

To prove the identity for $\nabla^\star\nabla$, for any $[Z] \in \mathbb{P}^2$, we need a transverse geodesic frame $[v_i = \frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})\xi, i = 1,2,3,4]$ near the Reeb orbit $\pi^{-1}_{5,4}[Z]$. Because $\xi[\eta(\frac{\partial}{\partial x_i})] = 0$ i.e. $\eta(\frac{\partial}{\partial x_i})$ is independent of θ_β, and that the connection is also pullback from $\mathbb{P}^2$, for each i, we find

$$(\nabla_{v_i}\nu)_{-k} = (\nabla_{[\frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})\xi]}\nu)_{-k} = \nabla_{[\frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})\xi]}\nu_k = \nabla_{v_i}\nu_k \text{ in the domain of } v_i.$$

Because $\nabla^\star\nabla\nu = \nabla_{v_i}\nabla_{v_i}\nu$ on the Reeb orbit $\pi^{-1}_{5,4}[Z]$, in view of Remark 17.8,

$$\begin{aligned} & (\nabla^\star\nabla\nu)_{-k}|_{\pi^{-1}_{5,4}|[Z]} = (\nabla^\star\nabla\nu|_{\pi^{-1}_{5,4}[Z]})_{-k} = (\nabla_{v_i}\nabla_{v_i}\nu|_{\pi^{-1}_{5,4}[Z]})_{-k} \\ = & (\nabla_{v_i}\nabla_{v_i}\nu)_{-k}|_{\pi^{-1}_{5,4}[Z]} = [\nabla_{v_i}\nabla_{v_i}(\nu)_{-k}]|_{\pi^{-1}_{5,4}[Z]} \\ = & [\nabla^\star\nabla(\nu)_{-k}]|_{\pi^{-1}_{5,4}[Z]}. \end{aligned}$$

The proof is complete. □

17.5 Some algebro-geometric calculations

Let $\omega_{O(1)}$ be $\frac{\sqrt{-1}}{2\pi}$ times the curvature form of the standard metric on $O(1) \to \mathbb{P}^2$. Then $\omega_{O(1)}$ represents $c_1[O(1)]$, and $\frac{d\eta}{2} = \pi\omega_{O(1)}$ (cf. [15, page 142 and 30], watch out the difference of our scaling from the one therein). Throughout this article, we call $\pi\omega_{O(1)}$ ($\frac{d\eta}{2}$) the Fubini-Study form, and denote it by ω_{FS}. The same applies to $\mathbb{P}^n$ as well (still let $\eta \triangleq d^c \log r$, r is the distance to the origin in $\mathbb{C}^{n+1}$).

We need the formula for the sheaf cohomologies for Theorem A and 9.2.

Lemma 17.10. *Let E be a holomorphic Hermitian vector bundle on $\mathbb{P}^2$. For any integer k, we have*

$$\begin{aligned} h^1[\mathbb{P}^2, (EndE)(k)] &= h^0[\mathbb{P}^2, (EndE)(k)] + h^0[\mathbb{P}^2, (EndE)(-k-3)] \\ & \quad +2rc_2(E) - (r-1)c_1^2(E) - \frac{r^2(k+1)(k+2)}{2}. \end{aligned}$$

Proof of Lemma 17.10: Since $c_1(EndE) = 0$, by [21, II, (1.10)], we compute

$$\begin{aligned} & ch[\mathbb{P}^2, (EndE)(k)] = ch[\mathbb{P}^2, O(k)] \cdot ch[\mathbb{P}^2, EndE] \qquad (184) \\ = & \{1 + c_1[O(k)] + \frac{c_1^2[O(k)]}{2}\}\{r^2 + c_1(EndE) + \frac{1}{2}[c_1^2(EndE) - 2c_2(EndE)]\}. \\ = & r^2 + kr^2[\omega_{O(1)}] - c_2(EndE) + \frac{r^2k^2}{2}[\omega_{O(1)}]^2. \end{aligned}$$

The well known formula for Todd class states (for example, see [21, page 288]):

$$Td(\mathbb{P}^2) = 1 + \frac{3[\omega_{O(1)}]}{2} + [\omega_{O(1)}]^2. \qquad (185)$$

We compute

$$\begin{aligned} & Td(\mathbb{P}^2) \cdot ch[(EndE)(k)] \\ = & r^2 + (\frac{3r^2}{2} + kr^2)[\omega_{O(1)}] - c_2(EndE) + [\frac{r^2k^2}{2} + r^2 + \frac{3kr^2}{2}][\omega_{O(1)}]^2. \qquad (186) \end{aligned}$$

Because $\int_{\mathbb{P}^2}[\omega_{O(1)}]^2 = \int_{\mathbb{P}^2}\{c_1[O(1)]\}^2 = 1$, using Hirzebruch Riemann-Roch theorem, we integrate (186) over $\mathbb{P}^2$ to obtain

$$\chi[\mathbb{P}^2,\ (EndE)(k)] = \int_{\mathbb{P}^2} Td(\mathbb{P}^2) \cdot ch[\mathbb{P}^2, (EndE)(k)] = \frac{r^2k^2}{2} + r^2 + \frac{3kr^2}{2} - c_2(EndE).$$

$\square$

Proof of Lemma 14.2: We only prove the formula for $h^1[\mathbb{P}^2,\ (EndT'\mathbb{P}^2)(l)]$, the formula for $h^0[\mathbb{P}^2,\ \{End_0(T'\mathbb{P}^2)\}(l)]$ thereupon follows by Riemann-Roch (see Lemma 17.10).

On $\mathbb{P}^2$, we tensor the Euler-Sequence

$$0 \to O \to O^{\oplus 3}(1) \to T'\mathbb{P}^2 \to 0$$

by the sheaf $\Omega^1(l)$, the local freeness of $\Omega^1(l)$ yields the exactness of the following.

$$0 \to \Omega^1(l) \to [\Omega^1(l+1)]^{\oplus 3} \to (EndT'\mathbb{P}^2)(l) \to 0. \qquad (187)$$

Hence we have the following exact sequence of cohomologies

$$... \to H^1[\mathbb{P}^2, \Omega^1(l+1)]^{\oplus 3}] \to H^1[\mathbb{P}^2, (EndT'\mathbb{P}^2)(l)] \to H^2[\mathbb{P}^2, \Omega^1(l)] \to ... \qquad (188)$$

By Bott formula of sheaf cohomology on complex projective spaces (see [31, Section 1.1]), when $l \geq 0$, both $H^1[\mathbb{P}^2, \Omega^1(l+1)]^{\oplus 3}]$ and $H^2[\mathbb{P}^2, \Omega^1(l)]$ vanish. Then $H^1[\mathbb{P}^2, (EndT'\mathbb{P}^2)(l)]$ vanishes if $l \geq 0$.

When $l = -1$, $H^1[\mathbb{P}^2, (\Omega^1)^{\oplus 3}] = \{H^1[\mathbb{P}^2, \Omega^1]\}^{\oplus 3} = \mathbb{C}^3$, $H^2[\mathbb{P}^2, \Omega^1(-1)] = 0$. Thus $H^1[\mathbb{P}^2, (EndT'\mathbb{P}^2)(-1)] = \mathbb{C}^3$. By Serre-duality, we find $H^1[\mathbb{P}^2, (EndT'\mathbb{P}^2)(-2)] = \mathbb{C}^3$ and $H^1[\mathbb{P}^2, (EndT'\mathbb{P}^2)(l)] = 0$ if $l \leq -3$. $\square$

17.6 Kähler identity for vector bundles

The usual Kähler identity says that on a Kähler manifold, the Laplace-Beltrami operator (on functions) is twice of the $\bar{\partial}-$Laplacian. The Lemma below is a straight-forward generalization to bundle case. Though we do not know whether it is stated explicitly in literature, the proof is completely routine. Please see a related calculation in [21, III.1].

Lemma 17.11. *Let Ξ be a holomorphic Hermitian vector bundle over a Kähler manifold (X,ω) and A denote the Chern connection. Then*

$$\nabla_A^\star\nabla_A = 2\partial_A^\star\bar{\partial}_A + 2\pi\cdot\frac{\sqrt{-1}}{2\pi}F_A\lrcorner\omega. \tag{189}$$

Consequently, in the setting of the first sentence in Theorem A and the convention for the Kähler metric in the first paragraph of Appendix 17.5 (above the proof of Lemma 17.10), we consider the Fubini-Study form ω_{FS}. On the twisted endomorphism bundle $(EndE)(l)$, under the tensor product of A and the standard connection on $O(l)$ (the twisted connection), suppressing the subscripts for the connection as usual, we have

$$\nabla^\star\nabla = 2\partial^\star\bar{\partial} + 2nl\cdot Id. \tag{190}$$

In particular, when $n=2$,

$$\nabla^\star\nabla = 2\partial^\star\bar{\partial} + 4l\cdot Id. \tag{191}$$

Proof of Lemma 17.11: At an arbitrary point $p\in X$, let $(z_j,\ j=1,..,n)$ be a Kähler geodesic coordinate for the metric ω. By definition, we have for any section φ of Ξ that

$$\nabla^\star\nabla\varphi = -2\Sigma_j(\varphi_{j\bar{j}}+\varphi_{\bar{j}j}) = -4\Sigma_j\varphi_{\bar{j}j} + 2\Sigma_j F_{A,j\bar{j}}\cdot\varphi = 2\partial_A^\star\bar{\partial}_A\varphi + 2\Sigma_j F_{A,j\bar{j}}\cdot\varphi \text{ at } p. \tag{192}$$

Please compare it to the usual Kähler identity in [15, Chap 0.7, page 106]. To complete the proof of (189), it suffices to observe that $\Sigma_j F_{A,j\bar{j}} = \frac{\sqrt{-1}}{2}F_A\lrcorner\omega$.

To prove (190), based on (189), we contract the following by ω_{FS}.

$$F_A = [F_E,\cdot]\otimes Id_{O(l)} + Id_E\otimes F_{O(l)}. \tag{193}$$

The Hermitian Yang-Mills condition says that $[F_E\lrcorner\omega_{FS},\cdot]$ acts by $0-$endomorphism on $EndE$, using $c_1[O(l)] = [\omega_{O(1)}]$, the following holds as endomorphisms on $(EndE)(l)$.

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}F_A\lrcorner\omega_{FS} &= \frac{\sqrt{-1}}{2\pi}Id_E\otimes(F_{O(l)}\lrcorner\omega_{FS}) = nId\Big(\frac{\int_{\mathbb{P}^n}c_1[O(l)]\wedge\omega_{FS}^{n-1}}{\int_{\mathbb{P}^n}\omega_{FS}^n}\Big) = \frac{(nl)Id}{\pi}\Big(\frac{\int_{\mathbb{P}^n}\omega_{O(1)}^n}{\int_{\mathbb{P}^n}\omega_{O(1)}^n}\Big)\\ &= \frac{(nl)Id}{\pi}.\end{aligned}$$

We should notice that the π factor in $\omega_{FS}=\pi\omega_{O(1)}$ produces the "π" in the denominator of the last line above. The proof of (190) is complete. □

The above Kähler identity relates the space of holomorphic sections to a certain eigenspace of the rough Laplacian.

Lemma 17.12. *(A holomorphic section is an eigensection of the rough Laplacian) In the setting of the first sentence in Theorem A, for any nonnegative integer l,*

$$\mathbb{E}_{4l}\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2} = H^0[\mathbb{P}^2,(End_0E)(l)]. \tag{194}$$

Moreover, the isomorphism "$=$" above is an actual equality: a holomorphic section of $(End_0E)(l)$ is an eigensection of $\nabla^\star\nabla|_{(End_0E)(l)\to\mathbb{P}^2}$ with respect to the eigenvalue $4l$, and vice versa.

The proof is straight-forward by formula (191).

17.7 On Killing reductive homogeneous spaces

Proof of Lemma 11.3: For any $V, X, Y \in \mathfrak{g}$, the usual Koszul formula [32, page 25] says

$$\begin{aligned} 2\langle \nabla_{V^\star} X^\star,\ Y^\star\rangle &= V^\star\langle X^\star,\ Y^\star\rangle - Y^\star\langle V^\star,\ X^\star\rangle + X^\star\langle Y^\star,\ V^\star\rangle \\ &\quad + \langle [V^\star, X^\star],\ Y^\star\rangle - \langle [X^\star, Y^\star], V^\star\rangle + \langle [Y^\star, V^\star], X^\star\rangle. \end{aligned} \tag{195}$$

Because $V^\star,\ X^\star,\ Y^\star$ are Killing vector fields, we find

$$\begin{aligned} V^\star\langle X^\star,\ Y^\star\rangle &= \langle [V^\star, X^\star],\ Y^\star\rangle + \langle X^\star,\ [V^\star, Y^\star]\rangle, \\ Y^\star\langle V^\star,\ X^\star\rangle &= \langle [Y^\star, V^\star],\ X^\star\rangle + \langle V^\star,\ [Y^\star, X^\star]\rangle, \\ X^\star\langle Y^\star,\ V^\star\rangle &= \langle [X^\star, Y^\star],\ V^\star\rangle + \langle Y^\star,\ [X^\star, V^\star]\rangle. \end{aligned}$$

Plugging the above into (195), we find

$$2\langle \nabla_{V^\star} X^\star,\ Y^\star\rangle = \langle [V^\star, X^\star],\ Y^\star\rangle - \{\langle [X^\star, [Y^\star,\ V^\star]\rangle + \langle V^\star,\ [Y^\star, X^\star]\rangle\}. \tag{196}$$

Next, for any $V, X, Y \in \mathfrak{m}$, we show that the condition of Killing homogeneous space implies

$$\langle X^\star, [Y^\star,\ V^\star]\rangle + \langle V^\star,\ [Y^\star, X^\star]\rangle = 0 \text{ at } eK. \tag{197}$$

[23, Proposition 2.1] says that $[X^\star, Y^\star] = -[X, Y]^\star$ at eK for any $X,\ Y \in \mathfrak{g}$. Then at eK,

$$\begin{aligned} & \langle X^\star, [Y^\star,\ V^\star]\rangle + \langle V^\star,\ [Y^\star, X^\star]\rangle = -\langle X^\star, [Y,\ V]^\star\rangle - \langle V^\star,\ [Y, X]^\star\rangle \\ = \ & -\langle X, [[Y,\ V]]_{\mathfrak{m}}\rangle_{\mathfrak{m}} - \langle V,\ [[Y, X]]_{\mathfrak{m}}\rangle_{\mathfrak{m}} \\ = \ & -\langle X, [Y,\ V]\rangle_{\mathfrak{g}} - \langle V,\ [Y, X]\rangle_{\mathfrak{g}} \text{ (because } X,\ Y,\ V \in \mathfrak{m}, \text{ and } \mathfrak{m} \perp \mathfrak{k}) \\ = \ & 0 \qquad \text{(because } \langle , \rangle_{\mathfrak{g}} \text{ is a scalar multiple of the Killing form).} \end{aligned} \tag{198}$$

In row 2 of (198), the inner bracket $[Y, V]$ means the Lie bracket, while the outer means the projection to $\mathfrak{m}$ according to the reductive splitting. The identity (197) is proved.

For any $V, X \in \mathfrak{m}$, plugging (197) back into (196), because $Y \in \mathfrak{m}$ is also arbitrary, we find $\nabla_{V^\star} X^\star = \frac{1}{2}[V^\star, X^\star]$ at eK. Therefore, for any $V \in \mathfrak{m}$, $\nabla_{V^\star} V^\star = 0$ at eK. Because g acts as an isometry (thus it preserves the Levi-Civita connection), equation (122) holds at gK. □

17.8 The standard connection on $O(l) \to \mathbb{P}^2$: proof of Lemma 12.1

To prove Lemma 12.1, we need the $K-$invariant function corresponding to the local defining section of $O(-1)$.

Lemma 17.13. *In $U_{0,\mathbb{P}^2} = \{[Z_0, Z_1, Z_2] \in \mathbb{P}^2 | Z_0 \neq 0\}$, the trivialization $s_0 \triangleq (1, u_1, u_2)$ of $O(-1)$ corresponds to the Span$[1,0,0]-$valued function $\alpha = (\frac{1}{g_1^1}, 0, 0)$ on $SU(3)$, where g_{11} is the $(1,1)-$entry of $g \in SU(3)$. This means $s_0([g]) = (g, \alpha)$ for all $g \in \pi^{-1}U_{0,\mathbb{P}^2}$.*

Remark 17.14. α is obviously $S[U(1) \times U(2)]-$invariant i.e. $\alpha(gk) = k^{-1}\alpha(g)$ for any $k \in S[U(1) \times U(2)]$. Moreover, $U_{0,\mathbb{P}^2}$ is invariant under the action of $S[U(1) \times U(2)]$.

Proof of Lemma 17.13: For any $g \in SU(3)$, it suffices to compute at $g \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g_1^1 \\ g_1^2 \\ g_1^3 \end{bmatrix}$ that

$$(g, \alpha) = g \cdot \begin{bmatrix} \frac{1}{g_1^1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g_1^1 & g_2^1 & g_3^1 \\ g_1^2 & g_2^2 & g_3^2 \\ g_1^3 & g_2^3 & g_3^3 \end{bmatrix} \begin{bmatrix} \frac{1}{g_1^1} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{g_1^2}{g_1^1} \\ \frac{g_1^3}{g_1^1} \end{bmatrix} \triangleq \begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix}. \qquad \square$$

***Proof of Lemma** 12.1:* On the universal bundle, the induced connection $d_{induced}$ is $SU(3)$ invariant, so is the standard connection d_{Chern}. It suffices to verify that they coincide at the base point $o \in \mathbb{P}^2$. The standard connection on $O(-1)$ yields that $d_{Chern}s_0 = (\partial \log \phi_0)s_0$. Consequently, by the Kähler potential ϕ_0 above (10), we find $d_{Chern}s_0 = 0$ at o. All the elements in $\mathfrak{m}_{\mathbb{P}^2}$ have vanishing $(1,1)-$entry (see (130)). Therefore, for any $X \in \mathfrak{m}_{\mathbb{P}^2}$, $\alpha(e^{tX}) = \begin{bmatrix} 1+O(t^2) \\ 0 \\ 0 \end{bmatrix}$. This implies $X(\alpha) = 0$ at $e \in SU(3)$. Lemma 17.13 means that $s_0 = (g, \alpha)$. Because $\mathfrak{m}_{\mathbb{P}^2}$ is horizontal, we find $d_{induced,X}s_0 = 0$ at o for any $X \in \mathfrak{m}_{\mathbb{P}^2}$ as well. Thus, the induced connection coincides with the Chern connection at the base point. $\square$

17.9 The horizontal distribution of the Fubini-Study connection on the holomorphic tangent bundle: proof of Lemma 12.2

For any $X \in \mathfrak{m}_{\mathbb{P}^2}$, let $X^{\star,\mathbb{C}}$ be the projection of the real vector field $X^\star$ to $T'\mathbb{P}^2$ (see the material from (131) to (132) for the projection, and see (119) for the definition of $X^\star$). To prove Lemma 12.2, we need another form for the vector fields $X_1^{\star,\mathbb{C}}$, $Y_1^{\star,\mathbb{C}}$, $X_3^{\star,\mathbb{C}}$, $Y_3^{\star,\mathbb{C}}$.

***Formula** 17.15.* In view of the basis (130) of $\mathfrak{m}_{\mathbb{P}^2}$,

$$X_1^{\star,\mathbb{C}} = \pi_{5,4,\star}(Z_1\frac{\partial}{\partial Z_0} - Z_0\frac{\partial}{\partial Z_1}). \text{ In } U_{0,\mathbb{P}^2},\ X_1^{\star,\mathbb{C}} = -(1+u_1^2)\frac{\partial}{\partial u_1} - u_1u_2\frac{\partial}{\partial u_2}.$$
$$Y_1^{\star,\mathbb{C}} = \sqrt{-1}\pi_{5,4,\star}(Z_1\frac{\partial}{\partial Z_0} + Z_0\frac{\partial}{\partial Z_1}). \text{ In } U_{0,\mathbb{P}^2},\ Y_1^{\star,\mathbb{C}} = \sqrt{-1}(1-u_1^2)\frac{\partial}{\partial u_1} - \sqrt{-1}u_1u_2\frac{\partial}{\partial u_2}.$$
$$X_3^{\star,\mathbb{C}} = \pi_{5,4,\star}(Z_2\frac{\partial}{\partial Z_0} - Z_0\frac{\partial}{\partial Z_2}). \text{ In } U_{0,\mathbb{P}^2},\ X_3^{\star,\mathbb{C}} = -u_1u_2\frac{\partial}{\partial u_1} - (1+u_2^2)\frac{\partial}{\partial u_2}.$$
$$Y_3^{\star,\mathbb{C}} = \sqrt{-1}\pi_{5,4,\star}(Z_2\frac{\partial}{\partial Z_0} + Z_0\frac{\partial}{\partial Z_2}). \text{ In } U_{0,\mathbb{P}^2},\ Y_3^{\star,\mathbb{C}} = -\sqrt{-1}u_1u_2\frac{\partial}{\partial u_1} + \sqrt{-1}(1-u_2^2)\frac{\partial}{\partial u_2}.$$

Consequently, $s_1^\star = -\pi_{5,4,\star}(Z_0\frac{\partial}{\partial Z_1})$, $s_2^\star = -\pi_{5,4,\star}(Z_0\frac{\partial}{\partial Z_2})$. In $U_{0,\mathbb{P}^2}$, $s_1^\star = -\frac{\partial}{\partial u_1}$, $\quad s_2^\star = -\frac{\partial}{\partial u_2}$.

Proof of Formula 17.15: We verify that $e^{tX_1} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus, when t is sufficiently small with respect to u_1, the following holds on $\mathbb{P}^2$.

$$e^{tX_1}\begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos t + (\sin t)u_1 \\ -\sin t + (\cos t)u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-\sin t + (\cos t)u_1}{\cos t + (\sin t)u_1} \\ \frac{u_2}{\cos t + (\sin t)u_1} \end{bmatrix}. \tag{199}$$

Then the identity

$$X_1^{\star,\mathbb{C}} = \frac{d}{dt}|_{t=0}e^{tX_1}\begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -(1+u_1^2) \\ -u_1u_2 \end{bmatrix} = -(1+u_1^2)\frac{\partial}{\partial u_1} - u_1u_2\frac{\partial}{\partial u_2}$$
$$= \quad -\pi_{5,4,\star}(Z_0\frac{\partial}{\partial Z_1}) + \pi_{5,4,\star}(Z_1\frac{\partial}{\partial Z_0}) \tag{200}$$

holds in $U_{0,\mathbb{P}^2}$. While the vector "$\begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix}$" above means a point in "$U_{0,\mathbb{P}^2} \subset \mathbb{P}^2$, the vector "$\begin{bmatrix} 0 \\ -(1+u_1^2) \\ -u_1u_2 \end{bmatrix}$" above means a $(1,0)$ tangent vector (at the point).

By continuity of both $X_1^{\star,\mathbb{C}}$ and $-\pi_{5,4,\star}(Z_0\frac{\partial}{\partial Z_1})+\pi_{5,4,\star}(Z_1\frac{\partial}{\partial Z_0})$, they are identical everywhere on $\mathbb{P}^2$. Employing the following identities of matrix exponentials,

$$e^{tY_1}=\begin{bmatrix}\cos t & \sqrt{-1}\sin t & 0\\ \sqrt{-1}\sin t & \cos t & 0\\ 0 & 0 & 1\end{bmatrix},\ e^{tX_3}=\begin{bmatrix}\cos t & 0 & \sin t\\ 0 & 1 & 0\\ -\sin t & 0 & \cos t\end{bmatrix},$$
$$e^{tY_3}=\begin{bmatrix}\cos t & 0 & \sqrt{-1}\sin t\\ 0 & 1 & 0\\ \sqrt{-1}\sin t & 0 & \cos t\end{bmatrix}, \tag{201}$$

similar computations as (199) and (200) show that in $U_{0,\mathbb{P}^2}$,

$$Y_1^{\star,\mathbb{C}}=\sqrt{-1}(1-u_1^2)\frac{\partial}{\partial u_1}-\sqrt{-1}u_1u_2\frac{\partial}{\partial u_2},\ X_3^{\star,\mathbb{C}}=-u_1u_2\frac{\partial}{\partial u_1}-(1+u_2^2)\frac{\partial}{\partial u_2},$$
$$Y_3^{\star,\mathbb{C}}=-\sqrt{-1}u_1u_2\frac{\partial}{\partial u_1}+\sqrt{-1}(1-u_2^2)\frac{\partial}{\partial u_2}. \tag{202}$$

As below (200), the 3 formulas respectively for $Y_1^{\star,\mathbb{C}},X_3^{\star,\mathbb{C}},Y_3^{\star,\mathbb{C}}$ follow by continuity. $\square$

Proof of Lemma *12.2:* Similarly to the proof of Lemma 12.1, because both connections are left invariant, it suffices to show that they are identical at the base point o.

The Fubini-Study co-variant derivatives of both $\frac{\partial}{\partial u_1}$ and $\frac{\partial}{\partial u_2}$ are 0 at o. Using Formula 17.15, we find

$$\nabla^{FS}s_1^\star=\nabla^{FS}s_2^\star=0\ \text{ at } o. \tag{203}$$

In view of the correspondence in Lemma 11.6, at $e\in SU(3)$, for any $X,Y\in\mathfrak{m}_{\mathbb{P}^2}$, we compute the ordinary derivative

$$[Y\widetilde{\mathfrak{m}}_{\mathbb{P}^2}(X)](e)=-[[Y,X]]_{\mathfrak{m}_{\mathbb{P}^2}}.$$

On the right hand side of the above, the inner bracket is the Lie bracket, the outer one is the projection to $\mathfrak{m}_{\mathbb{P}^2}$.

We straight-forwardly verify $[\mathfrak{m}_{\mathbb{P}^2},\mathfrak{m}_{\mathbb{P}^2}]\subseteq s[u(1)\times u(2)]$. Then $[Y\widetilde{\mathfrak{m}}_{\mathbb{P}^2}(X)](e)=0$. The correspondence (124) and Kobayashi-Nomizu formula (126) again yields that

$$(\nabla_Y^{induced}X^\star)|_o=0.$$

On the complexification, this means for any $s\in\mathfrak{m}_{\mathbb{P}^2}^{(1,0)}$, $\nabla^{induced}s^\star=0$ at o.

Then $\nabla^{induced}$ coincides with ∇^{FS} at the base point o. By $SU(3)-$invariance, they coincide everywhere on $\mathbb{P}^2$. $\square$

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